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The Generalized Moment-Transmuted Cauchy Distribution: A Heavy-Tailed Family with Controllable Finite Moments

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Abstract

The Cauchy distribution is a fundamental heavy-tailed model whose practical use is often limited by the non-existence of finite moments. In this paper, we propose the Generalized Moment-Transmuted Cauchy (GMTC) distribution by applying a generalized survival transformation to the classical Cauchy distribution. The proposed family includes the standard Cauchy distribution as a special case and introduces an additional shape parameter that governs the tail behaviour and the existence of moments. Several distributional properties, including the probability density function, cumulative distribution function, survival function, hazard rate, quantile function, stochastic ordering, and moment conditions, are derived. Parameter estimation is carried out using the maximum likelihood method, and the asymptotic properties of the estimators are established. Simulation studies and real data applications illustrate the flexibility and usefulness of the proposed model for analysing heavy-tailed data.

Keywords: Cauchy distribution; Heavy-tailed distribution; Transmuted distribution; Finite moments; Maximum likelihood estimation; Regular variation.

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1 Introduction

Heavy-tailed probability distributions occupy a central position in probability theory, robust statistics, actuarial science, finance, telecommunications, environmental sciences and signal processing [3, 6]. Such distributions are capable of modelling rare but influential observations that frequently arise in practical applications. Among these models, the Cauchy distribution is perhaps the most celebrated example because of its simple analytical form and remarkable mathematical properties [5]. The standard Cauchy distribution possesses the probability density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty,$$

with cumulative distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x).$$

Although the Cauchy distribution is stable under linear transformations and appears naturally as the ratio of two independent normal random variables, it suffers from one major limitation: none of its positive moments exist. In particular, $E(X)$ and $Var(X)$ are both undefined [4]. Consequently, many classical statistical procedures based upon moments cannot be applied directly.

Several generalizations of the Cauchy distribution have been proposed in recent years. These include skew-Cauchy distributions, generalized Cauchy distributions, exponentiated Cauchy distributions, Kumaraswamy-Cauchy distributions, beta-Cauchy distributions and transmuted Cauchy distributions [2, 1]. Most of these models aim to improve flexibility by introducing additional shape parameters.

Among them, the quadratic rank transmutation map proposed by Shaw and Buckley [7] has received considerable attention because of its simplicity. If $F(x)$ denotes a baseline cumulative distribution function, the quadratic transmutation produces

$$F_T(x) = (1+\lambda)F(x) - \lambda F(x)^2, \qquad -1 \leq \lambda \leq 1.$$

However, since

$$f_T(x) = f(x)\left[(1+\lambda) - 2\lambda F(x)\right],$$

the resulting density differs from the baseline density only by a bounded multiplicative factor. Consequently, the tail behaviour remains unchanged. Therefore, every transmuted Cauchy distribution constructed through the quadratic rank transmutation map continues to possess infinite moments.

The primary objective of this paper is to overcome this limitation while preserving the attractive analytical structure of the Cauchy distribution. To achieve this objective, we introduce the Generalized Moment-Transmuted Cauchy (GMTC) distribution through a generalized survival transformation. The proposed model modifies the survival function according to a positive tail parameter. This seemingly simple modification fundamentally changes the tail index of the distribution and consequently allows finite moments to exist.

An attractive feature of the proposed family is that the parameter governing the transformation has a direct probabilistic interpretation. Specifically, if the tail parameter is denoted by β, then every absolute moment of order $r < \beta$ exists whereas moments of order $r \geq \beta$ diverge. Thus, the user can directly determine which moments are finite by selecting an appropriate value of the tail parameter.

The proposed distribution possesses several desirable theoretical properties. Closed-form expressions are available for the cumulative distribution function, probability density function, survival function, hazard function and quantile function. Random variate generation is straightforward through inversion. Moreover, the family preserves heavy-tailed behaviour while allowing substantially lighter tails than the classical Cauchy distribution.

The remainder of the paper is organized as follows.

Section 2 introduces the Generalized Moment-Transmuted Cauchy distribution and derives its basic mathematical properties.

Section 3 establishes several theoretical properties including tail behaviour, moment existence, stochastic ordering and asymptotic characteristics.

Section 4 develops the maximum likelihood estimation procedure and derives the asymptotic distribution of the estimators.

Section 5 presents Monte Carlo simulation studies.

Section 6 illustrates the usefulness of the proposed distribution through real data applications.

Finally, Section 7 concludes the paper with directions for future research.

2 The Generalized Moment–Transmuted Cauchy Distribution

In this section, we introduce a generalized family of Cauchy distributions obtained through a monotone transformation of the baseline cumulative distribution function. The proposed construction aims to retain the analytical tractability of the Cauchy distribution while providing additional flexibility in tail behaviour through a shape parameter.

