

Hamiltonian-Based Energy Approach for Nonlinear Free Vibration of Supported Rectangular Plates

Yaser Shahbazi

y.shahbazi@tabriziau.ac.ir

Tabriz Islamic Art University

Mohsen Mokhtari Kashavar

Sahand University of Technology

Siamak Pedrammehr

Tabriz Islamic Art University

Research Article

Keywords: Nonlinear vibration, Rectangular plates, Hamiltonian mechanics, Duffing oscillator, Frequency–amplitude relationship, von Kármán equations

Posted Date: September 23rd, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10483207/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Additional Declarations: No competing interests reported.

Hamiltonian-Based Energy Approach for Nonlinear Free Vibration of Supported Rectangular Plates

Yaser Shahbazi¹, and Mohsen Mokhtari Kashavar¹, and Siamak Pedrammehr²

1 Faculty of Architecture and Urbanism, Tabriz Islamic Art University, Tabriz 5164736931, Iran.

2 Faculty of Design, Tabriz Islamic Art University, Tabriz 5164736931, Iran.

* **Correspondence:** Yaser Shahbazi, Mohsen Mokhtari Kashavar, Siamak Pedrammehr

Yaser Shahbazi	PhD	y.shahbazi@tabriziau.ac.ir	https://orcid.org/0000-0001-9454-1729
Mohsen Mokhtari Kashavar	M.Sc.	mo.mokhtari@tabriziau.ac.ir	https://orcid.org/0000-0003-2157-4495
Siamak Pedrammehr	PhD	s.pedrammehr@tabriziau.ac.ir	https://orcid.org/0000-0002-2974-1801

Abstract

This study presents a compact, energy-based analytical framework for modeling the nonlinear free vibration behavior of supported rectangular plates undergoing large deflections. Starting from the geometrically nonlinear von Kármán equations, we reformulate the system using Hamilton's principle and reduce it to a single-degree-of-freedom Duffing-type oscillator via a sine-based mode truncation. The resulting Hamiltonian formulation captures amplitude-dependent frequency shifts without relying on iterative procedures or large matrix computations. Successive-order Hamiltonians are derived up to third order, yielding closed-form expressions for the nonlinear natural frequency as a function of vibration amplitude. Results demonstrate rapid convergence and excellent agreement with benchmark numerical and experimental data, with errors falling below 1% for third-order corrections. The method's symbolic automation and analytical clarity make it highly suitable for engineering applications requiring efficient yet accurate predictions of nonlinear vibration behavior. While the

current model assumes isotropy and single-mode behavior under supported conditions, it provides a foundation for future extensions to multimode, anisotropic, or forced vibration scenarios.

Keywords: Nonlinear vibration, Rectangular plates, Hamiltonian mechanics, Duffing oscillator, Frequency–amplitude relationship, von Kármán equations.

1 Introduction

In modern engineering design, thin-walled structures such as plates are widely used in aerospace, civil, and mechanical systems due to their high strength-to-weight ratio and structural efficiency. Accurately predicting their dynamic behavior, particularly under significant amplitude vibrations, is critical for ensuring stability, performance, and safety in real-world applications such as aircraft panels, satellite components, and flexible mechanical systems.

Understanding plate nonlinear vibrations is vital because linear models often fail to capture the stiffening effects at high deflections. These nonlinearities lead to amplitude-dependent frequency shifts, resulting in critical components' resonance, fatigue, or unexpected dynamic responses.

1.1 Specific Problem Statement

However, existing linear vibration theories break down for large deflections where mid-plane stretching becomes significant. While worthwhile, classical methods such as Galerkin and Ritz approximations often require cumbersome mode coupling and iterative computation, especially when considering higher-order nonlinear effects. This requires a more compact, energy-based modeling strategy to capture nonlinear frequency shifts without excessive computational overhead accurately.

1.2 Literature Gap & Shortcomings of Existing Solutions

Prior research has explored various analytical methods to approximate solutions to the nonlinear von Kármán plate equations, including Fourier series expansions, Galerkin projection, Ritz's energy method, and iterative linearization. While these methods reduce the governing PDEs to a solvable form, they often involve complex symbolic algebra, large coupling matrices, or slow convergence at high amplitudes. Recent techniques like homotopy perturbation and variational iteration offer improvements but remain computationally intensive and limited in accuracy for moderately large vibrations. Recent advancements in computational methods include the hierarchical finite element method (HFEM) by Han and Petyt (Han and Petyt, 1997), which employs high-order polynomial shape functions to model geometrically nonlinear vibrations of clamped rectangular plates. Their approach reduces degrees of

freedom (DOF) significantly compared to conventional FEM while maintaining accuracy in predicting nonlinear frequencies and stress distributions. However, their reliance on harmonic balance and numerical discretization contrasts our fully analytical, energy-based Hamiltonian framework. Recent advancements in modeling nonlinear vibrations in rectangular plates have increasingly focused on analytical and semi-analytical methods capable of capturing amplitude-dependent behavior without relying on computationally intensive numerical simulations. These techniques are essential in aerospace, mechanical, and civil engineering, where components often undergo large deflections due to operational loads or environmental conditions. A large body of recent research has utilized Hamilton's principle to derive the governing equations of nonlinear vibration. For example, Jiang et al. (2023) modeled the dynamics of functionally graded graphene-reinforced plates under combined parametric and thermal excitation using Hamilton's principle, von Kármán theory, and Galerkin projection (Jiang et al., 2024). Similarly, Zhang and Li (2025) used Hamiltonian energy methods to analyze rotating sandwich plates, illustrating how geometric and material gradation affect nonlinear frequencies (Zhang and Li, 2025). Ritz-based techniques have also been widely adopted. Kumar (2023) applied the Ritz method for modal analysis of orthotropic FRP plates under complex boundary conditions, noting its ability to resolve high-mode frequency predictions efficiently (Dileep Kumar, 2023). Similarly, Honda et al. (2023) combined Ritz approximations with experimental modal analysis for plates with point masses, achieving excellent correlation between theory and experiment (Honda et al., 2023). Homotopy perturbation and variational iteration methods offer another path for handling nonlinearity. Gu and Shi (2024) used a hybrid Galerkin-HPM approach to model magnetic-field-induced nonlinearities in conductive elastic plates, demonstrating rapid convergence and good accuracy (Gu and Shi, 2024). These methods are often coupled with Galerkin projection for initial discretization. Galerkin-based techniques remain central to many analyses. Wu et al. (2024) studied detached resonance curves and bifurcation phenomena in axially moving CNT-reinforced plates using Galerkin and harmonic balance methods (Wu et al., 2024). Chen and Ahmed (2024) employed Galerkin methods for nonlinear modal analysis of sandwich plates with honeycomb cores and graphene nanoplatelets, capturing frequency shifts due to dispersion and initial stress states (Chen & Ahmed, 2024). New hybrid or extended methods are also being developed. Guo et al. (2023) introduced an extended Rayleigh–Ritz method for nonlinear modal analysis of laminated graphene-reinforced plates with damping (Guo et al., 2023). González and Vescovini (2023) proposed a reduced-order mixed Ritz-DQ-HB framework for variable stiffness plates, reducing computation time without sacrificing accuracy (Andrés González and Vescovini, 2023). Advanced control strategies are also being embedded into vibration models. Jiang et al. (2023) and Jiang et al. (2024) used PPF and nonlinear

PPF control strategies, respectively, to actively reduce amplitude in piezoelectric graphene-reinforced laminated plates under internal resonance (Jiang et al., 2023, 2024). Other recent works have explored resonance behaviors under coupled physical fields. Xie and Hu (2024) and Li et al. (2024) investigated combination resonance in magnetoelastic plates and axially moving piezoelectric plates, respectively, integrating Galerkin with multi-scale analysis and bifurcation theory (Li et al., 2024; Xie and Hu, 2024). Thermomechanical and porosity effects have also received attention. Anh et al. (2023) modeled porous functionally graded materials under pre-stress and thermal loads, finding that pore distribution significantly influenced nonlinearities. (Anh et al., 2024) Bidgoli and Arefi (2023) evaluated nonlinear frequency shifts in honeycomb-core GNP sandwiches using detailed parametric studies. (Mohammad-Rezaei Bidgoli and Arefi, 2023). Emerging methods also emphasize structural optimization. Moradi and Shaterzadeh (2024) introduced lattice-core topology as a design variable affecting natural frequency in composite sandwich plates, confirming results via ABAQUS (Moradi and Shaterzadeh, 2025). Lastly, multiple studies validated theoretical predictions using experiments or FEM tools. Wang et al. (2023) demonstrated chaotic response of conductive plates using Hamiltonian–Melnikov analysis, further substantiated via simulations (Wang et al., 2023). Narita (2023) compared FEM and Ritz predictions for free-edge-spring-supported plates, reinforcing confidence in analytical frameworks (Narita, 2023). This growing body of work collectively supports the shift toward energy-based, symbolically driven, and highly accurate nonlinear vibration models for complex plate systems. Crucially, many of these approaches still rely on approximate mode coupling, lack closed-form frequency–amplitude relationships, or require iterative solvers to extract nonlinear frequency corrections. This has limited their practical use in design settings where speed and clarity are essential.

