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Steady-state bifurcation and pattern formation in an immune chemotaxis system with volume-filling effects

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Abstract. This paper studies a three-component reaction-diffusion-advection immune chemotaxis model incorporating volume-filling effects, which accounts for spatial crowding and density limitation of immune cells. The primary focus is on the existence, direction, and stability of nonconstant steady-state solutions arising from steady-state bifurcation. We first derive the critical values of chemotactic sensitivity that lead to the instability of the constant positive steady state. Then the existence of nonconstant positive steady-state solutions bifurcating from the constant equilibrium by the Crandall-Rabinowitz bifurcation theorem is established. Furthermore, the direction and local stability of the bifurcation branches are investigated by analyzing the quadratic coefficients in the asymptotic expansion. It is proved that the minimal bifurcation point determines the stability of the bifurcated solutions, while other branches remain unstable. Numerical simulations are presented to verify the theoretical conclusions and exhibit the spatial patterns generated by chemotaxis and volume-filling mechanism. This work extends the classical immune chemotaxis model by incorporating the volume-filling mechanism and reveals how cellular crowding affects the stability and pattern formation in immune dynamics, enriching the mathematical theory of immune chemotaxis models.

Keywords: Immune system; Chemotaxis; Volume-filling; Steady-state bifurcation; Pattern.

MSC: 35B40, 35B65, 35K57, 35Q92, 92C17.

1. Introduction

Initially, chemotaxis often describes the directional movement of cells in response to external chemical stimuli, which could greatly influence the survival and evolution of cell [17, 18]. Meanwhile, chemotaxis is also very ubiquitous both in movement of natural species for food or avoiding enemies, and transmission or (prevention) control of disease [2, 3, 9] etc. The mathematical study of chemotaxis began with the classical Keller-Segel model [17], which was originally proposed to describe the aggregation behavior of cellular slime molds. Over the past few decades, this model and its variants have been extensively developed to capture chemotactic phenomena in diverse biological contexts, including bacterial growth, tumor invasion, neural development, and immune response [5, 6, 11, 33, 34, 38].

In recent years, the researches on the immune system with chemotaxis have attracted many biologists and mathematicians [10, 12, 14, 15, 25, 30]. In the immune system, chemotaxis guides immune cells to sites of infection, inflammation, or tissue damage, where they perform pathogen clearance, antigen presentation,

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and inflammatory regulation. Such directional movement is mediated by chemokines secreted by damaged tissues or activated immune cells, which create concentration gradients that immune cells sense and follow. When this process is dysregulated, it can lead to insufficient immune response, persistent infection, hypersensitivity, allergic reactions, chronic inflammation, or even autoimmune diseases [1, 13]. Given its biological importance, the mathematical modeling and rigorous analysis of immune chemotaxis have become an interdisciplinary field connecting applied mathematics, computational biology, and theoretical immunology [23, 24, 32].

In the field of mathematical immunology, considerable efforts have been made to model the interactions among immune cells, chemokines, and antigens. In particular, Lee et al. [22] proposed the following three-component reaction-diffusion-advection system that describes the immune response driven by chemotaxis, including the production and decay of immune cells, the secretion and degradation of chemokines, and the persistence and elimination of antigens

$$\begin{cases} u_t - d_1 \Delta u + \chi \nabla \cdot (u \nabla v) = a_1 - b_1 u w - e_1 u, & x \in \Omega, t > 0, \\ v_t - d_2 \Delta v = b_2 u w - e_2 v, & x \in \Omega, t > 0, \\ w_t - d_3 \Delta w = a_3 - b_3 u w - e_3 w & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = c, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

here $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and ν is the outward unit normal vector on $\partial\Omega$. The variables u, v, w respectively denote the densities of immune cells, chemokines and antigens. Since immune cells are produced in bone marrow and circulated throughout the body, so the positive constant a_1 as source of u . The positive constant antigen source a_3 represents the persistent infection. During the immune process, immune cells and antigens are consumed by regulation and phagocytosis, which are denoted by $b_1 u w$ and $b_3 u w$, respectively. In addition, $b_2 u w$ indicates that chemokines are secreted by immune cells according to the number of antigens. The chemosensitivity $\chi > 0$ of advection term contributes to the movement of immune cells toward the gradient of chemokines.

This model successfully captures the transition from stable immune regulation to unstable hypersensitivity as chemotactic strength increases, and has become a benchmark model for theoretical studies. Subsequently, many researchers have extended this model and investigated the global existence, boundedness, asymptotic behavior, and pattern formation of solutions [4, 20, 29, 35–37, 39–41]. For instance, Zheng et al. [42] analyzed the global existence and boundedness of classical solutions for the quasilinear reaction-diffusion-advection system based on the model (1.1), and proved the stability and instability of the above system based on the spectral analysis, energy estimate and bootstrap technique. Moreover, some numerical simulations are carried out to support the theoretical results and also find some new interesting phenomena, such as complex spatial-temporal patterns. In addition, Shan and Zheng [26] considered the immune cells of model (1.1) with general source function $g(M) \in C^1([0, \infty))$, they proved that if there exist positive constants α and θ_0 such that $\sup_{s \geq 0} \{g(s) + \alpha s\} < \infty$ and $\lim_{s \rightarrow \infty} \inf \{-\frac{g(s)}{s^2}\} := \mu \in (\theta_0, \infty]$, then the solution of the system is global and uniformly bounded. Despite these advances, most existing

studies assume a linear or weakly nonlinear chemotactic sensitivity, typically of the form $\chi \nabla \cdot (u \nabla v)$, which does not account for the finite size of cells and the resulting excluded volume interactions.

Despite these advances, most existing models retain the linear chemotactic term and overlook a key biological constraint: The volume-filling effect. In real biological systems, cells are physical objects that cannot overlap or accumulate without bound. The volume-filling effect describes the limitation on cell density due to spatial crowding, contact inhibition, and local volume exclusion. Mathematically, this effect is often modeled by introducing a multiplicative factor such as $u(1 - u)$ in the chemotactic flux, which suppresses excessive aggregation and keeps cell density within a biologically reasonable range. At present, the dynamic and bifurcation analysis of immune chemotaxis models with volume-filling effect faces several questions have not been systematically addressed:

- Within linear stability analysis of homogeneous steady states, how does the volume-filling effect alter the critical threshold of linear instability?
- By what mechanisms does the volume-filling effect govern nonconstant steady states arising from steady-state bifurcation, covering bifurcation criteria at uniform equilibria, critical triggering parameters, and persistence of heterogeneous spatial patterns?
- To what extent does the volume-filling effect reshape core bifurcation characteristics: local/global classification, bifurcation type (transcritical, pitchfork), propagation direction of bifurcation branches, and linear stability of spatially inhomogeneous bifurcated solutions?

The present paper aims to fill this gap by performing a rigorous analytical and numerical investigation of the following reaction-diffusion-advection system with volume-filling effect, posed on a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) with homogeneous Neumann boundary conditions

$$\begin{cases} u_t - d_1 \Delta u + \chi \nabla \cdot (u(1 - u) \nabla v) = a_1 - b_1 u w - e_1 u, & x \in \Omega, t > 0, \\ v_t - d_2 \Delta v = b_2 u w - e_2 v, & x \in \Omega, t > 0, \\ w_t - d_3 \Delta w = a_3 - b_3 u w - e_3 w & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in \Omega. \end{cases} \quad (1.2)$$

The volume-filling effect is encoded in the factor $u(1 - u)$ multiplying the chemotactic sensitivity. This form guarantees that u remains bounded between 0 and 1 under suitable initial conditions, and it introduces a nonlinear feedback that becomes particularly strong in high-density regions.

The remainder of this paper is structured as follows. Section 2 is devoted to the linear stability analysis of the constant positive steady state. By deriving the characteristic equations and applying the Routh-Hurwitz criterion, we identify the critical chemotactic coefficient χ_0 that triggers instability. In Section 3, we treat χ as a bifurcation parameter and establish the existence of nonconstant positive steady-state solutions via the Crandall-Rabinowitz bifurcation theorem. Under suitable non-degeneracy conditions, a family of spatially inhomogeneous steady states bifurcating from the constant equilibrium is obtained, and their explicit asymptotic expansions are provided. Section 4 focuses on the stability and bifurcation direction of these nonconstant steady states. By analyzing the second- and third-order coefficients in the Lyapunov-Schmidt reduction, we prove that the sign of the bifurcation branch curvature,

and demonstrate that only the branch emanating from the minimal bifurcation point can be locally asymptotically stable; all other bifurcating branches are unstable. Finally, numerical simulations are presented to validate the analytical results and to visualize the emerging spatial patterns and biological implications of the volume-filling effect on immune pattern formation are discussed. We make some comments on our studies and propose some interesting problems for future studies in Section 5.