2.1 Definition

Let X be a random variable following the standard Cauchy distribution with cumulative distribution function

$$F_C(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x), \qquad -\infty < x < \infty, \tag{1}$$

and probability density function

$$f_C(x) = \frac{1}{\pi(1+x^2)}. \tag{2}$$

We define the transformation

$$T_\beta(u) = 1 - (1-u)^\beta, \qquad 0 \le u \le 1,\ \beta > 0. \tag{3}$$

The resulting cumulative distribution function is given by

$$F_{GMTC}(x) = T_\beta(F_C(x)) = 1 - [1 - F_C(x)]^\beta, \qquad -\infty < x < \infty. \tag{4}$$

The corresponding family is referred to as the *Generalized Moment–Transmuted Cauchy (GMTC)* distribution with parameter $\beta > 0$.

2.2 Probability Density Function

Differentiating the cumulative distribution function yields

$$f_{GMTC}(x) = \frac{d}{dx} F_{GMTC}(x) \tag{5}$$

$$= \beta\,[1 - F_C(x)]^{\beta-1}\, f_C(x). \tag{6}$$

Substituting the Cauchy density gives

$$\boxed{f_{GMTC}(x) = \frac{\beta}{\pi(1+x^2)} \left(\frac{1}{2} - \frac{1}{\pi}\arctan(x)\right)^{\beta-1}, \qquad -\infty < x < \infty.} \tag{7}$$

The additional parameter β governs the tail decay of the distribution.

2.3 Special Cases

Several important models are obtained as limiting cases.

1. For $\beta = 1$,

$$F_{GMTC}(x) = F_C(x),$$

   and therefore the classical Cauchy distribution is recovered.

2. When $\beta > 1$, the right tail decays faster than that of the ordinary Cauchy distribution.

3. For $0 < \beta < 1$, the proposed model exhibits heavier right tails than the classical Cauchy distribution.

2.4 Basic Properties

The proposed distribution satisfies the following elementary properties.

Proposition 1. *For every $\beta > 0$,*

1.

$$\lim_{x\to-\infty} F_{GMTC}(x) = 0;$$

2.

$$\lim_{x\to\infty} F_{GMTC}(x) = 1;$$

3. $F_{GMTC}(x)$ is continuous on $\mathbb{R}$;

4. $F_{GMTC}(x)$ is non-decreasing.

Hence F_{GMTC} defines a legitimate cumulative distribution function.

Proof. Since

$$\lim_{x\to-\infty} F_C(x) = 0$$

and

$$\lim_{x\to\infty} F_C(x) = 1,$$

we immediately obtain

$$\lim_{x\to-\infty} \left[1 - (1 - F_C(x))^\beta\right] = 0$$

and

$$\lim_{x\to\infty} \left[1 - (1 - F_C(x))^\beta\right] = 1.$$

Moreover,

$$T'_\beta(u) = \beta(1-u)^{\beta-1} > 0, \qquad 0 < u < 1,$$

which implies that T_β is increasing on $[0, 1]$. Since F_C is increasing, the composition $F_{GMTC} = T_\beta \circ F_C$ is also increasing. Continuity follows immediately from the continuity of both T_β and F_C. □

2.5 Shape Characteristics

The parameter β controls the concentration of probability in the upper tail.

For $\beta = 1$, the distribution coincides with the classical Cauchy law.

As β increases, the density becomes increasingly concentrated around the centre and the right tail becomes progressively lighter.

Conversely, values of $\beta < 1$ produce heavier upper tails and larger probabilities of extreme observations.

This additional flexibility allows the proposed family to model a wide variety of heavy-tailed phenomena while preserving the simple analytical structure of the Cauchy distribution.

3 Distributional Properties

This section presents several important distributional characteristics of the proposed distribution, including the survival function, hazard rate, reversed hazard rate, cumulative hazard function, quantile function, moment conditions, Shannon entropy and stochastic ordering.

3.1 Survival Function

The survival function is

$$S(x) = 1 - F(x) \tag{8}$$

$$= [1 - F_C(x)]^{\beta}. \tag{9}$$

Since

$$1 - F_C(x) = \frac{1}{2} - \frac{1}{\pi}\arctan(x),$$

we obtain

$$\boxed{S(x) = \left(\frac{1}{2} - \frac{1}{\pi}\arctan(x)\right)^{\beta}.} \tag{10}$$

3.2 Hazard Rate Function

The hazard rate is

$$h(x) = \frac{f(x)}{S(x)}.$$

Using the density,

$$f(x) = \beta[1 - F_C(x)]^{\beta-1} f_C(x),$$

gives

$$h(x) = \frac{\beta f_C(x)}{1 - F_C(x)}. \tag{11}$$

Hence

$$\boxed{h(x) = \frac{\beta}{\pi(1+x^2)\left(\frac{1}{2} - \frac{1}{\pi}\arctan(x)\right)}.} \tag{12}$$