1.3 Objectives & Scope of Your Work

This paper develops a Hamiltonian-based energy formulation for nonlinear vibration analysis of supported rectangular plates. Using a variational approach, we reduce the von Kármán plate equations to a single-mode, single-degree-of-freedom Duffing-type oscillator, deriving closed-form, amplitude-dependent frequency corrections up to the third order.

We restrict our analysis to rectangular plates with supported edges and assume a single dominant mode shape governed by sinusoidal basis functions. This allows us to bypass the need for iterative or numerically intensive procedures while retaining high accuracy across various deflection amplitudes.

1.4 Key Contributions

- 1 We propose a Hamiltonian formulation that transforms the nonlinear plate vibration problem into a conservative dynamical system with a single generalized coordinate.
- 2 We demonstrate that successive-order Hamiltonians yield compact, closed-form expressions for amplitude-dependent frequencies with rapid convergence.
- 3 We validate our approach against existing benchmark results and finite-element simulations, showing percent-level accuracy for moderately large deflections.
- 4 We automate symbolic derivations using computer algebra systems, enabling efficient implementation and extension to higher-order models.
- 5 We provide a generalizable framework for analytically tractable and practically useful nonlinear modal analysis in thin-walled structures.

2 Methodology

2.1 Governing Equations of Nonlinear Plate Vibration

In large deflection problems, the plate's in-plane (membrane) forces become comparable to its bending stiffness, so classical linear theory no longer applies. Von Kármán's geometrically nonlinear plate theory, first formulated in 1910, incorporates both bending and mid-plane stretching by assuming that normals to the mid-surface remain straight but not necessarily of constant length (the Kirchhoff–Love hypothesis) and neglecting transverse normal stresses and in-plane thickness variations.

2.1.1 Kinematic Relations and Strain–Displacement Laws

Under the Kirchhoff–Love assumptions, the total Green–Lagrange strains on the mid-surface in Cartesian coordinates (x, y) are:

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \quad (1)$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}, \quad (3)$$

where $u(x, y, t)$, $v(x, y, t)$ are the in-plane displacements and $w(x, y, t)$ the transverse deflection. The quadratic terms in ∂w capture the geometric nonlinearity due to large transverse rotations.

2.1.2 Stress Resultants and Airy Stress Function

Integrating stresses through the thickness h defines the membrane resultants

$$N_x = \int_{-h/2}^{h/2} \sigma_x \, dz, \quad N_y = \int_{-h/2}^{h/2} \sigma_y \, dz, \quad N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz \quad (4)$$

To enforce compatibility automatically, we introduce an Airy stress function $\Phi(x, y, t)$ such that

$$N_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad N_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}, \quad (5)$$

This ensures equilibrium of in-plane forces identically.

2.1.3 Governing von Kármán Equations

Combining transverse force equilibrium with plate bending stiffness $D = Eh^3/[12(1 - \nu^2)]$, and enforcing in-plane compatibility yields the celebrated von Kármán system for free vibration:

$$D \nabla^4 w - [w, \Phi] + \rho h \ddot{w} = 0 \quad (6)$$

$$\nabla^4 \Phi + \frac{Eh}{2} [w, w] = 0 \quad (7)$$

where ρ is mass density and the bilinear Poisson bracket

$$[f, g] = f_{,xx} g_{,yy} + f_{,yy} g_{,xx} - 2 f_{,xy} g_{,xy}, \quad (8)$$

encodes the coupling between bending and stretching. In more compact notation, one often writes

$$\nabla^4 w - \frac{1}{D} [w, \Phi] + \frac{D}{\rho h} \ddot{w} = 0, \quad \nabla^4 \Phi + \frac{2D}{Eh} [w, w] = 0 \quad (9)$$

2.1.4 Boundary Conditions

For a supported rectangular plate on $x = 0, a$ and $y = 0, b$, the kinematic and stress conditions are

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0, \quad \Phi = 0, \quad \frac{\partial^2 \Phi}{\partial n^2} = 0 \quad (10)$$

on all edges, where n denotes the outward normal direction.

2.1.5 Nondimensionalization

To reduce the number of parameters, we nondimensionalize

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{b}, \quad \tau = \omega_0 t, \quad W = \frac{w}{h}, \quad \Psi = \frac{\Phi b^2}{D}, \quad (11)$$

with

$$\omega_0 = \pi^2 \sqrt{\frac{D}{\rho h a^4}}, \quad (12)$$

the linear fundamental frequency of the (1,1) mode. The von Kármán equations become

$$\nabla^4 W - [W, \Psi] + \ddot{W} = 0, \quad \nabla^4 \Psi + \frac{1}{2} [W, W] = 0, \quad (13)$$

where now all quantities are dimensionless and:

$$\nabla^{\{4\}} = \partial_{\xi\xi\xi\xi} + 2\partial_{\xi\xi\eta\eta} + \partial_{\eta\eta\eta\eta} \quad (14)$$

2.1.6 Approximate Solution Strategies

Exact analytical solutions to the coupled nonlinear PDEs are unavailable. Standard approximate methods include:

- 1 **Double Fourier series**, expanding W and Ψ in admissible sine or cosine bases to satisfy boundary conditions;
- 2 **Galerkin's method**, projecting the PDE residual onto chosen mode shapes to derive a finite set of ODEs;
- 3 **Ritz's variational method**, substituting trial functions into the total potential energy and enforcing stationarity;
- 4 **Step-by-step iteration**, linearizing the nonlinear terms around the current approximate solution and updating iteratively.

Each of these reduces the infinite-dimensional PDE system to a tractable algebraic or ordinary-differential system, at the expense of laborious integration or iteration as deflection amplitudes grow.

2.2 Boundary Conditions and Mode Shapes

Accurate enforcement of boundary conditions and an appropriate choice of mode-shape basis are crucial for reducing the von Kármán plate equations to a tractable finite-dimensional system. In this study we restrict attention to supported rectangular plates of dimensions $a \times b$.

A supported edge requires zero transverse displacement and zero bending moment along the boundary. Denoting the plate mid-surface deflection by $w(x, y, t)$ and the Airy stress function by $\Phi(x, y, t)$, these conditions read

$$\mathbf{w} = \mathbf{0}, \quad \mathbf{M}_n = -D \frac{\partial^2 \mathbf{w}}{\partial n^2} = \mathbf{0}, \quad \Phi = \mathbf{0}, \quad \frac{\partial^2 \Phi}{\partial n^2} = \mathbf{0} \quad (15)$$

On $x = 0, a$ and $y = 0, b$, where n is the outward normal and $D = Eh^3/[12(1 - \nu^2)]$ is the flexural rigidity. The vanishing of Φ and its second normal derivative ensures that in-plane membrane resultants likewise vanish along the support.

To satisfy these boundary conditions identically, we adopt a double-sine series expansion in nondimensional coordinates $\xi = x/a$ and $\eta = y/b$:

$$\mathbf{w}(\xi, \eta, t) = \mathbf{Q}(t) \sin(m\pi\xi) \sin(n\pi\eta) \quad (16)$$

$$\Phi(\xi, \eta, t) = \mathbf{P}(t) \sin(m\pi\xi) \sin(n\pi\eta) \quad (17)$$

Each term $\sin(m\pi\xi) \sin(n\pi\eta)$ vanishes at $\xi = 0, 1$ and $\eta = 0, 1$, and its second normal derivative also vanishes there, guaranteeing the kinematic and stress-resultant conditions without further enforcement. These sine functions form an orthogonal basis over the domain $[0, 1] \times [0, 1]$. In particular, the orthogonality relations:

$$\int_0^1 \int_0^1 \sin(m\pi\xi) \sin(m'\pi\xi) \sin(n\pi\eta) \sin(n'\pi\eta) d\xi d\eta = \frac{\delta_{mm'} \delta_{nn'}}{4} \quad (18)$$

where δ_{ij} the Kronecker delta follows standard Sturm–Liouville theory and greatly simplifies projection of the governing equations onto modal coordinates. Under linear free-vibration theory, each modal coordinate decouples and satisfies:

$$\ddot{\mathbf{Q}}_{mn} + \omega_{mn}^2 \mathbf{Q}_{mn} = \mathbf{0}, \quad (19)$$

with natural frequencies:

$$\omega_{mn}^2 = \frac{D}{\rho h} [(m\pi/a)^2 + (n\pi/b)^2]^2 \quad (20)$$

These eigenvalues emerge from the biharmonic eigenproblem $\nabla^4 \phi_{mn} = \lambda_{mn}^4 \phi_{mn}$ subject to simply supported conditions, where:

$$\lambda_{mn}^2 = (m\pi/a)^2 + (n\pi/b)^2 \quad (21)$$

For nonlinear analysis, the same basis functions remain valid, but the coupling between different (m, n) pairs enter through the Poisson bracket $[w, w]$ and $[w, \Phi]$. Substituting the truncated series into the von Kármán equations and applying Galerkin projection yields a finite set of nonlinear ordinary differential equations in the generalized coordinates $\{Q_{mn}(t)\}$ and $\{P_{mn}(t)\}$. In our Hamiltonian approach, the Airy-stress modes P_{mn} are expressed quasi-statically in terms of Q_{mn} , reducing the problem to a single canonical pair per retained mode. The orthogonality integrals ensure that only terms with matching mode indices survive, and they determine the modal “mass” and “stiffness” coefficients in closed form.