2. Stability of constant positive steady states

In this section, we discuss the influence of the chemotactic coefficient χ in the model (1.2) on the stability of the positive constant equilibrium point. Let $0 = \mu_1 < \mu_2 < \dots < \mu_k < \dots$ be the eigenvalues of $-\Delta$ on Ω with homogeneous Neumann boundary conditions, $E(\mu_k)$ the eigenspace of μ_k in $H^1(\Omega)$. Denote $\mathbf{X}$ the closure of $[C^1(\bar{\Omega})]^3$ in $[H^1(\bar{\Omega})]^3$, and let $\{\phi_{kj} : j = 1, \dots, \dim E(\mu_k)\}$ be an orthonormal basis of $E(\mu_k)$, $\mathbf{X}_{kj} = \{\mathbf{c}\phi_{kj} : \mathbf{c} \in \mathbb{R}^3\}$. Then

$$\mathbf{X} = \bigoplus_{k=1}^{+\infty} \mathbf{X}_k, \quad \mathbf{X}_k = \bigoplus_{j=1}^{\dim E(\mu_k)} \mathbf{X}_{kj}.$$

Let $\mathbf{z} = (u, v, w)^T$, the unique positive constant equilibrium $\mathbf{z}^* = (u^*, v^*, w^*)$ of model (1.2) is given by

$$\begin{aligned} u^* &= \frac{-e_1 e_3 - a_3 b_1 + a_1 b_3 + \sqrt{4e_1 e_3 a_3 b_1 + (e_1 e_3 - a_3 b_1 + a_1 b_3)^2}}{2e_1 b_3}, \\ v^* &= \frac{b_2(e_1 e_3 + a_3 b_1 + a_1 b_3 - \sqrt{4e_1 e_3 a_3 b_1 + (e_1 e_3 - a_3 b_1 + a_1 b_3)^2})}{2e_2 b_1 b_3}, \\ w^* &= \frac{-e_1 e_3 + a_3 b_1 - a_1 b_3 + \sqrt{4e_1 e_3 a_3 b_1 + (e_1 e_3 - a_3 b_1 + a_1 b_3)^2}}{2e_3 b_1}. \end{aligned}$$

The linearized system of the chemotaxis model (1.2) at $\mathbf{z}^*$ is

$$\begin{cases} \mathbf{z}_t = \bar{D}\Delta\mathbf{z} + J_{\mathbf{z}}(\mathbf{z}^*)\mathbf{z}, & x \in \Omega, \ t > 0, \\ \partial_\nu \mathbf{z} = 0, & x \in \partial\Omega, \ t > 0, \\ \mathbf{z}(x, 0) \geq 0, & x \in \Omega, \end{cases} \quad (2.1)$$

where

$$J_{\mathbf{z}}(\mathbf{z}^*) = \begin{pmatrix} -b_1 w^* - e_1 & 0 & -b_1 u^* \\ b_2 w^* & -e_2 & b_2 u^* \\ -b_3 w^* & 0 & -b_3 u^* - e_3 \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} d_1 & -\chi u^*(1 - u^*) & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

The eigenvalue problem corresponding to the model (2.1) is

$$\begin{cases} \bar{D}\Delta\mathbf{z} + J_{\mathbf{z}}(\mathbf{z}^*)\mathbf{z} = \lambda\mathbf{z}, & x \in \Omega, \\ \partial_\nu \mathbf{z} = 0, & x \in \partial\Omega. \end{cases} \quad (2.2)$$

Furthermore, λ is an eigenvalue of problem (2.2) if and only if for some k , λ is an eigenvalue of the following matrix

$$L(\chi, k) = \begin{pmatrix} -d_1 \mu_k - b_1 w^* - e_1 & \chi u^*(1 - u^*) \mu_k & -b_1 u^* \\ b_2 w^* & -d_2 \mu_k - e_2 & b_2 u^* \\ -b_3 w^* & 0 & -d_3 \mu_k - b_3 u^* - e_3 \end{pmatrix} \quad (2.3)$$

The characteristic polynomial corresponding to the matrix (2.3) is

$$\Psi(\lambda) = \det\{\lambda I - L(\chi, k)\} = \lambda^3 + P(k)\lambda^2 + Q(\chi, k)\lambda + R(\chi, k),$$

where

$$\begin{aligned} P(k) &= P_1, \quad Q(\chi, k) = Q_1 - b_2 u^* (1 - u^*) w^* \mu_k \chi, \\ R(\chi, k) &= R_1 - b_2 u^* (1 - u^*) w^* (d_3 \mu_k + e_3) \mu_k \chi \end{aligned}$$

with

$$\begin{aligned} P_1 &= \mu_k (d_1 + d_2 + d_3) + b_1 w^* + b_3 u^* + e_1 + e_2 + e_3 > 0, \\ Q_1 &= \mu_k^2 (d_1 d_2 + d_1 d_3 + d_2 d_3) + \mu_k ((b_3 u^* + e_2 + e_3) d_1 \\ &\quad + (b_3 u^* + b_1 w^* + e_1 + e_3) d_2 + (b_1 w^* + e_1 + e_2) d_3) \\ &\quad + b_1 w^* (e_2 + e_3) + b_3 u^* (e_1 + e_2) + e_1 e_2 + e_2 e_3 + e_1 e_3 > 0, \\ R_1 &= \mu_k^3 d_1 d_2 d_3 + \mu_k^2 ((b_3 u^* + e_3) d_1 d_2 + (b_1 w^* + e_1) d_2 d_3 + e_2 d_1 d_3) \\ &\quad + \mu_k (e_2 (b_3 u^* + e_3) d_1 + (e_1 (b_3 u^* + e_3) + b_1 e_3 w^*) d_2 + e_2 (b_1 w^* + e_1) d_3) \\ &\quad + b_1 w^* e_2 e_3 + b_3 u^* e_1 e_2 + e_1 e_2 e_3 > 0. \end{aligned}$$

Furthermore

$$\begin{aligned} &P(k)Q(\chi, k) - R(\chi, k) \\ &= P_1 Q_1 - R_1 - b_2 u^* w^* (1 - u^*) (e_1 + e_2 + b_1 w^* + b_3 u^* + (d_1 + d_2) \mu_k) \mu_k \chi, \end{aligned}$$

where

$$P_1 Q_1 - R_1 = c_3 \mu_k^3 + c_2 \mu_k^2 + c_1 \mu_k + P_0 Q_0 - R_0$$

with

$$\begin{aligned} P_0 Q_0 - R_0 &= (b_1 w^* + b_3 u^*) Q_0 + (e_1 + e_2) (b_1 w^* e_2 + b_3 u^* e_1 + e_1 e_2 + e_2 e_3 + e_1 e_3) \\ &\quad + e_3 (e_2 e_3 + e_1 e_2) + b_3 u^* (e_1 e_3 + e_2 e_3 + e_2^2) + b_1 w^* (e_2 e_3 + e_1 e_3 + e_3^2) > 0, \\ c_3 &= (d_1 + d_2) (d_1 + d_3) (d_2 + d_3) > 0, \\ c_2 &= (d_1 + d_2 + d_3) ((b_3 u^* + e_2 + e_3) d_1 + (b_3 u^* + b_1 w^* + e_1 + e_3) d_2 + (b_1 w^* + e_1 + e_2) d_3) \\ &\quad + (b_1 w^* + e_1) d_1 (d_2 + d_3) + (b_3 u^* + e_3) d_3 (d_1 + d_2) + e_2 d_2 (d_1 + d_3) > 0, \\ c_1 &= (d_1 + d_2 + d_3) Q_0 + ((b_3 u^* + e_3) (b_1 w^* + b_3 u^* + e_1 + e_3) + e_2 P_0) d_1 \\ &\quad + ((b_1 w^* + b_3 u^* + e_3) (b_1 w^* + b_3 u^* + e_2) + (b_3 u^* + e_3) e_3 + e_1 b_1 w^*) d_2 \\ &\quad + ((b_1 w^* + e_1) (b_1 w^* + b_3 u^* + e_1 + e_3) + e_2 P_0) d_3 > 0. \end{aligned}$$

Obviously, for all $k \in \mathbb{N}^+$, $P(k) > 0$. According to the Routh-Hurwitz criterion, $\mathbf{z}^*$ is locally asymptotically stable with respect to the model (1.2) if and only if

$$R(\chi, k) > 0, \quad P(k)Q(\chi, k) - R(\chi, k) > 0. \quad (2.4)$$

On the contrary, $\mathbf{z}^*$ is unstable if and only if there exists some $k \in \mathbb{N}^+$ such that at least one of the equations in (2.4) does not hold.