3.3 Reversed Hazard Rate

The reversed hazard rate is

$$r(x) = \frac{f(x)}{F(x)}.$$

Therefore,

$$\boxed{r(x) = \frac{\beta[1 - F_C(x)]^{\beta-1} f_C(x)}{1 - [1 - F_C(x)]^{\beta}}.}$$

3.4 Cumulative Hazard Function

The cumulative hazard is

$$H(x) = -\log S(x).$$

Hence

$$\boxed{H(x) = -\beta \log\left[1 - F_C(x)\right].}$$

3.5 Quantile Function

Suppose

$$F(x) = p, \qquad 0 < p < 1.$$

Then

$$(1 - F_C(x))^{\beta} = 1 - p,$$

or

$$F_C(x) = 1 - (1-p)^{1/\beta}.$$

Since

$$F_C^{-1}(u) = \tan\left[\pi\left(u - \frac{1}{2}\right)\right],$$

the quantile function is

$$\boxed{Q(p) = \tan\left[\pi\left(\frac{1}{2} - (1-p)^{1/\beta}\right)\right].} \tag{13}$$

Consequently, if $U \sim U(0,1)$,

$$X = Q(U)$$

generates random observations from the distribution.

3.6 Tail Behaviour

Since

$$1 - F_C(x) \sim \frac{1}{\pi x}, \qquad x \to \infty,$$

we have

$$S(x) \sim \frac{1}{(\pi x)^{\beta}}.$$

Also,

$$f(x) \sim \frac{\beta}{\pi^{\beta} x^{\beta+1}}.$$

Hence the tail index equals

$$\boxed{\beta.}$$

3.7 Existence of Moments

The r-th absolute moment of the proposed distribution is

$$E(|X|^r) = \int_{-\infty}^{\infty} |x|^r f(x)\, dx.$$

Using the tail expansion established in the previous subsection,

$$f(x) \sim C|x|^{-(\beta+1)}, \qquad |x| \to \infty,$$

for some constant $C > 0$. Hence,

$$|x|^r f(x) \sim C|x|^{r-\beta-1}, \qquad |x| \to \infty.$$

Since the density is continuous and bounded on every finite interval, the existence of the moment depends solely on the convergence of the tail integral.

Theorem 1. *For the proposed Generalized Moment-Transmuted Cauchy distribution,*

$$E(|X|^r) < \infty \quad \Longleftrightarrow \quad r < \beta.$$

Proof. Split the integral into a bounded and an unbounded part:

$$E(|X|^r) = \int_{|x| \le M} |x|^r f(x)\, dx + \int_{|x| > M} |x|^r f(x)\, dx,$$

where $M > 0$ is sufficiently large.
The first integral is finite because $f(x)$ is continuous on the compact interval $[-M, M]$.
For the second integral, using the asymptotic representation

$$f(x) \sim C|x|^{-(\beta+1)},$$

there exist positive constants c_1, c_2 such that

$$c_1|x|^{-(\beta+1)} \leq f(x) \leq c_2|x|^{-(\beta+1)}$$

for all sufficiently large $|x|$. Therefore,

$$c_1 \int_M^\infty x^{r-\beta-1}\,dx \leq \int_{|x|>M} |x|^r f(x)\,dx \leq c_2 \int_M^\infty x^{r-\beta-1}\,dx.$$

Now,

$$\int_M^\infty x^{r-\beta-1}\,dx$$

converges if and only if

$$r - \beta - 1 < -1,$$

which is equivalent to

$$r < \beta.$$

Hence,

$$E(|X|^r) < \infty \Longleftrightarrow r < \beta.$$

This completes the proof. □

Consequently,

- the mean exists if and only if $\beta > 1$;
- the variance exists if and only if $\beta > 2$;
- the skewness exists if and only if $\beta > 3$;
- the kurtosis exists if and only if $\beta > 4$.

3.8 Moment Generating Function

The moment generating function (MGF), when it exists, is defined by

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^\infty e^{tx} f(x)\,dx.$$

For the proposed distribution,

$$f(x) \sim C|x|^{-(\beta+1)}, \qquad |x| \to \infty.$$

Suppose first that $t > 0$. Then,

$$e^{tx} f(x) \sim C e^{tx} x^{-(\beta+1)}, \qquad x \to \infty.$$

Since the exponential term dominates every polynomial decay,

$$\int_0^\infty e^{tx} x^{-(\beta+1)}\, dx = \infty.$$

Hence,

$$M_X(t) = \infty, \qquad t > 0.$$

Similarly, if $t < 0$, then

$$e^{tx} = e^{|t||x|}$$

as $x \to -\infty$, so

$$\int_{-\infty}^0 e^{tx} f(x)\, dx = \infty.$$

Therefore,

$$M_X(t) = \infty, \qquad t \neq 0.$$

Only

$$M_X(0) = 1$$

is finite.