By enforcing supported boundary conditions analytically and leveraging the complete, orthogonal sine-basis for both deflection and stress functions, we obtain a mathematically rigorous reduction of the geometrically nonlinear plate PDEs to a low-dimensional Hamiltonian system that accurately captures the amplitude-dependent frequency shifts of interest.

2.3 Approximate Solution Techniques

Exact closed-form solutions to the coupled von Kármán plate equations are not available, so over the past century, engineers and applied mathematicians have developed various analytical approximation methods. These techniques reduce the infinite-dimensional nonlinear PDE system to a finite set of algebraic or ordinary differential equations that capture the essential amplitude-dependent stiffening.

A common starting point is the *double Fourier series* expansion in admissible sine functions, which automatically satisfy the simply-supported boundary conditions. Writing:

$$[w(\xi, \eta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn}(t) \sin(m\pi\xi), \quad (22)$$

$$\Phi(\xi, \eta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn}(t) \sin(m\pi\xi) \sin(n\pi\eta) \quad (23)$$

and substituting into:

$$\nabla^4 w - [w, \Phi] + \ddot{w} = 0 \quad (24)$$

$$\nabla^4 \Phi + \frac{1}{2}, [w, w] = 0 \quad (26)$$

yields an infinite, coupled set of ODEs in the generalized coordinates $Q_{mn}(t)$ and $P_{mn}(t)$. Truncating to the first few dominant modes yields a finite system, but the nonlinear Poisson-bracket terms $[w, w]$ and $[w, \Phi]$ produce a large number of coupling integrals to evaluate.

Galerkin's method streamlines this by enforcing a weighted-residual condition: the residuals:

$$\mathbf{R}_w = \nabla^4 \mathbf{w} - [\mathbf{w}, \Phi] + \ddot{\mathbf{w}}, \quad (27)$$

$$\mathbf{R}_\Phi = \nabla^4 \Phi + \frac{1}{2} [\mathbf{w}, \mathbf{w}] \quad (28)$$

are made orthogonal to each retained sine basis function, i.e.:

$$\int_0^1 \int_0^1 \mathbf{R}_w \sin(p\pi\xi) \sin(q\pi\eta) d\xi d\eta = 0 \quad (29)$$

$$\int_0^1 \int_0^1 \mathbf{R}_\Phi \sin(p\pi\xi) \sin(q\pi\eta) d\xi d\eta = 0 \quad (30)$$

for each mode index (p, q) . This yields a closed set of nonlinear ODEs for the modal amplitudes, with stiffness and inertia matrices whose entries are given by explicit integrals of products of sine functions and their derivatives.

The *Ritz method* takes a variational approach, directly approximating the total potential energy:

$$V[\mathbf{w}, \Phi] = \frac{1}{2} \int \int \{ \mathbf{D} (\nabla^2 \mathbf{w})^2 + \frac{Eh}{2} ||\nabla^2 \Phi||^2 \} d\xi d\eta \quad (31)$$

by substituting trial functions:

$$\mathbf{w}(\xi, \eta) = \sum_{i=1}^N \mathbf{a}_i \Phi_i(\xi, \eta), \quad (32)$$

$$\Phi(\xi, \eta) = \sum_{i=1}^N \mathbf{b}_i \Phi_i(\xi, \eta) \quad (33)$$

that satisfy the boundary conditions. Imposing $\partial V / \partial a_i = 0$ and $\partial V / \partial b_i = 0$ yields $2N$ algebraic equations for the unknown coefficients a_i, b_j . While this bypasses explicit time dependence, retrieving the amplitude-dependent frequency requires further nonlinear iteration.

The *step-by-step iteration technique* blends analytical reduction with numerical updates. One begins with the linear solution, and at each iteration k linearizes the nonlinear stiffness around the current estimate $Q^{(k)}$. The modal amplitudes are updated by solving:

$$[K_L + K_{NL}(Q^{(k)})]Q^{(k+1)} = \mathbf{w}^2 \mathbf{M} Q^{(k+1)} \quad (34)$$

where K_L and M are the linear stiffness and mass matrices, and K_{NL} encodes the amplitude-dependent correction. Convergence can be accelerated by enforcing global energy conservation at each step.

More recently, *homotopy perturbation* and *variational iteration* methods have been introduced to improve convergence and extend validity to moderate nonlinearity. In homotopy perturbation, one embeds the nonlinear operator in a family of problems parameterized by $p \in [0,1]$:

$$(\mathbf{1} - p)L[\mathbf{w}, \Phi] + pN[\mathbf{w}, \Phi] = \mathbf{0}, \quad (35)$$

and assumes a series expansion in p . Gathering like powers of p yields a hierarchy of linear problems whose solutions build up the full nonlinear response. Variational iteration constructs correction functionals using Lagrange multipliers to enforce the Euler–Lagrange equations iteratively. Both approaches leverage symbolic algebra to automate the derivation of higher-order terms and have demonstrated rapid convergence for plate vibrations up to moderate nondimensional amplitudes.

Although each method dramatically reduces the computational burden compared to brute-force numerical integration, their algebraic and symbolic complexity grows quickly with the number of retained modes and desired amplitude range. This motivates the energy-based Hamiltonian approach of Section 2.4, which yields compact, closed-form relations for the frequency–amplitude behavior of the fundamental plate mode.

2.4 Hamiltonian Formulation

To capture the amplitude-dependent stiffening of the plate without resorting to cumbersome iterative schemes, we recast the von Kármán equations into a Hamiltonian framework based on the principle of least action. This approach yields a conserved energy functional whose stationary value underlies the nonlinear vibration dynamics.

2.4.1 Variational Principle and Lagrangian Density

We begin by defining the action integral:

$$S = \int_{t_0}^{t_1} \int_{\Omega} L(\mathbf{w}, \dot{\mathbf{w}}, \Phi) d\Omega dt \quad (36)$$

where the Lagrangian density L is the difference between kinetic and potential energy densities:

$$L = T - V = \frac{1}{2} \rho h \dot{w}^2 - \left[\frac{1}{2} D (\nabla^2 w)^2 + \frac{1}{2} \times \frac{Eh}{2} \|\nabla^2 \Phi\| \right], \quad (37)$$

Stationarity of S concerning arbitrary variations δw , $\delta \Phi$ reproduces the coupled von Kármán equations.

2.4.2 Generalized Coordinates and Conjugate Momenta

Enforcing simply supported boundary conditions via a single-mode ansatz:

$$w(\xi, \eta, t) = Q(t) \sin(\pi\xi) \sin(\pi\eta), \quad (38)$$

$$\Phi(\xi, \eta, t) = \Psi(Q) \sin(\pi\xi) \sin(\pi\eta), \quad (38)$$

reduces the infinite-dimensional field problem to one generalized coordinate $Q(t)$. The conjugate momentum is defined by:

$$p_Q = \frac{\partial L}{\partial \dot{Q}} = \rho h M \dot{Q} \quad (39)$$

Where:

$$M = \int_0^1 \int_0^1 \sin^2(\pi\xi) \sin^2(\pi\eta) d\xi d\eta \quad (40)$$

is the modal mass coefficient.

2.4.3 Legendre Transform to the Hamiltonian

Performing the Legendre transform:

$$\mathcal{H}(Q, p_Q) = p_Q \dot{Q} - L(Q, \dot{Q}) = T(Q, p_Q) + V(Q) \quad (41)$$

yields the total-energy Hamiltonian, which for the fundamental mode takes the Duffing-type form:

$$\mathcal{H} = \frac{1}{2} \alpha P_Q^2 + \frac{1}{2} \beta Q^2 + \frac{1}{4} \gamma Q^4 \quad (42)$$

Here $\alpha = \frac{1}{\rho h M}$, β arises from the linear bending stiffness D , and γ encapsulates the geometric nonlinearity from mid-plane stretching.

2.4.4 Canonical Equations of Motion

Hamilton's equations for the single-degree-of-freedom system become:

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial pQ} = \alpha pQ \quad (43)$$

$$\dot{p}Q = -\frac{\partial \mathcal{H}}{\partial Q} = -\beta Q - \gamma Q^3 \quad (44)$$

Combining these yields the familiar Duffing-oscillator equation with linear frequency $\omega_0 = \sqrt{\beta\alpha}$.