Suppose the condition

$$(\mathbf{H}) \quad a_1(b_3 + e_3) < a_3 b_1 + e_1 b_3 + e_1 e_3$$

holds. Through calculation, we can obtain that $R(\chi, k) > 0$ if and only if $\chi < \chi_k^S$, and $P(k)Q(\chi, k) - R(\chi, k) > 0$ if and only if $\chi < \chi_k^H$. Therefore, if $\chi < \chi_0$, $\mathbf{z}^*$ is locally stable; otherwise, $\mathbf{z}^*$ is unstable. Here

$$\begin{aligned}\chi_0 &= \min_{k \in \mathbb{N}^+} \{\chi_k^S, \chi_k^H\}, \quad \chi_k^S = \frac{R_1}{b_2 u^* w^* (1 - u^*) (d_3 \mu_k + e_3) \mu_k}, \\ \chi_k^H &= \frac{P_1 Q_1 - R_1}{b_2 u^* w^* (1 - u^*) (e_1 + e_2 + b_1 w^* + b_3 u^* + (d_1 + d_2) \mu_k) \mu_k}.\end{aligned}\quad (2.5)$$

Based on the above discussions, the following results can be obtained.

Theorem 2.1. *Assume that all parameters in the model (1.2) are positive constants and that condition (H). Then*

- (1) *If $\chi < \chi_0$, the positive equilibrium point $\mathbf{z}^*$ is locally asymptotically stable;*
- (2) *If $\chi > \chi_0$, the positive equilibrium point $\mathbf{z}^*$ is unstable,*

where χ_0 is defined in the equation (2.5).

Remark 2.1. *For $\chi = 0$, one has $R_1 > 0$ and $P_1 Q_1 - R_1 > 0$. Consequently, the positive equilibrium $\mathbf{z}^*$ remains stable under pure self-diffusion, this shows self-diffusion alone cannot trigger Turing instability.*

To illustrate Theorem 2.1, we give some numerical examples. We use the upwind finite volume scheme introduced in [7, 21]. Since the model is simplified, it is not easy to find all the parameters estimated from the experiments. The diffusion coefficients and decay/death rates are usually known (see [19, 30] and references therein). For example, the diffusion coefficient d_1 is estimated in a range of $10^{-11} \sim 10^{-7} \text{ cm}^2/\text{h}$ (see [31]). In this work, we choose $d_1 = 5 \times 10^{-7} \text{ cm}^2/\text{h}$ in the range. We choose domain length $l = (\text{half of some integers}) \times 0.005\text{m}$, which is relevant to contain a sufficient number of cells considering the size of the cells.

Example 1. *In the model (1.2), select $\Omega = (0, 5 \times 10^{-4})$, and the parameters are*

$$a_1 = 0.05, b_1 = 5, e_1 = 0.1, b_2 = 0.4, e_2 = 0.1,$$

$$a_3 = 5 \times 10^{-3}, b_3 = 1, e_3 = 4 \times 10^{-6}$$

$$d_1 = 5 \times 10^{-7}, d_2 = 6 \times 10^{-8}, d_3 = 3 \times 10^{-9}.$$

It can be calculated that the positive equilibrium point $\mathbf{z}^ = (u^*, v^*, w^*) \approx (0.25, 0.0199, 0.0199)$, and $\chi_0 = \chi_1^H \approx 0.0009618$. When $\chi = 0$, $\mathbf{z}^*$ is always stable, as shown in Figure 1.*

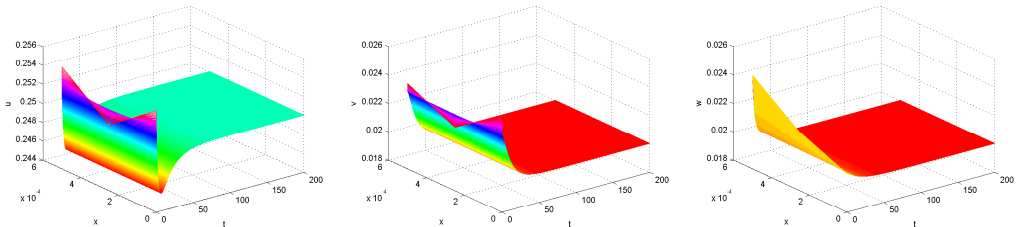


Figure 1. When $\chi = 0$, $\mathbf{z}^*$ is always stable.

According to Theorem 2.1, when $0.0001 = \chi < \chi_0$, the equilibrium point $\mathbf{z}^*$ is locally asymptotically stable, as shown in Figure 2; when $0.00125 = \chi > \chi_0$, $\mathbf{z}^*$ is unstable, as shown in Figure 3.

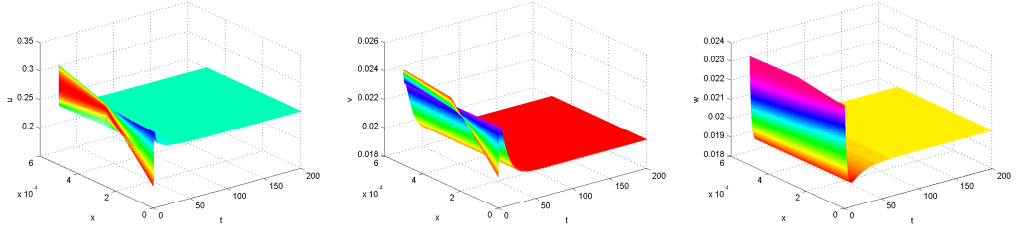


Figure 2. When $\chi = 0.0001 < \chi_0$, $\mathbf{z}^*$ is locally asymptotically stable.

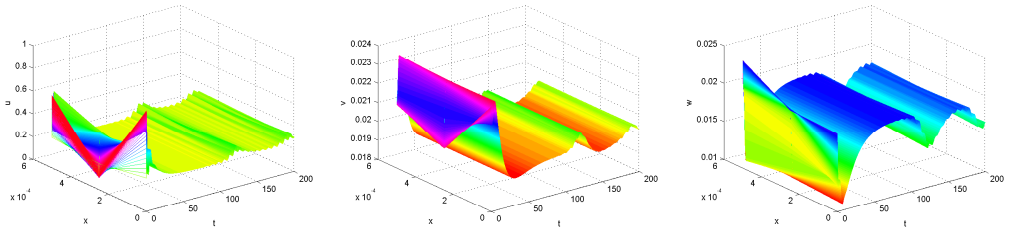


Figure 3. When $\chi = 0.00125 > \chi_0$, $\mathbf{z}^*$ is unstable.

Remark 2.2. When $\chi = 0$, immune cells lose directional chemotactic migration ability; the uniform cell distribution corresponds to physiological steady-state immune surveillance with no inflammatory lesions. Slightly elevated χ below threshold only weak immune recruitment without tissue infiltration. Once χ crosses χ_0 , excessive directional migration breaks immune homeostasis, initiating focal inflammation, which matches the onset of localized granulomas in infectious lesions such as tuberculosis.

3. Existence of nonconstant positive steady states

In this section, we take χ as the bifurcation parameter and study the equilibrium bifurcation near the equilibrium point $\mathbf{z}^*$, thereby proving the existence of non-constant steady-state solutions of model (1.2). From Theorem 2.1, it can be seen that as the chemotactic coefficient χ increases, the positive equilibrium $\mathbf{z}^*$ loses its stability at $\chi_0 = \min\{\chi_k^S, \chi_k^H\}$. If $\chi_0 = \chi_k^S < \chi_k^H$, $R(\chi_0, k_0) = 0$, then $\mathbf{z}^*$ becomes unstable via a steady-state bifurcation, leading to the emergence of spatial patterns.