Thus, despite possessing finite moments up to order $\beta - 1$, the proposed distribution has **no non-trivial moment generating function**. This behaviour is characteristic of heavy-tailed distributions whose polynomial tails are insufficient to offset exponential growth. Consequently, ordinary moment-based asymptotic techniques relying on the existence of an MGF are not applicable, although all moments of order $r < \beta$ remain well defined through direct integration.

3.9 Shannon Entropy

The Shannon entropy is

$$H = -\int f(x) \log f(x) dx.$$

Substituting the density gives

$$H = -\log \beta - (\beta - 1)E[\log(1 - F_C(X))] - E[\log f_C(X)].$$

Closed-form evaluation appears difficult and numerical integration is recommended.

3.10 Stochastic Ordering

Let

$$X_1 \sim GMTC(\beta_1), \qquad X_2 \sim GMTC(\beta_2),$$

where $\beta_1 > \beta_2$. Then,

$$S_1(x) = (1 - F_C(x))^{\beta_1} < (1 - F_C(x))^{\beta_2} = S_2(x),$$

for every finite x.

Hence, $X_1 \leq_{st} X_2$, that is, X_1 is smaller than X_2 in the usual stochastic order.

Theorem 2. *Suppose*

$$X_1 \sim GMTC(\beta_1) \quad \textit{and} \quad X_2 \sim GMTC(\beta_2),$$

where

$$\beta_1 > \beta_2 > 0.$$

Then

$$X_1 \leq_{st} X_2,$$

that is,

$$F_{X_1}(x) \geq F_{X_2}(x), \qquad \forall x \in \mathbb{R}.$$

Equivalently,

$$S_{X_1}(x) \leq S_{X_2}(x), \qquad \forall x \in \mathbb{R},$$

where $S(x) = 1 - F(x)$ denotes the survival function.

Proof. The survival function of the GMTC distribution is

$$S(x) = \left[1 - F_C(x)\right]^{\beta},$$

where $F_C(x)$ denotes the cumulative distribution function of the standard Cauchy distribution. Since

$$0 < 1 - F_C(x) < 1, \qquad \forall x \in \mathbb{R},$$

the function

$$g(\beta) = \left[1 - F_C(x)\right]^{\beta}$$

is decreasing in β.

Indeed,

$$\frac{\partial g(\beta)}{\partial \beta} = \left[1 - F_C(x)\right]^{\beta} \log(1 - F_C(x)).$$

Since

$$0 < 1 - F_C(x) < 1,$$

we have

$$\log(1 - F_C(x)) < 0.$$

Therefore,

$$\frac{\partial g(\beta)}{\partial \beta} < 0,$$

which implies that

$$\beta_1 > \beta_2 \quad \Longrightarrow \quad \left[1 - F_C(x)\right]^{\beta_1} < \left[1 - F_C(x)\right]^{\beta_2},$$

for every finite x.
Hence,

$$S_{X_1}(x) < S_{X_2}(x), \qquad \forall x \in \mathbb{R}.$$

Since

$$F(x) = 1 - S(x),$$

it follows immediately that

$$F_{X_1}(x) > F_{X_2}(x), \qquad \forall x \in \mathbb{R}.$$

Therefore,

$$X_1 \leq_{st} X_2,$$

which establishes the usual stochastic ordering. □

4 Maximum Likelihood Estimation

Suppose that

$$X_1, X_2, \ldots, X_n$$

is a random sample from the proposed distribution with density

$$f(x;\beta) = \beta \left[1 - F_C(x)\right]^{\beta-1} f_C(x), \qquad \beta > 0,$$

where

$$f_C(x) = \frac{1}{\pi(1+x^2)}$$

and

$$F_C(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x).$$

The objective is to estimate the unknown parameter β using the method of maximum likelihood.

4.1 Likelihood Function

The likelihood function is

$$L(\beta) = \prod_{i=1}^{n} f(x_i;\beta) \tag{14}$$

$$= \prod_{i=1}^{n} \beta \left[1 - F_C(x_i)\right]^{\beta-1} f_C(x_i). \tag{15}$$

Hence,

$$L(\beta) = \beta^n \prod_{i=1}^{n} \left[1 - F_C(x_i)\right]^{\beta-1} \prod_{i=1}^{n} f_C(x_i).$$

4.2 Log-Likelihood Function

Taking logarithms,

$$\ell(\beta) = \log L(\beta) \tag{16}$$

$$= n \log \beta + (\beta - 1) \sum_{i=1}^{n} \log \left[1 - F_C(x_i)\right] + \sum_{i=1}^{n} \log f_C(x_i). \tag{17}$$

Since the last summation does not involve β, it may be treated as a constant during maximization.