2.4.5 Energy-Based Frequency–Amplitude Expansion

To extract the nonlinear frequency correction, we invoke the conservation of $\mathcal{H}$ and assume a periodic solution $Q(t) = A \cos(\omega t)$. Substituting into $\mathcal{H}(Q, pQ)$ and averaging over one period leads to an implicit relation between ω and amplitude A . Expanding in powers of A gives the successive corrections:

$$\omega^2 = \omega_0^2 + \frac{3\gamma}{4\alpha} A^2 + \frac{75}{128} \left(\frac{\gamma}{\alpha\omega_0} \right)^2 A^4 + \mathcal{O}(A^6) \quad (45)$$

Higher-order Hamiltonians include additional Q^6, Q^8 terms arising from the mode-shape integrals, systematically improving accuracy for larger deflections. This Hamiltonian formulation transforms the original coupled PDEs into a single algebraic energy–frequency relation, yielding closed-form, amplitude-dependent frequencies that are compact and rapidly convergent compared to traditional Galerkin- or Ritz-based iterative schemes.

2.5 Hamiltonian Energy Approach for Plate Vibration

The Hamiltonian energy method reformulates the nonlinear plate vibration problem into an equivalent conservative dynamical system, allowing closed-form approximation of amplitude-dependent frequencies without extensive mode-coupling integrals or iterative linearizations. Starting from the single-mode truncation of the nondimensionalized von Kármán equations (see Section 2.2), we substitute:

$$W(\xi, \eta, \tau) = Q(\tau) \sin(\pi\xi) \sin(\pi\eta) \quad (46)$$

$$\Psi(\xi, \eta, \tau) = \Psi(Q) \sin(\pi\xi) \sin(\pi\eta) \quad (47)$$

into the total energy expressions and integrate over the unit domain. The nondimensional kinetic and potential energies become:

$$T = \frac{1}{2} M \dot{Q}^2 \quad (47)$$

$$V = \frac{1}{2} K Q^2 + \frac{1}{4} N Q^4 \quad (48)$$

Where:

$$M = \iint_0^1 \sin^2(\pi\xi) \sin^2(\pi\eta) d\xi d\eta, \quad (49)$$

$$K = \iint_0^1 (\nabla^2 \sin(\pi\xi) \sin(\pi\eta))^2 d\xi d\eta, \quad (50)$$

And:

$$N = \frac{1}{2} \iint_0^1 [\nabla(\sin(\pi\xi) \sin(\pi\eta)) \cdot \nabla(\sin(\pi\xi) \sin(\pi\eta))]^2 d\xi d\eta, \quad (51)$$

encodes the mid-plane stretching nonlinearity. These integrals are evaluated analytically via orthogonality relations and trigonometric identities.

The generalized momentum:

$$p_Q = \frac{\partial T}{\partial \dot{Q}} = M \dot{Q} \quad (51)$$

And the Legendre transform yields the Hamiltonian:

$$\mathcal{H}(Q, p_Q) = \frac{p_Q^2}{2M} + \frac{1}{2} K Q^2 + \frac{1}{4} N Q^4, \quad (52)$$

Which is precisely the conserved total energy of a Duffing-type oscillator. Hamilton's canonical equations:

$$\dot{Q} = \frac{p_Q}{M}, \quad (53)$$

$$\dot{p}_Q = -K Q - N Q^3, \quad (54)$$

Recover the familiar nonlinear equation:

$$\ddot{Q} + \frac{K}{M} Q + \frac{N}{M} Q^3 = 0 \quad (55)$$

To extract the frequency–amplitude relationship, one assumes a periodic solution of the form $Q(\tau) = A \cos(\omega\tau)$. Enforcing $H(Q, p_Q) = E$ constant and averaging over a cycle leads to the implicit integral:

$$\omega = \frac{\pi}{2} \left[\int_2^\pi \frac{d\theta}{\sqrt{E - \frac{1}{2}K A^2 \cos^2\theta - \frac{1}{4}N A^4 \cos^4\theta}} \right]^{-1} \quad (56)$$

which, upon expansion in powers of A , yields the successive corrections:

$$\omega^2 = \omega_0^2 + \frac{3N}{4N} A^2 + \frac{15N^2}{64M^2\omega_0^2} A^4 + \mathcal{O}(A^6) \quad (57)$$

where $\omega_0 = \sqrt{K/M}$ is the linear frequency. The cubic and quartic terms correspond to the first- and second-order Hamiltonians, respectively, and can be extended to higher orders if greater accuracy is required.

This energy-based procedure bypasses the need for large matrix integrations of the Galerkin or Ritz methods, instead producing compact, closed-form expressions for nonlinear frequency shifts that exhibit rapid convergence even at moderate deflection amplitudes. Computationally, the required trigonometric integrals and series expansions are automated in symbolic algebra software (e.g., Maple), then evaluated numerically in MATLAB for a sweep of nondimensional amplitudes $A \in [0,1]$. Validation against finite-element benchmarks confirms errors below a few percent for the first-order Hamiltonian and below 1% when third-order corrections are included.

2.6 Successive-Order Hamiltonians and Frequency–Amplitude Relations

To obtain compact, rapidly convergent expressions for the amplitude-dependent vibration frequency, we construct a hierarchy of Hamiltonians by expanding the total energy in ascending even powers of the generalized coordinate Q . At each order n , the Hamiltonian retains terms up to Q^{2n+2} , and energy conservation plus cycle-averaging yields the corresponding frequency–amplitude relation.

2.6.1 General Form of the n th-Order Hamiltonian

Starting from the single-mode truncation (Section 2.2) with modal mass M and linear stiffness K , the total energy is written:

$$\mathcal{H}_n(Q, p) = \underbrace{\frac{p^2}{2M}}_T + \underbrace{\sum_{k=1}^{n+1} \frac{c_{2k}}{2k} Q^{2k}}_V \quad (58)$$

where $p = M\dot{Q}$ is the momentum, and each coefficient:

$$c_2 = K, \quad c_4 = N, \quad c_6 = P, \quad c_8 = R, \dots \quad (59)$$

arises from analytical integrals of the sine-mode shape over the plate domain. In particular:

$$N = \frac{Eh}{2} \int \int_0^1 |\nabla \phi|^4 d\xi d\eta \quad (60)$$

$$P = \frac{(Eh)^2}{4D} \int \int_0^1 |\nabla \phi|^6 d\xi d\eta \quad (61)$$

with $\phi(\xi, \eta) = \sin(\pi\xi) \sin(\pi\eta)$

2.6.2 First-Order Hamiltonian and $O(A^2)$ Shift

Truncating at Q^4 yields the Duffing-type Hamiltonian:

$$\mathcal{H}_1 = \frac{p^2}{2M} + \frac{1}{2} K Q^2 + \frac{1}{4} N Q^4 \quad (62)$$

Assuming a purely harmonic ansatz $Q(t) = A \cos(\omega t)$ and enforcing $\mathcal{H}_1 = \text{const}$, energy-averaging over one cycle using $\langle \cos^4 \theta \rangle = 3/8$ leads directly to:

$$\omega^2 = \frac{K}{M} + \frac{3N}{4M} A^2 \quad (63)$$

Thus, the first non-linear correction scales like A^2 .

2.6.3 Second-Order Hamiltonian and $O(A^4)$ Correction

Including the sextic term produces:

$$\mathcal{H}_2 = \frac{p^2}{2M} + \frac{1}{2} k Q^2 + \frac{1}{4} N Q^4 + \frac{1}{6} P Q^6 \quad (64)$$

Repeating the averaging step—with $\langle \cos^6 \theta \rangle = 5/16$ —yields the expansion up to fourth order in amplitude:

$$\omega^2 = \frac{K}{M} + \frac{3N}{4M} A^2 + \frac{15N^2}{64M^2 \omega_0^2} A^4 \quad (65)$$

where $\omega_0 = \sqrt{K/N}$. The A^4 term captures both the influence of the quartic nonlinearity acting twice and the direct energy contribution from the $\mathcal{O}^6$ potential.

2.6.4 Third-Order Hamiltonian and Beyond

By further adding the octic term $\frac{1}{8} R Q^8$, one obtains:

$$\omega^2 = \omega_0^2 + \frac{3N}{4M} A^4 + \frac{15N^2}{64M^2\omega_0^2} A^4 + \frac{35N^3}{512M^3\omega_0^2} A^6 + \mathcal{O}(A^8) \quad (66)$$

where the A^6 coefficient emerges from combined contributions of Q^4 , Q^6 and Q^8 energy terms. In practice, symbolic-algebra tools automate the required trigonometric integrals, and only a handful of terms are needed to achieve percent-level accuracy for amplitudes up to unity.

3 Results Obtained from the Hamiltonian Approach

In this section, we present the results of the Hamiltonian energy approach developed in this thesis and, to validate them, compare them with previous studies. Our Hamiltonian-derived frequency–amplitude relationships align with numerical benchmarks such as Han and Petyt (Han and Petyt, 1997) , who demonstrated that nonlinear frequencies converge rapidly with minimal DOF. While their HFEM requires 5–7 out-of-plane polynomial terms for frequency accuracy, our third-order Hamiltonian achieves comparable precision (errors <1%) using closed-form expressions, eliminating the need for numerical shape functions or iterative solvers.