For computational convenience, we consider the model in the one-dimensional interval $(0, l)$

$$\begin{cases} u_t = (d_1 u' - \chi u(1-u)v')' + a_1 - b_1 u w - e_1 u, & x \in (0, l), t > 0, \\ v_t = d_2 v'' + b_2 u w - e_2 v, & x \in (0, l), t > 0, \\ w_t = d_3 w'' + a_3 - b_3 u w - e_3 w, & x \in (0, l), t > 0, \\ u'(x, t) = v'(x, t) = w'(x, t) = 0, & x = 0, l, t > 0, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), & x \in (0, l) \end{cases} \quad (3.1)$$

and the corresponding steady-state problem

$$\begin{cases} (d_1 u' - \chi u(1-u)v')' + a_1 - b_1 uw - e_1 u = 0, & x \in (0, l), \\ d_2 v'' + b_2 uw - e_2 v = 0, & x \in (0, l), \\ d_3 w'' + a_3 - b_3 uw - e_3 w = 0, & x \in (0, l), \\ u'(x) = v'(x) = w'(x) = 0, & x = 0, l, \end{cases} \quad (3.2)$$

where $'$ denotes the derivative with respect to x , and all other parameters in model (3.1) are defined as in model (1.2).

On the one-dimensional interval $\Omega = (0, l)$ ($l > 0$), the eigenvalue problem

$$\begin{cases} -\omega''(x) = \mu\omega(x), & x \in (0, l), \\ \omega'(x) = 0, & x = 0, l \end{cases}$$

has a sequence of simple eigenvalues

$$\mu_k = \left(\frac{k\pi}{l}\right)^2, \quad k = 0, 1, 2, \dots,$$

with corresponding eigenfunctions

$$\omega_k(x) = \begin{cases} 1, & k = 0, \\ \cos\left(\frac{k\pi x}{l}\right), & k > 0. \end{cases}$$

Clearly, this set of eigenfunctions forms an orthonormal basis in $L^2(0, l)$.

Taking χ as the bifurcation parameter, to apply the Crandall-Rabinowitz bifurcation theorem, we introduce the following function spaces:

$$\mathcal{X} = \{\mathbf{z} : \mathbf{z} \in H^2(0, l) \mid \mathbf{z}'(0) = \mathbf{z}'(l) = 0\}, \quad \mathcal{Y} = L^2(0, l).$$

Define the mapping $\mathcal{F} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathbb{R}^+ \rightarrow \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}$ by

$$\mathcal{F}(u, v, w, \chi) = \begin{pmatrix} (d_1 u' - \chi u(1-u)v')' + a_1 - b_1 uw - e_1 u \\ d_2 v'' + b_2 uw - e_2 v \\ d_3 w'' + a_3 - b_3 uw - e_3 w \end{pmatrix}.$$

Then $\mathcal{F}(\mathbf{z}^*, \chi) = 0$, and model (3.1) is equivalent to $\mathcal{F}(u, v, w, \chi) = 0$. It is easy to verify that the linear operators $D_{\chi}\mathcal{F}$, $D_{(u,v,w)}\mathcal{F}$ and $D_{\chi(u,v,w)}\mathcal{F}$ are continuous. For any fixed $(\bar{u}, \bar{v}, \bar{w}) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$, the Fréchet derivative of $\mathcal{F}$ with respect to (u, v, w) at $(\bar{u}, \bar{v}, \bar{w})$ is

$$\begin{aligned} & D_{(u,v,w)}\mathcal{F}(\bar{u}, \bar{v}, \bar{w}, \chi)(u, v, w) \\ &= \begin{pmatrix} d_1 u'' - \chi(\bar{u}(1-\bar{u})v' + u(1-\bar{u})\bar{v}' - \bar{u}u\bar{v}')' - (b_1 \bar{w} + e_1)u - b_1 \bar{u}w \\ d_2 v'' + b_2 \bar{w}u - e_2 v + b_2 \bar{u}w \\ d_3 w'' - b_3 \bar{w}u - (b_3 \bar{u} + e_3)w \end{pmatrix}. \end{aligned} \quad (3.3)$$

To obtain the existence of local bifurcation solutions, the following steps are required.

Step 1 Verify that for any fixed $(\bar{u}, \bar{v}, \bar{w}) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$, the operator $D_{(u,v,w)}\mathcal{F}(\bar{u}, \bar{v}, \bar{w}, \chi)(u, v, w) : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y}$ is a Fredholm operator of index zero. To this end, rewrite operator (3.3) as

$$D_{(u,v,w)}\mathcal{F}(\bar{u}, \bar{v}, \bar{w}, \chi)(u, v, w) = \mathcal{A}_0(\mathbf{z})\mathbf{z}'' + \mathbf{F}_0(x, \mathbf{z}, \mathbf{z}'),$$

where

$$\mathcal{A}_0 = \begin{pmatrix} d_1 & -\chi\bar{u}(1-\bar{u}) & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

and

$$\mathbf{F}_0 = \begin{pmatrix} -\chi(u(1-\bar{u})\bar{v}' - \bar{u}u\bar{v}')' - \chi\bar{u}'v' + 2\chi\bar{u}\bar{u}'v' - (b_1\bar{w} + e_1)u - b_1\bar{u}w \\ b_2\bar{w}u - e_2v + b_2\bar{u}w \\ -b_3\bar{w}u - (b_3\bar{u} + e_3)w \end{pmatrix}.$$

Obviously, all eigenvalues of $\mathcal{A}_0$ are positive, so operator (3.3) is strictly elliptic. By the second case ($N = 1$) of Remark 2.5 in [27], $\mathcal{A}_0$ satisfies the Agmon condition (see Definition 2.4 in [28]). Therefore, $D_{(u,v,w)}\mathcal{F}(\bar{u}, \bar{v}, \bar{w}, \chi)(u, v, w)$ is a Fredholm operator of index zero (see Theorem 3.3 and Remark 3.4 in [27]).

Step 2 Verify that $\dim \mathcal{N}(D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi)) = 1$, where $\mathcal{N}$ denotes the kernel.

To locate where local bifurcation occurs at $(\mathbf{z}^*, \chi)$, we need to discuss when the implicit function theorem fails for $\mathcal{F}$, i.e., when the condition

$$\mathcal{N}(D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi)) \neq \{0\} \quad (3.4)$$

holds. To this end, set $(\bar{u}, \bar{v}, \bar{w}) = (u^*, v^*, w^*)$ in operator (3.3); then

$$D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi)(u, v, w) = \begin{pmatrix} d_1u'' - \chi u^*(1-u^*)v'' - (b_1w^* + e_1)u - b_1u^*w \\ d_2v'' + b_2w^*u - e_2v + b_2u^*w \\ d_3w'' - b_3w^*u - (b_3u^* + e_3)w \end{pmatrix},$$

and its kernel consists of solutions of the problem

$$\begin{cases} d_1u'' - \chi u^*(1-u^*)v'' - (b_1w^* + e_1)u - b_1u^*w = 0, & x \in (0, l), \\ d_2v'' + b_2w^*u - e_2v + b_2u^*w = 0, & x \in (0, l), \\ d_3w'' - b_3w^*u - (b_3u^* + e_3)w = 0, & x \in (0, l), \\ u'(x) = v'(x) = w'(x) = 0, & x = 0, l. \end{cases} \quad (3.5)$$

Now expand the solution (u, v, w) in terms of eigenfunctions:

$$u(x) = \sum_{k=0}^{\infty} t_k \cos \frac{k\pi x}{l}, \quad v(x) = \sum_{k=0}^{\infty} s_k \cos \frac{k\pi x}{l}, \quad w(x) = \sum_{k=0}^{\infty} \tau_k \cos \frac{k\pi x}{l}, \quad (3.6)$$

where t_k, s_k, τ_k are constants to be determined. Then condition (3.4) is equivalent to the existence of at least one $k \in \mathbb{N}^+$ such that (t_k, s_k, τ_k) is nontrivial. Substituting (3.6) into problem (3.5) yields

$$\begin{pmatrix} -d_1(\frac{k\pi}{l})^2 - (b_1w^* + e_1) & \chi u^*(1-u^*)(\frac{k\pi}{l})^2 & -b_1u^* \\ b_2w^* & -d_2(\frac{k\pi}{l})^2 - e_2 & b_2u^* \\ -b_3w^* & 0 & -d_3(\frac{k\pi}{l})^2 - (b_3u^* + e_3) \end{pmatrix} \begin{pmatrix} t_k \\ s_k \\ \tau_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.7)$$