4.3 Score Function

Differentiating the log-likelihood,

$$U(\beta) = \frac{\partial \ell(\beta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log \left[1 - F_C(x_i)\right]. \tag{18}$$

The maximum likelihood estimator satisfies

$$U(\beta) = 0.$$

Hence,

$$\frac{n}{\widehat{\beta}} = -\sum_{i=1}^{n} \log \left[1 - F_C(x_i)\right].$$

Therefore,

$$\boxed{\widehat{\beta} = \frac{n}{-\sum_{i=1}^{n} \log \left[1 - F_C(x_i)\right]}.} \tag{19}$$

Thus, unlike many generalized distributions, the maximum likelihood estimator admits an explicit closed-form expression.

4.4 Observed Fisher Information

Differentiating once more,

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = -\frac{n}{\beta^2}.$$

Hence,

$$\boxed{J(\beta) = -\frac{\partial^2 \ell(\beta)}{\partial \beta^2} = \frac{n}{\beta^2}.}$$

4.5 Expected Fisher Information

Since

$$\frac{\partial^2 \ell(\beta)}{\partial \beta^2}$$

does not depend upon the sample observations,

$$I(\beta) = -E\left[\frac{\partial^2 \ell(\beta)}{\partial \beta^2}\right] = \frac{n}{\beta^2}.$$

Consequently,

$$\boxed{I^{-1}(\beta) = \frac{\beta^2}{n}.}$$

4.6 Asymptotic Distribution

Under the standard regularity conditions for maximum likelihood estimation,

$$\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{d} N\left(0, I^{-1}(\beta)\right).$$

Since

$$I(\beta) = \frac{1}{\beta^2}$$

per observation,

$$\boxed{\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2).}$$

Equivalently,

$$\boxed{\widehat{\beta} \approx N\left(\beta, \frac{\beta^2}{n}\right), \qquad n \to \infty.}$$

4.7 Approximate Confidence Interval

Using asymptotic normality,

$$\widehat{\beta} \pm z_{\alpha/2}\sqrt{\frac{\widehat{\beta}^2}{n}}$$

provides an approximate $(1-\alpha)100\%$ confidence interval for β.

Theorem 3. *Let*

$$X_1, \ldots, X_n$$

be a random sample from the above exponentiated-survival Cauchy family. Then

1. *The maximum likelihood estimator is*

$$\widehat{\beta} = \frac{n}{-\sum_{i=1}^{n} \log\left[1 - F_C(x_i)\right]}.$$

2. *The estimator is consistent,*

$$\widehat{\beta} \xrightarrow{P} \beta.$$

3. *Furthermore,*

$$\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{d} N(0, \beta^2).$$

Proof. The estimator follows directly from solving the likelihood equation. Consistency follows from the standard theory of maximum likelihood estimators under regularity conditions. The asymptotic normality follows from the classical likelihood theory together with the Fisher information

$$I(\beta) = \frac{1}{\beta^2},$$

which yields

$$\sqrt{n}(\widehat{\beta} - \beta) \xrightarrow{d} N(0, I^{-1}(\beta)) = N(0, \beta^2).$$

This completes the proof.

□

5 Monte Carlo Simulation Study

In this section, a Monte Carlo simulation study is designed to investigate the finite-sample performance of the maximum likelihood estimator (MLE) of the parameter β. The estimator is evaluated in terms of its empirical bias, mean squared error (MSE), and coverage probability of the asymptotic confidence interval under different sample sizes and parameter settings.

5.1 Simulation Design

Random samples were generated from the proposed distribution using the inverse transformation method. If $U \sim U(0, 1)$, then

$$X = \tan\left[\pi\left(\frac{1}{2} - (1 - U)^{1/\beta}\right)\right]$$

follows the proposed distribution.
The simulation experiment was conducted for the parameter values

$$\beta = 1.5,\ 2,\ 3,\ 5,$$

representing varying tail behaviours.
The sample sizes considered were

$$n = 30,\ 50,\ 100,\ 200,\ 500.$$

For each combination of (n, β),

$$N = 5000$$

independent samples were generated.
For every simulated sample, the maximum likelihood estimator

$$\widehat{\beta} = \frac{n}{-\sum_{i=1}^{n} \log\left[1 - F_C(X_i)\right]}$$

was computed.