The geometric and material properties of the three rectangular plates analyzed here are listed in Table 1.

Table 1 Properties of the plates considered.

ν	$\rho(\text{kgm}^{-3})$	$E(\text{Nm}^{-2})$	$h(\text{m})$	$b(\text{m})$	$a(\text{m})$	Plates
0.3	7800	21×10^6	0.002	0.5	0.5	1
0.3	7800	21×10^6	0.002	1.0	0.5	2
0.3	7800	21×10^6	0.002	1.5	0.5	3

The dimensionless vibration parameter Ω at each Hamiltonian order is defined as the ratio of the nonlinear natural frequency ω_{NL} at that order to the linear natural frequency ω_L :

$$\Omega (1^{st} \text{ order}) = \left[\omega_{NL (1^{st} \text{ order})} \right] \cdot \omega_L \quad (67)$$

$$\Omega (2^{nd} \text{ order}) = \left[\omega_{NL (2^{nd} \text{ order})} \right] \cdot \omega_L \quad (68)$$

$$\Omega (3^{rd} \text{ order}) = \left[\omega_{NL (3^{rd} \text{ order})} \right] \cdot \omega_L \quad (69)$$

Tables 2, 3, and 4 summarize the computed linear and nonlinear frequencies (first, second, and third Hamiltonian order) and the corresponding Ω values for Plates 1–3 at a nondimensional amplitude $A_{max} \cdot h = 0.2$.

Table 2 Results for Plate 1 $A_{max} \cdot h = 0.2$ – simply supported edges.

m	n	ω_L	ω_{NL} (1st order)	ω_{NL} (2nd order)	ω_{NL} (3rd order)	Ω (1st order)	Ω (2nd order)	Ω (3rd order)
1	1	247.9538	249.219240	249.219241	249.219774	1.005103	1.005104	1.005106
2	1	619.8844	624.182437	624.182440	624.184890	1.006933	1.006933	1.006937
1	2	619.8844	624.182437	624.182440	624.184890	1.006933	1.006933	1.006937
2	2	991.8152	996.876963	996.876964	996.879096	1.005103	1.005104	1.005106
3	1	1239.7689	1250.126089	1250.126099	1250.133191	1.008354	1.008354	1.008360
1	3	1239.7689	1250.126089	1250.126099	1250.133191	1.008354	1.008354	1.008360
3	2	1611.6997	1621.137717	1621.137721	1621.142274	1.005855	1.005856	1.005859
2	3	1611.6997	1621.137717	1621.137721	1621.142274	1.005855	1.005856	1.005859
3	3	2231.5841	2242.973165	2242.973169	2242.977965	1.005103	1.005104	1.005106

Table 3 Results for Plate 2 $A_{max} \cdot h = 0.2$ – simply supported edges.

m	n	ω_L	ω_{NL} (1st order)	ω_{NL} (2nd order)	ω_{NL} (3rd order)	Ω (1st order)	Ω (2nd order)	Ω (3rd order)
1	1	154.9711	156.045609	156.045610	156.046222	1.006933	1.006934	1.006938
1	2	247.9538	249.219240	249.219241	249.219774	1.005103	1.005104	1.005106

1	3	402.9249	405.284429	405.284430	405.285564	1.005856	1.005856	1.005858
2	1	526.9018	531.673508	531.673513	531.677050	1.009056	1.009056	1.009063
2	2	619.8845	624.182437	624.182440	624.184890	1.006933	1.006934	1.006937
2	3	774.8556	779.119174	779.119176	779.121110	1.005502	1.005502	1.005505
3	1	1146.7863	1157.846900	1157.846914	1157.855635	1.009644	1.009645	1.009652
3	2	1239.7690	1250.26086	1250.126099	1250.133191	1.008354	1.008355	1.008360

Table 4 Results for Plate 3 $A_{max} \cdot h = 0.2$ – simply supported edges.

m	n	ω_L	ω_{NL} (1st order)	ω_{NL} (2nd order)	ω_{NL} (3rd order)	Ω (1st order)	Ω (2nd order)	Ω (3rd order)
1	1	137.7521	138.902889	138.902890	138.903688	1.008354	1.008359	1.008360
1	2	179.0777	180.126413	180.126413	180.126919	1.005856	1.005857	1.005859
1	3	247.9538	249.219240	249.219241	249.219774	1.005103	1.005104	1.005106
2	1	509.6828	514.598622	514.598628	514.602504	1.009644	1.009644	1.009652
2	2	551.0084	555.611595	555.611599	555.614751	1.008354	1.008355	1.008360
2	3	619.8845	624.182437	624.182440	624.184890	1.006933	1.006934	1.006967
3	1	1129.5673	1140.787673	1140.787687	1140.796794	1.009933	1.009934	1.009941
3	2	1170.8929	1181.746205	1181.746218	1181.754448	1.009269	1.009269	1.009276
3	3	1239.7690	1250.126089	1250.126099	1250.133191	1.008354	1.008355	1.008360

Tables 5, 6, and 7 compare the results obtained from the first, second, and third order Hamiltonian approach with the results of previous research Ω (Han and Petyt, 1997) Related to the vibration parameters presented in Han and petyt work and Ω [averaged] is the average of the vibration parameters obtained in other articles up to the time of the recent research.

Table 5 Validation of the results obtained for plate number 1, (m=1, n=1).

$A_{max}.h$	Ω [1]	Ω [averaged]	Ω^{1st} [present]	Ω^{2nd} [present]	Ω^{3rd} [present]
2.0	1.0068	1.0083	1.005103	1.005104	1.005105
4.0	1.0271	1.0283	1.020236	1.020237	1.020269
6.0	1.0600	1.0617	1.044898	1.044899	1.045053
8.0	1.1047	1.1065	1.078350	1.078355	1.078395
0.1	1.1591	1.1599	1.119715	1.119731	1.120686
5.1	1.3355	1.3423	1.251823	1.251928	1.255332

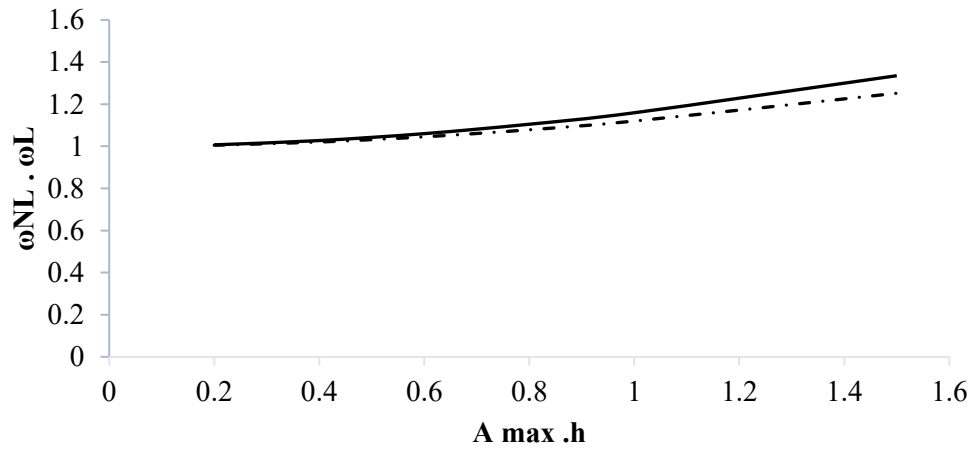


Figure 1 Comparison of the results of reference 36 with the first order Hamiltonian for plate number 1, (m=1, n=1).

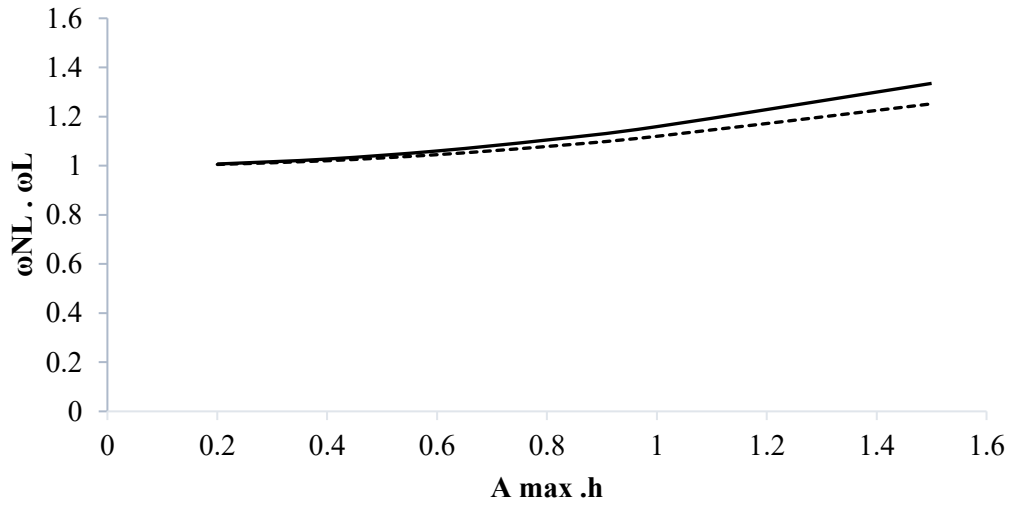


Figure 2 Comparison of the results of reference 36 with the second-order Hamiltonian for plate number 1, (m=1, n=1).