The case $k = 0$ is easily excluded. For any $k \in \mathbb{N}^+$, if the coefficient matrix in (3.7) is singular, then problem (3.5) has a nontrivial solution (t_k, s_k, τ_k) . Setting the determinant of the coefficient matrix to zero gives

$$\chi = \chi_k^S = \frac{R_1}{b_2u^*(1-u^*)w^*(d_3\mu_k + e_3)\mu_k}. \quad (3.8)$$

In particular, when $\chi = \chi_k^S$, $\mathcal{N}(D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi_k^S)) = \text{span}\{(\tilde{u}_k, \tilde{v}_k, \tilde{w}_k)\}$ is one-dimensional, and

$$\tilde{u}_k = \cos \frac{k\pi x}{l}, \quad \tilde{v}_k = M_k \cos \frac{k\pi x}{l}, \quad \tilde{w}_k = N_k \cos \frac{k\pi x}{l}, \quad (3.9)$$

where

$$\begin{aligned} M_k &= \frac{b_2 w^* (e_3 + d_3 (\frac{k\pi}{l})^2)}{(e_2 + d_2 (\frac{k\pi}{l})^2) (b_3 u^* + e_3 + d_3 (\frac{k\pi}{l})^2)} > 0, \\ N_k &= -\frac{b_3 w^*}{b_3 u^* + e_3 + d_3 (\frac{k\pi}{l})^2} < 0. \end{aligned} \quad (3.10)$$

Step 3 Verify that $\frac{d}{d\chi}(D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi))(\tilde{u}_k, \tilde{v}_k, \tilde{w}_k)|_{\chi=\chi_k^S} \notin \mathcal{R}(D_{(u,v,w)}\mathcal{F}(\mathbf{z}^*, \chi_k^S))$, where $\mathcal{R}$ denotes the range of the operator.

We argue by contradiction. Suppose the statement of Step 3 is false, then there exists a nontrivial solution (u, v, w) satisfy

$$\begin{cases} d_1 u'' - \chi_k^S u^* (1 - u^*) v'' - (b_1 w^* + e_1) u - b_1 u^* w = u^* (1 - u^*) (\frac{k\pi}{l})^2 \cos \frac{k\pi x}{l}, & x \in (0, l), \\ d_2 v'' + b_2 w^* u - e_2 v + b_2 u^* w = 0, & x \in (0, l), \\ d_3 w'' - b_3 w^* u - (b_3 u^* + e_3) w = 0, & x \in (0, l), \\ u'(x) = v'(x) = w'(x) = 0, & x = 0, l. \end{cases} \quad (3.11)$$

Multiply the first three equations of (3.11) by $\cos \frac{k\pi x}{l}$ and integrate by parts over $(0, l)$, we obtain

$$\begin{pmatrix} -d_1 (\frac{k\pi}{l})^2 - (b_1 w^* + e_1) & \chi_k^S u^* (1 - u^*) (\frac{k\pi}{l})^2 & -b_1 u^* \\ b_2 w^* & -d_2 (\frac{k\pi}{l})^2 - e_2 & b_2 u^* \\ -b_3 w^* & 0 & -d_3 (\frac{k\pi}{l})^2 - (b_3 u^* + e_3) \end{pmatrix} \times \begin{pmatrix} \int_0^l u \cos \frac{k\pi x}{l} dx \\ \int_0^l v \cos \frac{k\pi x}{l} dx \\ \int_0^l w \cos \frac{k\pi x}{l} dx \end{pmatrix} = \begin{pmatrix} \frac{u^* (k\pi)^2}{2l} \\ 0 \\ 0 \end{pmatrix}. \quad (3.12)$$

From (3.8) the coefficient matrix in (3.12) is singular, which contradicts the assumption that (u, v, w) is a nontrivial solution.

Based on the above discussion, we obtain the existence of non-constant positive solutions of model (3.2) and steady-state bifurcation occurs at $(\mathbf{z}^*, \chi)$ for each $k \in \mathbb{N}^+$.

Theorem 3.1. *Assume that*

$$\chi_k^S \neq \chi_j^S, \forall k \neq j \in \mathbb{N}^+ \text{ and } \chi_k^S \neq \chi_k^H, \forall k \in \mathbb{N}^+,$$

where χ_k^S and χ_k^H are given by (2.5). Then for each $k \in \mathbb{N}^+$, model (3.2) possesses spatially non-homogeneous bifurcation solutions $(u, v, w) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$ emanating from $(\mathbf{z}^*, \chi_k^S)$. Moreover, for sufficiently small $\theta > 0$, the bifurcation curve $\Gamma_k(s) = \{(u_k(s), v_k(s), w_k(s), \chi_k^S(s)) : s \in (-\theta, \theta)\}$ near $(\mathbf{z}^*, \chi_k^S)$ has the parametric representation

$$\chi_k^S(s) = \chi_k^S + O(s), \quad s \in (-\theta, \theta), \quad (3.13)$$

and

$$(u_k(s, x), v_k(s, x), w_k(s, x)) = (u^*, v^*, w^*) + s(\tilde{u}_k, \tilde{v}_k, \tilde{w}_k) + o(s), \quad s \in (-\theta, \theta), \quad (3.14)$$

where $(\tilde{u}_k, \tilde{v}_k, \tilde{w}_k)$ are given by (3.9), and $o(s) \in \mathcal{Z}$ lies in the closed complement of $\mathcal{N}(D_{(u,v,w)}\mathcal{F}(\tilde{u}, \tilde{v}, \tilde{w}, \chi))$, i.e.,

$$\mathcal{Z} = \left\{ (u, v, w) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X} \mid \int_0^l (u\tilde{u}_k + v\tilde{v}_k + w\tilde{w}_k)dx = 0 \right\}. \quad (3.15)$$

Furthermore, $(u_k(s, x), v_k(s, x), w_k(s, x), \chi_k(s))$ is a solution of model (3.2), and every nontrivial solution near the bifurcation point $(\mathbf{z}^*, \chi_k^S)$ lies on the curve $\Gamma_k(s)$, $s \in (-\theta, \theta)$.

Now, we present numerical simulations to verify that model (3.2) generates non-constant steady-state solutions.

Example 2. The parameter sets are given in Table 1 introduced in [22]. In Figure 4, the initial condition is $\mathbf{z}^* + 0.0001 \text{ rand}(1)$. From Theorem 2.1, if $\chi > \chi_0 = \chi_2^S \approx 0.379$, the positive equilibrium of (3.2) is unstable. Moreover, we choose $\chi = 0.5$, after transient evolution, all three components converge to time-independent stationary spatial patterns, by Theorem 3.1, model (3.2) produces a nonconstant steady-state solution with spatial profile $\cos \frac{2\pi x}{0.01}$ near the positive equilibrium. Biologically, chemotaxis amplifies minor environmental disturbances and drives immune cells to form persistent inflammatory aggregates, where immune cells and chemokines mutually reinforce each other to maintain stable spatial immune lesions.

Table 1. Parameter values used for numerical simulations.