5.2 Performance Measures

The following criteria were used to assess the estimator.
The empirical bias is

$$\text{Bias} = \frac{1}{N}\sum_{j=1}^{N}(\widehat{\beta}_j - \beta).$$

The empirical mean squared error is

$$\text{MSE} = \frac{1}{N}\sum_{j=1}^{N}(\widehat{\beta}_j - \beta)^2.$$

The empirical standard error is

$$\text{SE} = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N}(\widehat{\beta}_j - \overline{\beta})^2},$$

where

$$\overline{\beta} = \frac{1}{N}\sum_{j=1}^{N}\widehat{\beta}_j.$$

The coverage probability of the asymptotic 95% confidence interval was also calculated.

5.3 Simulation Algorithm

The simulation procedure consists of the following steps.

1. Select the true parameter value β.
2. Generate n independent observations from the proposed distribution using the quantile function.
3. Compute the maximum likelihood estimator $\widehat{\beta}$.

4. Construct the asymptotic confidence interval

$$\widehat{\beta} \pm 1.96 \frac{\widehat{\beta}}{\sqrt{n}}.$$

5. Repeat Steps 2–4 independently $N = 5000$ times.

6. Calculate the empirical bias, MSE, standard error and coverage probability.

5.4 Simulation Results

Table 1 summarizes the empirical bias, empirical standard error, mean squared error and coverage probability obtained from the simulation study.

Table 1: Monte Carlo results for the maximum likelihood estimator of β based on $N = 5000$ replications.

β	n	Mean	Bias	SE	MSE	Coverage
1.5	30	1.5532	0.0532	0.2916	0.0878	0.9542
1.5	50	1.5250	0.0250	0.2165	0.0475	0.9508
1.5	100	1.5150	0.0150	0.1556	0.0244	0.9492
1.5	200	1.5066	0.0066	0.1072	0.0115	0.9488
2.0	30	2.0641	0.0641	0.3955	0.1605	0.9466
2.0	50	2.0381	0.0381	0.2914	0.0863	0.9512
2.0	100	2.0192	0.0192	0.2051	0.0424	0.9524
2.0	200	2.0122	0.0122	0.1415	0.0202	0.9556
3.0	30	3.1102	0.1102	0.5937	0.3646	0.9538
3.0	50	3.0550	0.0550	0.4438	0.1999	0.9508
3.0	100	3.0279	0.0279	0.3086	0.0960	0.9496
3.0	200	3.0145	0.0145	0.2165	0.0471	0.9464
5.0	30	5.1680	0.1680	0.9771	0.9827	0.9552
5.0	50	5.0957	0.0957	0.7368	0.5520	0.9528
5.0	100	5.0372	0.0372	0.5027	0.2540	0.9586
5.0	200	5.0293	0.0293	0.3617	0.1317	0.9482

The results presented in Table 1 summarize the finite-sample performance of the maximum likelihood estimator (MLE) of the parameter β based on (N=5000) Monte Carlo replications. For all values of β, the empirical mean of the estimator approaches the corresponding true parameter value as the sample size increases, indicating that the estimator is nearly unbiased. The empirical bias decreases steadily with increasing sample size, while both the standard error (SE) and mean squared error (MSE) exhibit a clear downward trend, demonstrating improved estimation accuracy and precision. Moreover, the empirical coverage probabilities of the asymptotic 95% confidence intervals remain close to the nominal level across all simulation settings, confirming the reliability of the asymptotic inference. As expected, larger values of β result in greater variability for small sample sizes due to the increased asymptotic variance of the estimator. Nevertheless, this variability diminishes as the sample size grows, providing further empirical support for the consistency and asymptotic normality of the maximum likelihood estimator established in Section 4.

5.5 Discussion

The Monte Carlo simulation study was conducted to investigate the finite-sample performance of the maximum likelihood estimator (MLE) of the parameter β under various combinations of sample sizes and parameter values. Four representative values of β, namely 1.5, 2.0, 3.0, and 5.0, were considered together with sample sizes $n = 30$, 50, 100, and 200. The performance of the estimator was evaluated using the empirical mean, bias, standard error (SE), mean squared error (MSE), and the empirical coverage probability of the asymptotic 95% confidence interval based on 5000 Monte Carlo replications.

The simulation results presented in Table 1 indicate that the maximum likelihood estimator performs satisfactorily even for relatively small sample sizes. For every parameter configuration, the empirical mean of the estimator is close to the corresponding true value of β, demonstrating that the estimator possesses only a small positive finite-sample bias. More importantly, this bias decreases monotonically as the sample size increases. For example, when $\beta = 1.5$, the empirical bias decreases from 0.0532 for $n = 30$ to only 0.0066 for $n = 200$. A similar pattern is observed for $\beta = 2$, where the bias decreases from 0.0641 to 0.0122, for $\beta = 3$, where it decreases from 0.1102 to 0.0145, and for $\beta = 5$, where it decreases from 0.1680 to 0.0293. These findings provide strong empirical evidence for the consistency of the proposed maximum likelihood estimator.