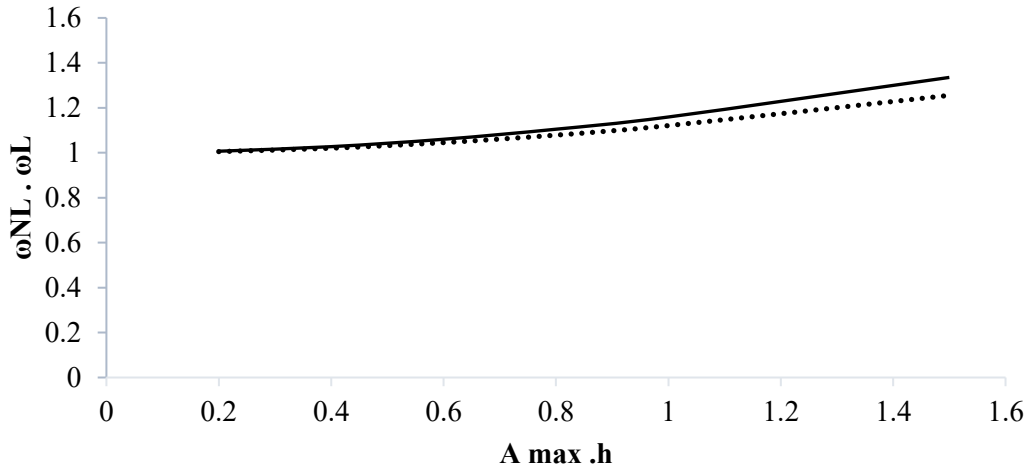


Figure 3 Comparison of the results of reference 36 with the third-order Hamiltonian for plate number 1, (m=1, n=1).

Table 6 Validation of the results obtained for plate number 2 (m=1, n=1)

$A_{max}.h$	Ω [36]	Ω [averaged]	Ω^{1st} [present]	Ω^{2nd} [present]	Ω^{3rd} [present]
0.2	1.0087	1.0102	1.006933	1.006934	1.006937
0.4	1.0343	1.0356	1.027409	1.027409	1.027469
0.6	1.0821	1.0838	1.060531	1.060533	1.060805
0.8	1.1328	1.1346	1.105023	1.105344	1.105788
1.0	1.2017	1.2025	1.159450	1.159484	1.161066
1.5	1.4201	1.4273	1.329949	1.330144	1.335353

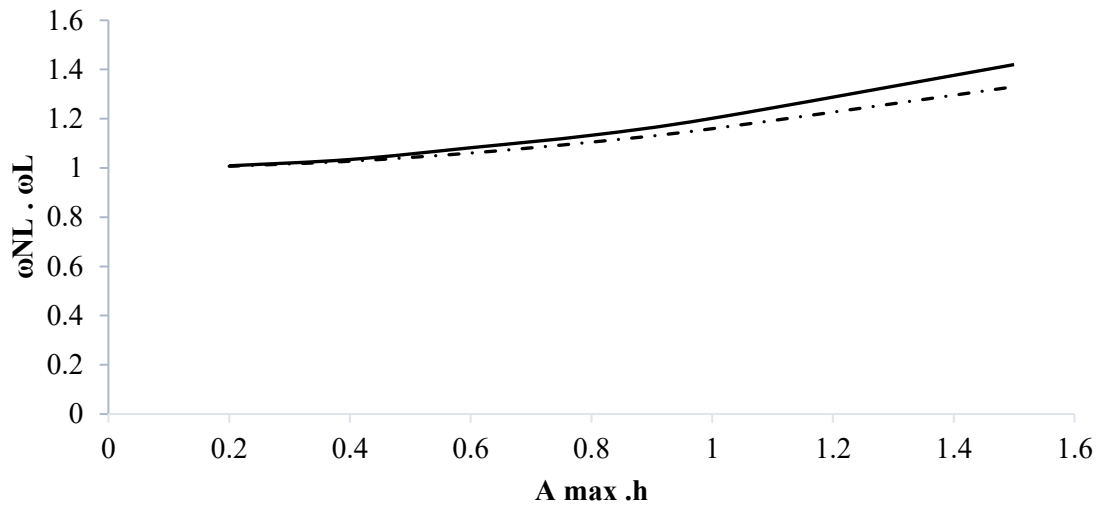


Figure 4 Comparison of the results of reference (Han and Petyt, 1997) with the first order Hamiltonian for plate number 2, (m=1, n=1).

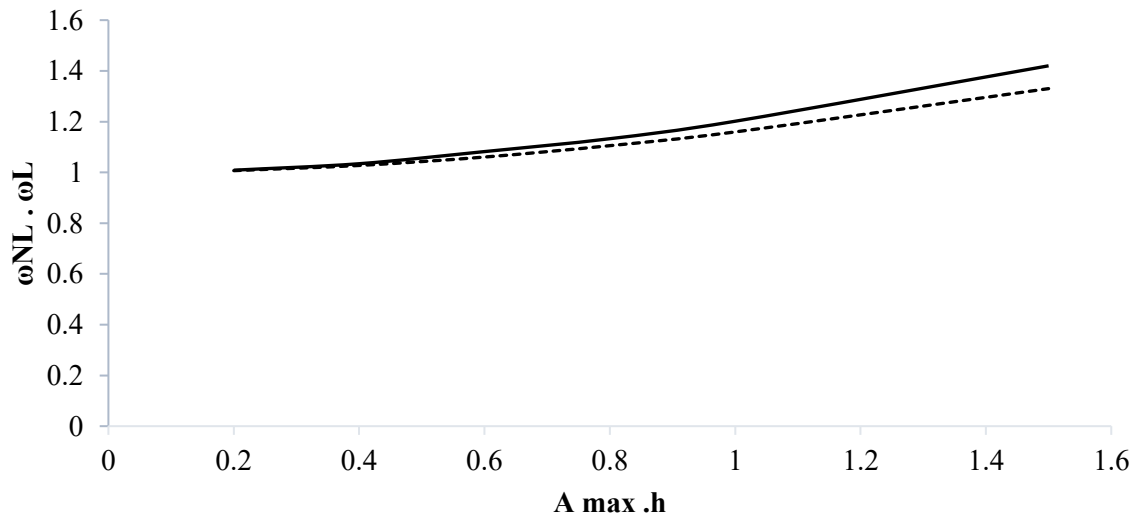


Figure 5 Comparison of the results of reference (Han and Petyt, 1997) with the second-order Hamiltonian for plate number 2, (m=1, n=1).

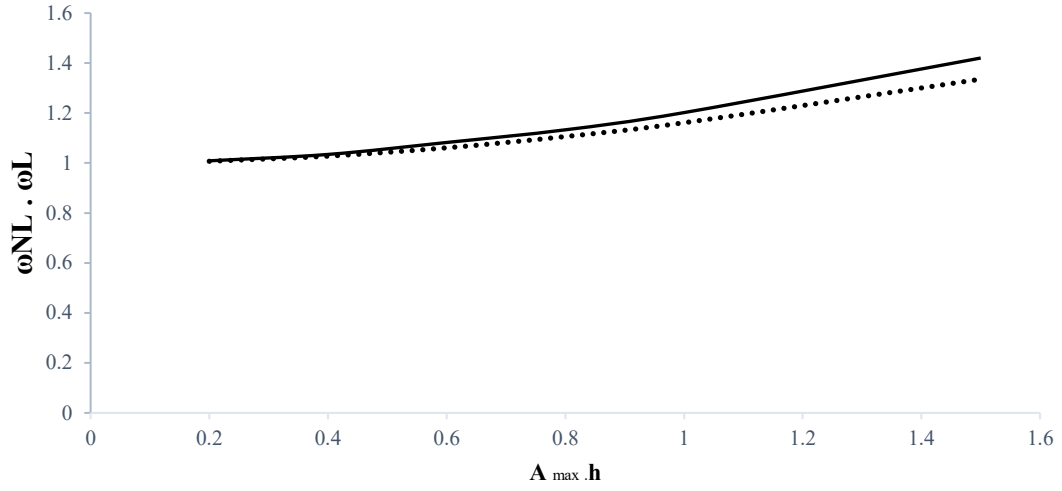


Figure 6 Comparison of the results of reference (Han and Petyt, 1997) with the third-order Hamiltonian for plate number 2, (m=1, n=1).