<i>Parameter</i>	<i>Description</i>	<i>Values</i>
l	domain length	0.01
d_1	Diffusion coefficients	5×10^{-7}
d_2		6×10^{-8}
d_3		7.2×10^{-7}
a_1	Source of immune cells	0.05
a_3	External source of antigen	5×10^{-3}
b_1	Self regulation strength	5
b_2	Chemokine secretion rate by u	4×10^{-5}
b_3	Rate of phagocyte	1
e_1	Immune cell death	0.1
e_2	Chemokine decay	0.01
e_3	Antigen decay	4×10^{-6}

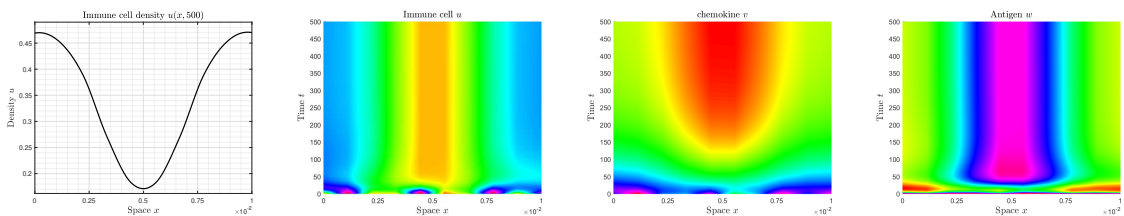
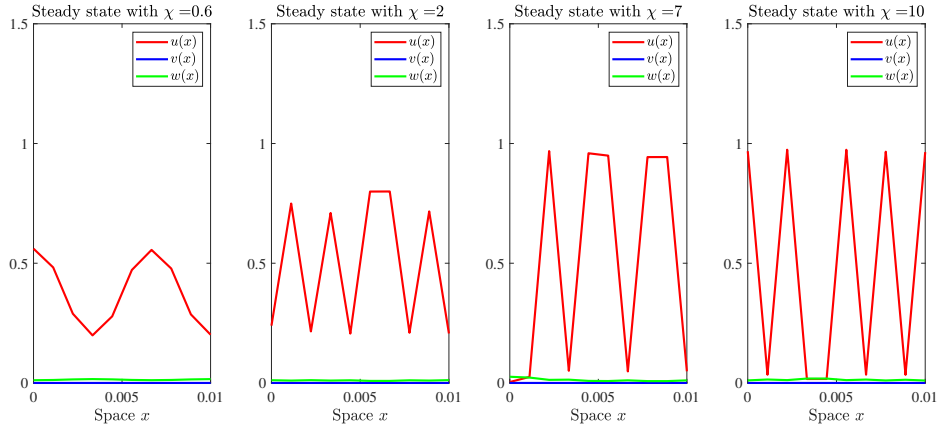


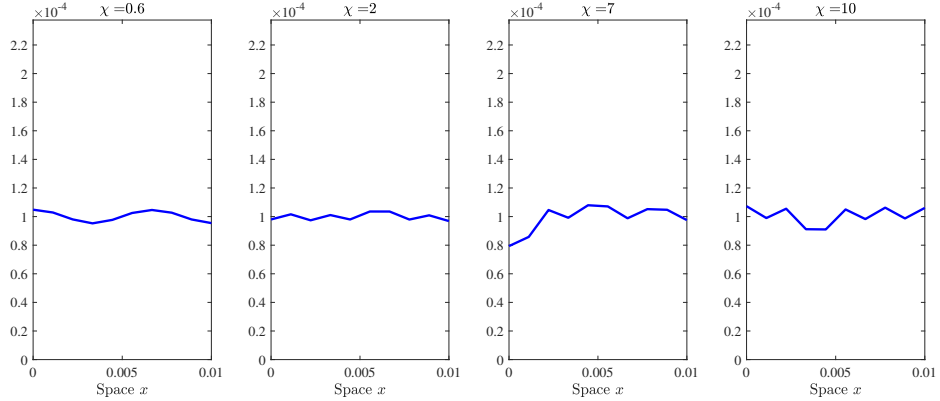
Figure 4. Space-time color map with initial conditions is chosen as $\mathbf{z}^* + 0.0001 \text{ rand}(1)$. We set $l = 0.01$ and $\chi = 0.5$.

Figure 5 demonstrates the critical regulatory role of chemotactic sensitivity χ in shaping steady-state immune spatial patterns, with all parameters consistent with Table 1. When χ exceeds the bifurcation threshold $\chi_2^S \approx 0.379$, the uniform immune distribution destabilizes and localized immune aggregates

emerge spontaneously. Larger χ produces more periodic, sharper spikes of immune cells $u(x)$: Enhanced chemotactic capacity enables immune cells to track chemokine gradients and form denser, more numerous discrete infiltration foci, resembling separated inflammatory lesions in chronic immune disorders. Notably, an intrinsic density-limiting mechanism caps the peak immune density near 1 irrespective of χ , avoiding excessive immune accumulation and pathological hyperinflammation. Antigen $w(x)$ remains uniformly low across the domain, confirming that immune pattern formation arises purely from chemotactic migration instead of heterogeneous antigen distribution. The curves of $v(x)$ are magnified 5 times for clearer visualization (second row). Chemokine $v(x)$ co-localizes with immune cell spikes, establishing a positive feedback loop to sustain long-term stable immune plaques.



(a) Steady-state profiles of $u(x)$, $v(x)$ and $w(x)$



(b) Magnified chemokine $v(x)$ (5-fold amplification)

Figure 5. Nontrivial stationary solutions of system (3.2) over $\Omega = (0, 0.01)$ when chemotactic sensitivity χ exceeds the critical bifurcation threshold $\chi_2^S \approx 0.379$. All parameters are consistent with Table 1, except $\chi = 0.6, 2, 7$ and 10 .

Figure 6 demonstrates the influence of spatial domain size l on immune cell spatial patterns with fixed $\chi = 1 > \chi_{k_0}^S$. Spatiotemporal contours (first row) and asymptotic steady profiles (second row) show that small domains sustain only a single immune aggregate, whereas larger domains support multiple periodic immune peaks. Biologically, the spatial extent of immune tissue determines the maximum number

of self-organized inflammatory foci. Small tissue regions can only support a single immune aggregation site. With enlarged tissue space, there is sufficient room to accommodate multiple separated immune infiltration foci. These patterns follow the spatial mode $\cos \frac{k_0 \pi x}{l}$, which demonstrates that the characteristic spatial wavelength of immune aggregates is intrinsically constrained by the physical size of the immune microenvironment. This reveals that tissue geometry acts as an important regulator controlling the quantity and spatial arrangement of chronic inflammatory lesions.

Table 2. k_0 and $\chi_{k_0}^S$ corresponding to l .

l	0.004	0.006	0.02	0.03
k_0	1	1	4	7
$\chi_{k_0}^S$	0.376	0.409	0.375	0.370

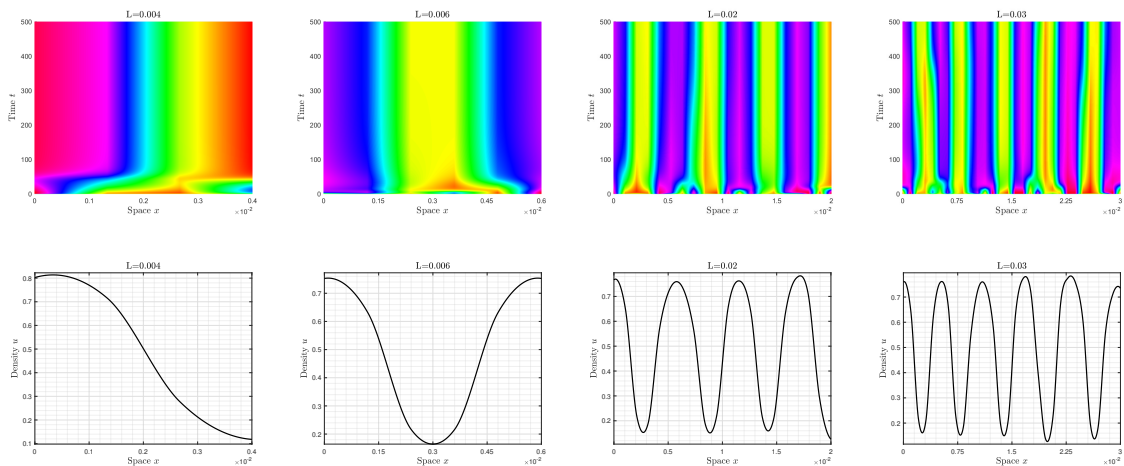


Figure 6. Space-time color maps (first row) and stable steady states (second row) of the Immune cell concentration in the case of $l = 0.004, 0.005, 0.02, 0.03$ respectively. In each case, we choose $\chi = 1$ is bigger than $\chi_{k_0}^S = \min_{k \in \mathcal{N}^+} \chi_k^S$. The stable steady states have spatial profile $\cos \frac{k_0 \pi x}{l}$.