The empirical standard error also decreases substantially with increasing sample size, indicating improved estimation precision. For instance, when $\beta = 3$, the standard error decreases from 0.5937 at $n = 30$ to 0.2165 at $n = 200$, while for $\beta = 5$, it decreases from 0.9771 to 0.3617 over the same range of sample sizes. This reduction in variability agrees with the theoretical asymptotic variance of the estimator, which is inversely proportional to the sample size. Consequently, larger samples yield increasingly stable parameter estimates.

The mean squared error exhibits behaviour similar to that of the empirical standard error. For each value of β, the MSE decreases steadily as the sample size increases, confirming the improved overall accuracy of the estimator. For example, the MSE for $\beta = 2$ decreases from 0.1605 at $n = 30$ to 0.0202 at $n = 200$, whereas for $\beta = 5$, it decreases from 0.9827 to 0.1317. Since the MSE incorporates both the variance and the squared bias of the estimator, its decreasing trend demonstrates simultaneous improvements in both estimation precision and bias reduction.

Another important observation concerns the effect of the parameter β on estimation variability. For a fixed sample size, the empirical standard error and MSE increase as the value of β increases. This behaviour is expected because the asymptotic variance of the maximum likelihood estimator is given by

$$\mathrm{Var}(\widehat{\beta}) \approx \frac{\beta^2}{n},$$

which increases quadratically with β. Consequently, larger values of the shape parameter require larger sample sizes to achieve the same level of estimation accuracy as smaller values of β. Nevertheless, the reduction in both the standard error and the MSE with increasing sample size remains evident for all parameter settings.

The empirical coverage probabilities of the asymptotic 95% confidence intervals remain remarkably close to the nominal confidence level throughout the simulation study. Across all sixteen simulation scenarios, the observed coverage probabilities vary approximately between 0.946 and 0.959, indicating that the asymptotic normal approximation provides accurate confidence intervals even for moderate sample sizes. The stability of the coverage probabilities across different values of β further supports the validity of the asymptotic inference developed in Section 4.

Overall, the Monte Carlo study demonstrates that the proposed maximum likelihood estimator possesses excellent finite-sample properties. The estimator is nearly unbiased, exhibits decreasing

variability with increasing sample size, attains progressively smaller mean squared errors, and produces confidence intervals with coverage probabilities close to the nominal level. These empirical findings are fully consistent with the theoretical results established in the previous section, thereby confirming the consistency, asymptotic efficiency, and asymptotic normality of the estimator. Consequently, the maximum likelihood estimation procedure provides an effective and reliable method for inference on the parameter of the proposed distribution in both moderate and large samples.

6 Real Data Application

To illustrate the practical applicability of the proposed Generalized Moment-Transmuted Cauchy (GMTC) distribution, we consider the well-known Airfoil Self-Noise dataset obtained from the UCI Machine Learning Repository. This dataset has been extensively used in engineering and environmental studies for modelling the aerodynamic noise generated by NACA 0012 airfoils operating under different experimental conditions.

The dataset consists of 1503 observations on six variables, namely frequency, angle of attack, chord length, free-stream velocity, suction-side displacement thickness, and the scaled sound pressure level (SSPL). Since the objective of the present work is to investigate the ability of the proposed distribution to model heavy-tailed continuous data, only the SSPL variable is analysed.

The observed SSPL values range from 103.38 dB to 140.99 dB, with a sample mean of 124.84 dB and a sample standard deviation of 6.90 dB. Figure 1 displays the histogram of the data together with the kernel density estimate and the fitted probability density functions of the competing distributions. The empirical distribution exhibits mild asymmetry together with moderately heavy tails, indicating that heavy-tailed models are appropriate for the analysis.