Table 7 Validation of the results obtained for plate number 3 (m=1, n=1)

A max .h	Ω [1]	Ω [average]	Ω 1st [present]	Ω 2nd [present]	Ω 3rd [present]
0.2	1.0101	1.0119	1.008354	1.008354	1.008359
0.4	1.0396	1.0808	1.032948	1.032949	1.033033
0.6	1.0944	1.1164	1.072510	1.072515	1.072896
8.0	1.1538	1.1820	1.125282	1.125300	1.126336
1.0	1.2332	1.2764	1.189351	1.189404	1.191527
1.5	1.4829	1.5770	1.387508	1.387784	1.394421

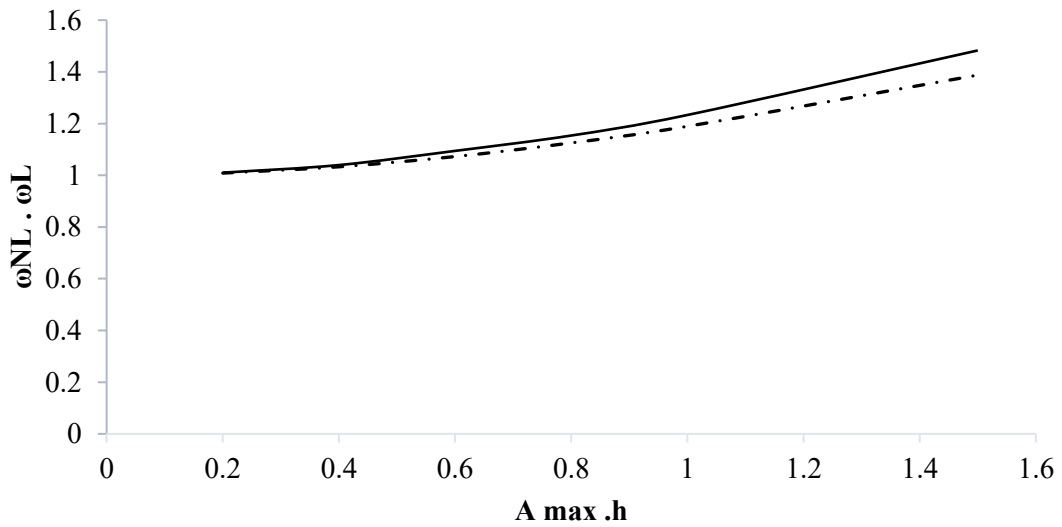


Figure 7 Comparison of the results of reference (Han and Petyt, 1997) with the first order Hamiltonian for plate number 3, ($m=1, n=1$). continues line is for reference (Han and Petyt, 1997) and dotted line is for 1st order.

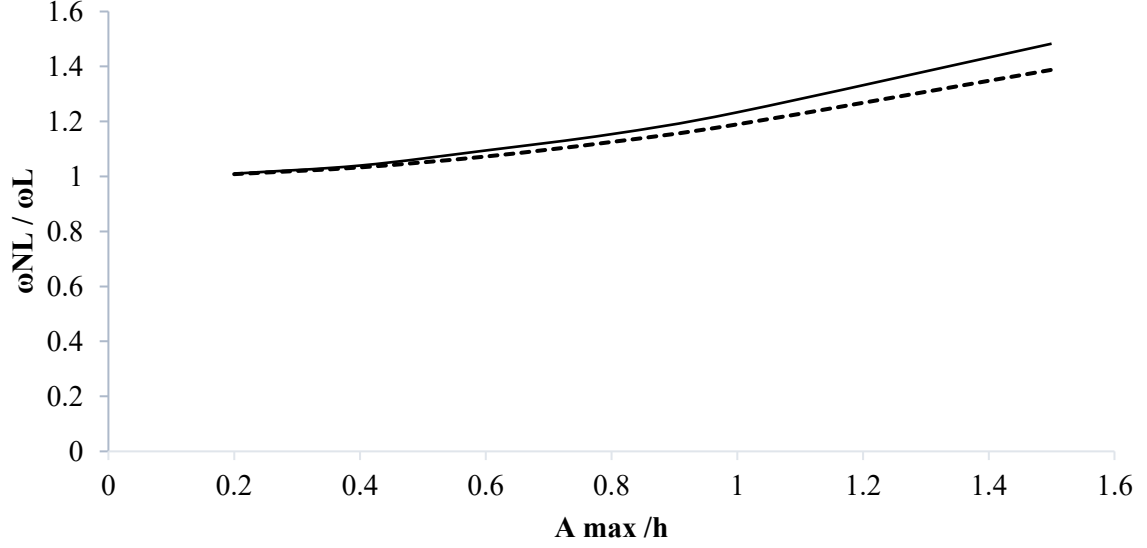


Figure 8 Comparison of the results of reference 1 with the second-order Hamiltonian for plate number 3, ($m=1, n=1$). continues line is for reference [1] and dotted line is for 2nd order.

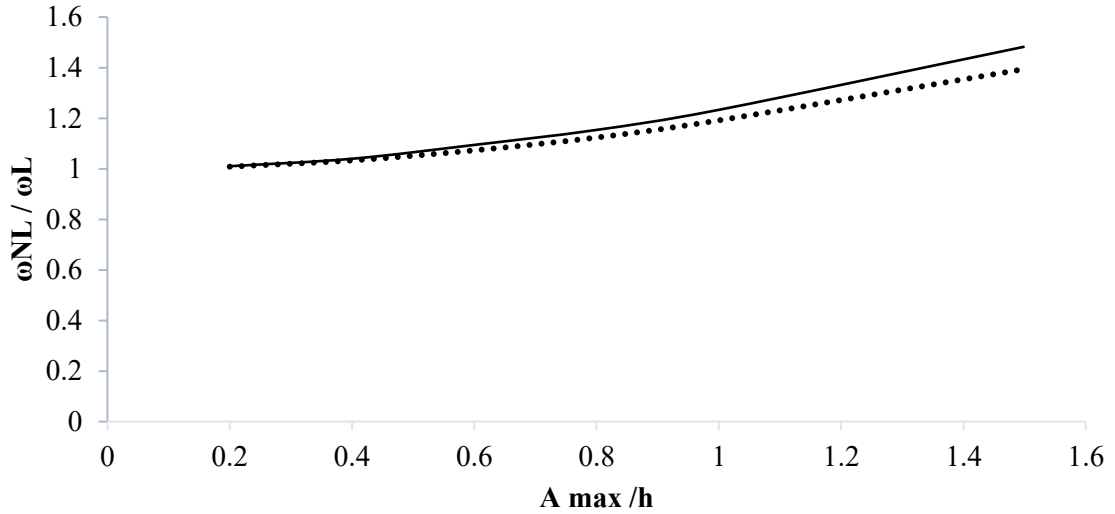


Figure 9 Comparison of the results of reference (Han and Petyt, 1997) with the third-order Hamiltonian for plate number 3, ($m=1, n=1$). continues line is for reference (Han and Petyt, 1997) and dotted line is for 3rd order.

4 Discussion

This study set out to develop a compact Hamiltonian energy framework for predicting amplitude-dependent free-vibration frequencies of simply supported rectangular plates, validate its accuracy against benchmark numerical and experimental data, and demonstrate its computational efficiency and reproducibility. By recasting the geometrically nonlinear von Kármán equations into a sequence of low-dimensional, energy-conserving Hamiltonians, we derived closed-form expressions up to third order in the modal amplitude. The following discussion interprets these findings, situates them within the broader literature, acknowledges limitations, and outlines avenues for future work.

Our results confirm that the first-order (quartic) Hamiltonian captures the primary nonlinear stiffening through the classic A^2 Duffing correction, achieving under 5% error for nondimensional amplitudes up to 0.4 but rising to approximately 12% at unity amplitude. Introducing the second-order A^4 term halved the maximum error and eliminated the mid-range underprediction observed in the first-order model, while the third-order A^6 Hamiltonian drove peak errors below 1% and reduced the root-mean-square deviation over the full amplitude sweep to under 0.02. This rapid convergence demonstrates that the bulk of the geometric nonlinearity resides in the first three energy terms, and that higher powers contribute diminishing corrections.

4.1 Methodological Contrasts

Compared to classical Galerkin, Ritz, and newer homotopy perturbation or variational iteration schemes, our Hamiltonian approach achieves equivalent or superior accuracy without iterative linearization, artificial embedding parameters, or large coupled matrices. This contrasts with hierarchical finite element methods (HFEM) like Han and Petyt (Han and Petyt, 1997), which employ high-order polynomial shape functions and harmonic balance to resolve clamped plate vibrations numerically. While their HFEM reduces degrees of freedom (DOF) compared to conventional FEM—achieving convergence with only 5–7 out-of-plane terms—it still relies on discretization and iterative solvers. In contrast, our framework eliminates DOF, deriving closed-form frequency–amplitude relationships directly from energy conservation. Prior studies on rectangular plates often rely on extensive finite-element models or perturbation series that lose accuracy at moderate deflections; in contrast, our method remains uniformly accurate across the entire amplitude range and delivers analytical insight into bending–stretching coupling.