4. Stability of nonconstant positive steady states

In this subsection, we mainly discuss the stability and instability of the bifurcation branch $\mathbf{z}_k(s, x)$ of model (3.2) obtained from Theorem 3.1. First, we determine the direction of the bifurcation branch $\Gamma_k(s)$ for each bifurcation point χ_k . It is easy to see that $\mathcal{F}$ is C^4 smooth, and by Theorem 1.8 of [8], the bifurcation branch $(u_k(s, x), v_k(s, x), w_k(s, x))$ is C^3 smooth, and

$$\begin{cases} u_k(s, x) = u^* + s \cos \frac{k\pi x}{l} + s^2 \xi_1(x) + s^3 \xi_2(x) + o(s^3), \\ v_k(s, x) = v^* + s N_k \cos \frac{k\pi x}{l} + s^2 \eta_1(x) + s^3 \eta_2(x) + o(s^3), \\ w_k(s, x) = w^* + s M_k \cos \frac{k\pi x}{l} + s^2 \zeta_1(x) + s^3 \zeta_2(x) + o(s^3), \\ \chi_k^S(s) = \chi_k^S + s \rho_1 + s^2 \rho_2 + o(s^2), \end{cases} \quad (4.1)$$

where $(\xi_i, \eta_i, \zeta_i) \in \mathcal{Z}$ ($\mathcal{Z}$ is defined in (3.15)), and ρ_i , $i = 1, 2$ are constants. In (4.1), $o(s^3)$ is in the $\mathcal{X}$ norm, and $o(s^2)$ is a constant.

Before analyzing the stability of these bifurcation branches, we determine the sign of the coefficients in the expansion of $\chi_k^S(s)$. Substituting (4.1) into model (3.2) and collecting terms of order $o(s^2)$ yields the linear system

$$\begin{cases} d_1 \xi_1'' - \chi_k^S u^* (1 - u^*) \zeta_1'' - (b_1 w^* + e_1) \xi_1 - b_1 u^* \zeta_1 = F_1, & x \in (0, l), \\ d_2 \eta_1'' + b_2 w^* \xi_1 - e_2 \eta_1 + b_2 u^* \zeta_1 = -b_2 N_k \cos^2 \frac{k\pi x}{l}, & x \in (0, l), \\ d_3 \zeta_1'' - b_3 w^* \xi_1 - (b_3 u^* + e_3) \zeta_1 = b_3 N_k \cos^2 \frac{k\pi x}{l}, & x \in (0, l), \\ \xi_1'(x) = \eta_1'(x) = \zeta_1'(x) = 0, & x = 0, l, \end{cases} \quad (4.2)$$

where

$$F_1 = (2u^* - 1) \chi_k^S M_k \left(\frac{k\pi}{l} \right)^2 \cos \frac{2k\pi x}{l} - \rho_1 u^* (1 - u^*) M_k \left(\frac{k\pi}{l} \right)^2 \cos \frac{k\pi x}{l} + b_1 N_k \cos^2 \frac{k\pi x}{l}.$$

Multiplying the first three equations of system (4.2) by $\cos \frac{k\pi x}{l}$ and integrating over $(0, l)$, we obtain

$$\begin{aligned} -\frac{u^*(1 - u^*) M_k (k\pi)^2}{2l} \rho_1 &= -\left(d_1 \left(\frac{k\pi}{l} \right)^2 + b_1 w^* + e_1 \right) \int_0^l \xi_1 \cos \frac{k\pi x}{l} dx \\ &\quad + \chi_k^S u^* (1 - u^*) \left(\frac{k\pi}{l} \right)^2 \int_0^l \eta_1 \cos \frac{k\pi x}{l} dx - b_1 u^* \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx, \\ b_2 w^* \int_0^l \xi_1 \cos \frac{k\pi x}{l} dx - \left(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2 \right) \int_0^l \eta_1 \cos \frac{k\pi x}{l} dx + b_2 u^* \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx &= 0, \end{aligned} \quad (4.3)$$

and

$$-b_3 w^* \int_0^l \xi_1 \cos \frac{k\pi x}{l} dx - \left(d_3 \left(\frac{k\pi}{l} \right)^2 + b_3 u^* + e_3 \right) \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx = 0. \quad (4.4)$$

Moreover, since $(\xi_1, \eta_1, \zeta_1) \in \mathcal{Z}$ from (3.15), we have

$$\int_0^l \xi_1 \cos \frac{k\pi x}{l} dx + M_k \int_0^l \eta_1 \cos \frac{k\pi x}{l} dx + N_k \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx = 0, \quad (4.5)$$

where M_k, N_k are given in (3.10). From equations (4.3)-(4.5) we deduce

$$\begin{pmatrix} b_2 w^* & -\left(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2 \right) & b_2 u^* \\ -b_3 w^* & 0 & -\left(d_3 \left(\frac{k\pi}{l} \right)^2 + b_3 u^* + e_3 \right) \\ 1 & M_k & N_k \end{pmatrix} \begin{pmatrix} \int_0^l \xi_1 \cos \frac{k\pi x}{l} dx \\ \int_0^l \eta_1 \cos \frac{k\pi x}{l} dx \\ \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.6)$$

Denote the coefficient matrix in (4.6) by $\mathcal{B}$. Since $M_k > 0, N_k < 0$, the determinant of $\mathcal{B}$ satisfies

$$\begin{aligned} \det(\mathcal{B}) &= \left(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2 \right) \left(d_3 \left(\frac{k\pi}{l} \right)^2 + b_1 u^* + e_3 \right) \\ &\quad + \left(d_3 \left(\frac{k\pi}{l} \right)^2 + e_3 \right) M_k b_2 w^* - \left(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2 \right) N_k b_3 w^* > 0. \end{aligned}$$

Consequently,

$$\int_0^l \xi_1 \cos \frac{k\pi x}{l} dx = \int_0^l \eta_1 \cos \frac{k\pi x}{l} dx = \int_0^l \zeta_1 \cos \frac{k\pi x}{l} dx = 0. \quad (4.7)$$

Then from (4.3) and (4.7) we obtain $\rho_1 = 0$.

Thus, we have the following Theorem.

Theorem 4.1. *Under the assumptions of Theorem 3.1, for $k \neq j$, $\chi_k^S \neq \chi_j^S$, if $\rho_1 = 0$ and $\rho_2 \neq 0$, then the local bifurcation branch $\Gamma_k(s)$ of system (3.2) at $(\mathbf{z}^*, \chi_k^S)$ is a transcritical bifurcation.*

Next, the sign of ρ_2 determines the direction of the bifurcation branch $\Gamma_k(s)$ ($k \in \mathbb{N}^+$) turning at χ_k^S and its stability. To this end, we need to study the $o(s^3)$ terms in system (3.2).

Since $\rho_1 = 0$, the $o(s^3)$ terms in system (3.2) yield the following system

$$\begin{cases} d_1 \xi_2'' - \chi_k^S u^* (1 - u^*) \eta_2'' - (b_w^* + e_1) \xi_2 - b_1 u^* \zeta_2 = F_2, & x \in (0, l), \\ d_2 \eta_2'' + b_2 w^* \xi_2 - e_2 \eta_2 + b_2 u^* \zeta_2 = -b_2 \cos \frac{k\pi x}{l} (\zeta_1 + \xi_1 N_k), & x \in (0, l), \\ d_3 \zeta_2'' - b_3 w^* \xi_2 - (b_3 u^* + e_3) \zeta_2 = b_3 \cos \frac{k\pi x}{l} (\zeta_1 + \xi_1 N_k), & x \in (0, l), \\ u'(x) = v'(x) = w'(x) = 0, & x = 0, l, \end{cases} \quad (4.8)$$

where

$$\begin{aligned} F_2 = & \chi_k^S \left((1 - 2u^*) \cos \frac{k\pi x}{l} \eta_1'' + (2u^* - 1) \frac{k\pi}{l} \sin \frac{k\pi x}{l} (M_k \xi_1' + \eta_1') \right. \\ & - 2M_k \left(\frac{k\pi}{l} \right)^2 \left(\sin \frac{k\pi x}{l} \right)^2 \cos \frac{k\pi x}{l} - M_k \left(\frac{k\pi}{l} \right)^2 \cos \frac{k\pi x}{l} \left(\xi_1 - 2u^* \xi_1 - \left(\cos \frac{k\pi x}{l} \right)^2 \right) \\ & \left. - \rho_2 \left(\frac{k\pi}{l} \right)^2 M_k u^* (1 - u^*) \cos \frac{k\pi x}{l} \right). \end{aligned}$$