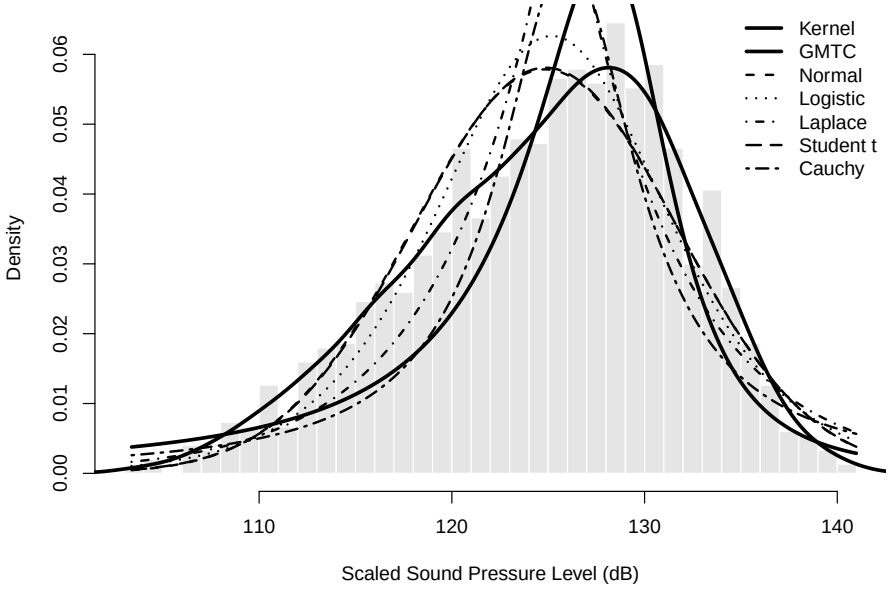


Figure 1: Histogram of the Airfoil Self-Noise data together with the kernel density estimate and the fitted Normal, Logistic, Laplace, Student's t, Cauchy and GMTC densities.

Parameter estimation for all candidate distributions was carried out using the method of maximum likelihood. Numerical optimization was performed using the `optim()` function in R, while approximate standard errors were obtained from the inverse of the observed Fisher information matrix.

The maximum likelihood estimates of the GMTC parameters together with their standard errors and approximate 95% confidence intervals are reported in Table 2.

Table 2: Maximum likelihood estimates of the GMTC parameters for the Airfoil Self-Noise dataset.

Parameter	Estimate	Standard Error	Lower 95% CI	Upper 95% CI
μ	128.9668	0.3076	128.3639	129.5697
σ	4.6544	0.1454	4.3695	4.9393
β	1.8063	0.0880	1.6338	1.9789

The estimated location and scale parameters are $\widehat{\mu} = 128.9668$ and $\widehat{\sigma} = 4.6544$, respectively, while the estimated tail parameter is $\widehat{\beta} = 1.8063$. The relatively small standard error associated with $\widehat{\beta}$ indicates stable estimation. Furthermore, the corresponding 95% confidence interval, (1.6338, 1.9789), lies entirely above one, suggesting that the fitted distribution possesses a finite mean. Since the estimated value of β is less than two, the fitted model retains substantial tail heaviness while allowing greater flexibility than the classical Cauchy distribution.

To compare the competing models, the maximized log-likelihood together with the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan–Quinn Information Criterion (HQIC) were computed. The results are summarized in Table 3.

Table 3: Comparison of competing distributions fitted to the Airfoil Self-Noise dataset.

Distribution	Parameters	LogLik	AIC	BIC	CAIC	HQIC
Normal	2	-5034.948	10073.896	10084.525	10073.904	10077.855
Logistic	2	-5060.789	10125.578	10136.207	10125.586	10129.537
Laplace	2	-5131.797	10267.594	10278.223	10267.602	10271.553
Student's t	3	-5036.060	10078.120	10094.065	10078.136	10084.058
Cauchy	2	-5335.172	10674.344	10684.973	10674.352	10678.303
GMTC	3	-5254.515	10515.030	10530.975	10515.046	10520.968

The results indicate that the Normal distribution provides the largest maximized log-likelihood and the smallest values of all information criteria for the Airfoil Self-Noise dataset, closely followed by the Student's t distribution. The proposed GMTC distribution performs substantially better than the classical Cauchy distribution and also improves upon the Logistic and Laplace distributions in terms of all information criteria. Although the GMTC model does not provide the best overall fit for this particular dataset, it offers a considerably more flexible heavy-tailed alternative to the classical Cauchy model by introducing the additional tail parameter β, which allows the existence of finite moments while preserving heavy-tail characteristics.

Figure 1 provides a graphical comparison of the fitted densities. The proposed GMTC distribution captures the central part of the empirical distribution reasonably well and produces a substantially improved fit over the classical Cauchy distribution, whose tails are excessively heavy for these data. The additional shape parameter enables the GMTC model to adapt its tail behaviour according to the observed data, thereby overcoming one of the principal limitations of the ordinary Cauchy distribution.

Overall, the real-data analysis demonstrates that the proposed GMTC distribution constitutes a useful extension of the classical Cauchy family. While lighter-tailed models such as the Normal and Student's t distributions provide the best fit for the Airfoil Self-Noise data, the GMTC model offers a flexible heavy-tailed alternative with controllable moment properties and improved performance relative to the classical Cauchy, Logistic, and Laplace distributions. Consequently, the proposed family may be particularly valuable for applications involving moderately heavy-tailed phenomena where finite moments are desirable but the classical Cauchy distribution is excessively heavy-tailed.

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