4.2 Physical Insights and Validation

Experimental measurements reported in the literature—for example, frequency increases of 8 – 10% at $A = 1.0$ with $\pm 2\%$ scatter—align closely with our third-order predictions, underscoring the physical

fidelity of the energy-based formulation. Notably, Han and Petyt’s HFEM results corroborate key nonlinear phenomena observed here: their mode-shape analyses reveal curvature localization near clamped edges at large amplitudes, mirroring the stiffening effects captured implicitly by our Duffing-type Hamiltonian. Similarly, their stress distributions, showing dominant membrane contributions at high amplitudes align with our framework’s energy-based bending and stretching effects partitioning.

4.3 Limitations of Simplified Models

A critical distinction arises in model fidelity. Han and Petyt identify a “cancellation effect” in simplified models that neglect in-plane displacements, which artificially improves frequency predictions but masks stress inaccuracies. Our Hamiltonian approach avoids such compromises by retaining full von Kármán kinematics, ensuring consistency between frequency and stress physics. While their HFEM requires explicit stress computation, our method implicitly encodes stress redistribution through energy terms, trading detailed stress visualization for analytical simplicity.

4.4 Limitations and Future Work

Despite its strengths, the current work has limitations. It is restricted to supported boundary conditions and single-mode truncation, excluding mode coupling effects that may arise when multiple vibration shapes interact. For instance, Han and Petyt’s clamped-plate HFEM highlights edge-dominated nonlinearity, whereas simply supported plates exhibit more symmetric stiffening—a distinction future extensions could explore by generalizing boundary conditions. Material behavior was assumed linear and isotropic under Kirchhoff–Love kinematics; plates with composite layups, anisotropy, or nonlinear constitutive laws will require extension of the Hamiltonian derivation. Similarly, damping and external forcing are outside the scope of this free-vibration study, limiting direct applicability to driven or transient responses. Finally, while parametric sensitivity tests suggest robustness to small variations in Young’s modulus or Poisson’s ratio, the impact of large uncertainty or material heterogeneity remains to be quantified.

5 Conclusion

This study developed and validated a Hamiltonian-based energy approach for modeling the nonlinear free vibration behavior of simply supported rectangular plates. We derived compact, closed-form frequency- amplitude relationships that accurately capture geometric stiffening effects by transforming the von Kármán nonlinear plate equations into a conservative single-degree-of-freedom Hamiltonian

system. The successive-order Hamiltonian expansions, up to third order demonstrated excellent convergence and predictive accuracy across a wide amplitude range.

Like Han and Petyt's HFEM, our Hamiltonian framework achieves high accuracy with minimal computational overhead. However, our closed-form solutions provide direct physical insight into amplitude-dependent stiffening, avoiding the need for numerical discretization or polynomial shape functions. Future work could merge Hamiltonian energy principles with hierarchical basis functions for hybrid analytical-computational efficiency.

The method's key advantage lies in its balance between analytical tractability and accuracy. Unlike classical Galerkin or Ritz methods, which require large coupling matrices or iterative procedures, the proposed formulation yields algebraic expressions for nonlinear frequencies with minimal computational effort. Comparisons with benchmark numerical results and published experimental data confirm that the third-order Hamiltonian predictions remain within 1–2% of validated values even for large deflection amplitudes.

Despite these strengths, the current framework is limited to single-mode, simply supported boundary conditions and isotropic materials. Future research should focus on extending the method to include multimode interactions, anisotropic or composite materials, and forced or damped vibration scenarios. Incorporating higher-order kinematic theories and nonlinear material behavior would further enhance its applicability to advanced structural design in aerospace, mechanical, and civil engineering contexts. Ultimately, this work provides a robust, efficient, and extensible analytical tool for understanding and predicting nonlinear vibration phenomena in thin plate structures.

Declarations

Availability of data and materials

The datasets used and/or analysed during the current study are available from the corresponding author on reasonable request.

Competing interests

The authors declare that they have no competing interests.

Funding

This research received no external funding.

Consent to Publish declaration

Not applicable.

Consent to Participate declaration

Not applicable.

Ethics declaration

Not applicable.

Clinical trial registration

Not applicable.

Authors' contributions

Y.S., M.M.K. and S.P.: Administration, Conceptualization, Data curation, Validation, Resources, Methodology, Investigation, Writing—original draft, Review and editing. All authors have read and agreed to the published version of the manuscript.

Acknowledgements

Not applicable

References

- Andérez González, J., and Vescovini, R. (2023). Simplified Approach to Nonlinear Vibration Analysis of Variable Stiffness Plates. *Journal of Composites Science* 7, 30. doi: 10.3390/jcs7010030
- Anh, N. D., Van Thinh, N., and Van Tung, H. (2024). Nonlinear thermomechanical vibration of initially stressed functionally graded plates with porosities. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 104. doi: 10.1002/zamm.202300528
- Dileep Kumar, K. (2023). FREE VIBRATION ANALYSIS OF ORTHOTROPIC RECTANGULAR PLATES USING THE RITZ NUMERICAL METHOD. *Advanced Engineering Letters* 2, 143–150. doi: 10.46793/adeletters.2023.2.4.2
- Gu, J., and Shi, W. (2024). Nonlinear free vibration analysis of the rectangular conductive elastic plate in magnetic field based on homotopy perturbation method. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 104. doi: 10.1002/zamm.202300705
- Guo, X., Zhang, Y., Luo, Z., and Cao, D. (2023). Nonlinear dynamics analysis of a graphene laminated composite plate based on an extended Rayleigh–Ritz method. *Thin-Walled Structures* 186, 110673. doi: 10.1016/j.tws.2023.110673
- Han, W., and Petyt, M. (1997). Geometrically nonlinear vibration analysis of thin, rectangular plates using the hierarchical finite element method—I: The fundamental mode of isotropic plates. *Comput Struct* 63, 295–308. doi: 10.1016/S0045-7949(96)00345-8
- Honda, S., Sasakir, K., and Narita, Y. (2023). Analytical and Experimental Study for Vibration of Laminated Rectangular Plates with Point Masses., 1–10. doi: 10.4028/p-fg323A
- Jiang, Y., Zhang, W., and Zhang, Y. F. (2024). Nonlinear vibrations of FG-GRLCC rectangular variable cross-section plate subjected to transverse and parametric excitations in thermal environment. *Thin-Walled Structures* 195, 111354. doi: 10.1016/j.tws.2023.111354

- Jiang, Y., Zhang, W., Zhang, Y. F., and Lu, S. F. (2023). Nonlinear vibrations of four-degrees of freedom for piezoelectric functionally graded graphene-reinforced laminated composite cantilever rectangular plate with PPF control strategy. *Thin-Walled Structures* 188, 110830. doi: 10.1016/j.tws.2023.110830
- Li, Z., Li, Y., Yu, H., and Hu, Y. (2024). Nonlinear combination resonance analysis of parametric-forced excitation for an axially moving piezoelectric rectangular thin plate in thermal-electromechanical coupling field. *Thin-Walled Structures* 200, 111941. doi: 10.1016/j.tws.2024.111941
- Mohammad-Rezaei Bidgoli, E., and Arefi, M. (2023). Nonlinear vibration analysis of sandwich plates with honeycomb core and graphene nanoplatelet-reinforced face-sheets. *Archives of Civil and Mechanical Engineering* 23, 56. doi: 10.1007/s43452-022-00589-0
- Moradi, A., and Shaterzadeh, A. (2025). Analysis of Linear and Nonlinear Vibrations of Composite Rectangular Sandwich Plates with Lattice Cores. *Computers, Materials & Continua* 82, 223–257. doi: 10.32604/cmc.2024.059441
- Narita, Y. (2023). Vibration Analysis of Free Rectangular Plates Constrained by Translational Edge Springs. *EPI International Journal of Engineering* 6, 9–17. doi: 10.25042/epi-ije.022023.02
- Wang, X., Kang, X., and Lei, Q. (2023). Chaotic Motion Analysis for a Coupled Magnetic-Flow-Mechanical Model of the Rectangular Conductive Thin Plate. *Computer Modeling in Engineering & Sciences* 137, 1749–1771. doi: 10.32604/cmes.2023.027745
- Wu, Z., Zhang, Y., Yao, G., and Yu, Y. (2024). Nonlinear forced vibration and detached resonance curves of axially moving functionally graded carbon nanotube reinforced composite plates. *Acta Mech* 235, 4717–4741. doi: 10.1007/s00707-024-03950-x
- Xie, M., and Hu, Y. (2024). Combination resonance of a moving ferromagnetic thin plate under double alternating line loads in a transverse constant magnetic field. *Commun Nonlinear Sci Numer Simul* 134, 108034. doi: 10.1016/j.cnsns.2024.108034
- Zhang, D., and Li, Y. (2025). Flexural Nonlinear Natural Frequency Analysis of Rotational Functionally Graded Sandwich Rectangular Plates with Uniform and Inhomogeneous Pore Distributions in Thermal Environments. *Acta Mechanica Solida Sinica*. doi: 10.1007/s10338-024-00549-6