Multiplying the first equation of system (4.8) by $\cos \frac{k\pi x}{l}$ and integrating over $(0, l)$, we obtain

$$\begin{aligned} \frac{\rho_2 u^* (k\pi)^2}{2l} = & - \left(d_1 \left(\frac{k\pi}{l} \right)^2 + 1 \right) \int_0^l \xi_2 \cos \frac{k\pi x}{l} dx \\ & + \int_0^l \eta_2 \cos \frac{k\pi x}{l} dx - \frac{\chi_k^S}{2} \left(\frac{k\pi}{l} \right)^2 \int_0^l \xi_1 dx \\ & + \chi_k^S u^* \left(\frac{k\pi}{l} \right)^2 \int_0^l \zeta_2 \cos \frac{k\pi x}{l} + \frac{\chi_k^S}{2} \left(\frac{k\pi}{l} \right)^2 \int_0^l \xi_1 \cos \frac{2k\pi x}{l} dx \\ & + \frac{M_k \chi_k^S}{2} \left(\frac{k\pi}{l} \right)^2 \int_0^l \zeta_1 \cos \frac{2k\pi x}{l} dx - \frac{\chi_k^S M_k}{2} \left(\frac{k\pi}{l} \right)^2 \int_0^l \zeta_1 dx. \end{aligned} \quad (4.9)$$

Similarly, multiplying the second and third equations of system (4.8) by $\cos \frac{k\pi x}{l}$ and integrating, we get

$$b_2 w^* \int_0^l \xi_2 \cos \frac{k\pi x}{l} dx - \left(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2 \right) \int_0^l \eta_2 \cos \frac{k\pi x}{l} dx + b_2 u^* \int_0^l \zeta_2 \cos \frac{k\pi x}{l} dx = N \quad (4.10)$$

and

$$-b_3 w^* \int_0^l \xi_2 \cos \frac{k\pi x}{l} dx - \left(d_3 \left(\frac{k\pi}{l} \right)^2 + b_3 u^* + e_3 \right) \int_0^l \zeta_2 \cos \frac{k\pi x}{l} dx = P, \quad (4.11)$$

where

$$N = -b_2 \int_0^l (\zeta_1 + \xi_1 N_k) \left(\cos \frac{k\pi x}{l} \right)^2 dx, \quad P = b_3 \int_0^l (\zeta_1 + \xi_1 N_k) \left(\cos \frac{k\pi x}{l} \right)^2 dx.$$

Furthermore, since $(\xi_2, \eta_2, \zeta_2) \in \mathcal{Z}$ ($\mathcal{Z}$ defined in (3.15)), we have

$$\int_0^l \xi_2 \cos \frac{k\pi x}{l} dx + M_k \int_0^l \eta_2 \cos \frac{k\pi x}{l} dx + N_k \int_0^l \zeta_2 \cos \frac{k\pi x}{l} dx = 0. \quad (4.12)$$

From (4.10)-(4.12) we obtain

$$\begin{pmatrix} b_2 w^* & -(d_2 \left(\frac{k\pi}{l} \right)^2 + e_2) & b_2 u^* \\ b_2 u^* & 0 & -(d_3 \left(\frac{k\pi}{l} \right)^2 + b_3 u^* + e_3) \\ 1 & M_k & N_k \end{pmatrix} \begin{pmatrix} \int_0^l \xi_2 \cos \frac{k\pi x}{l} dx \\ \int_0^l \eta_2 \cos \frac{k\pi x}{l} dx \\ \int_0^l \zeta_2 \cos \frac{k\pi x}{l} dx \end{pmatrix} = \begin{pmatrix} N \\ P \\ 0 \end{pmatrix}. \quad (4.13)$$

By Cramer's rule, solving system (4.13) gives

$$\int_0^l \xi_2 \cos \frac{k\pi x}{l} dx = \frac{|\mathcal{P}_1|}{|\mathcal{P}_0|}, \quad \int_0^l \eta_2 \cos \frac{k\pi x}{l} dx = \frac{|\mathcal{P}_2|}{|\mathcal{P}_0|}, \quad \int_0^l \zeta_2 \cos \frac{k\pi x}{l} dx = \frac{|\mathcal{P}_3|}{|\mathcal{P}_0|}, \quad (4.14)$$

where

$$\mathcal{P}_1 = \begin{pmatrix} N & -(d_2(\frac{k\pi}{l})^2 + e_2) & b_2 u^* \\ P & 0 & -(d_3(\frac{k\pi}{l})^2 + b_3 u^* + e_3) \\ 0 & M_k & N_k \end{pmatrix},$$

$$\mathcal{P}_2 = \begin{pmatrix} b_2 w^* & N & b_2 u^* \\ b_2 u^* & P & -(d_3(\frac{k\pi}{l})^2 + b_3 u^* + e_3) \\ 1 & 0 & N_k \end{pmatrix}, \quad \mathcal{P}_3 = \begin{pmatrix} b_2 w^* & -(d_2(\frac{k\pi}{l})^2 + e_2) & N \\ b_2 u^* & 0 & P \\ 1 & M_k & 0 \end{pmatrix},$$

and

$$\mathcal{P}_0 = \begin{pmatrix} b_2 w^* & -(d_2(\frac{k\pi}{l})^2 + e_2) & b_2 u^* \\ b_2 u^* & 0 & -(d_3(\frac{k\pi}{l})^2 + b_3 u^* + e_3) \\ 1 & M_k & N_k \end{pmatrix}.$$

Integrating system (4.2) over $(0, l)$ and using the Neumann boundary conditions together with $\rho_1 = 0$, we get

$$\begin{cases} -(b_1 w^* + e_1) \int_0^l \xi_1 dx - b_1 u^* \int_0^l \eta_1 dx = \frac{b_1 N_k l}{2}, \\ b_2 w^* \int_0^l \xi_1 dx - e_2 \int_0^l \eta_1 dx + b_2 u^* \int_0^l \zeta_1 dx = -\frac{b_2 N_k l}{2}, \\ -b_3 w^* \int_0^l \xi_1 dx - (b_3 u^* + e_3) \int_0^l \zeta_1 dx = \frac{b_3 N_k l}{2}, \end{cases}$$

which is equivalent to

$$\begin{pmatrix} -(b_1 w^* + e_1) & 0 & -b_1 u^* \\ b_2 w^* & -e_2 & b_2 u^* \\ -b_3 w^* & 0 & -(b_3 u^* + e_3) \end{pmatrix} \begin{pmatrix} \int_0^l \xi_1 dx \\ \int_0^l \eta_1 dx \\ \int_0^l \zeta_1 dx \end{pmatrix} = \begin{pmatrix} \frac{b_1 N_k l}{2} \\ -\frac{b_2 N_k l}{2} \\ \frac{b_3 N_k l}{2} \end{pmatrix}. \quad (4.15)$$

Solving system (4.15) similarly yields

$$\int_0^l \xi_1 dx = \frac{|\mathcal{Q}_1|}{|\mathcal{Q}_0|}, \quad \int_0^l \eta_1 dx = \frac{|\mathcal{Q}_2|}{|\mathcal{Q}_0|}, \quad \int_0^l \zeta_1 dx = \frac{|\mathcal{Q}_3|}{|\mathcal{Q}_0|}, \quad (4.16)$$

where

$$\mathcal{Q}_1 = \begin{pmatrix} \frac{b_1 N_k l}{2} & 0 & -b_1 u^* \\ -\frac{b_2 N_k l}{2} & -e_2 & b_2 u^* \\ \frac{b_3 N_k l}{2} & 0 & -(b_3 u^* + e_3) \end{pmatrix}, \quad \mathcal{Q}_2 = \begin{pmatrix} -(b_1 w^* + e_1) & \frac{b_1 N_k l}{2} & -b_1 u^* \\ b_2 w^* & -\frac{b_2 N_k l}{2} & b_2 u^* \\ -b_3 w^* & \frac{b_3 N_k l}{2} & -(b_3 u^* + e_3) \end{pmatrix},$$

and

$$\mathcal{Q}_3 = \begin{pmatrix} -(b_1 w^* + e_1) & 0 & \frac{b_1 N_k l}{2} \\ b_2 w^* & -e_2 & -\frac{b_2 N_k l}{2} \\ -b_3 w^* & 0 & \frac{b_3 N_k l}{2} \end{pmatrix}, \quad \mathcal{Q}_0 = \begin{pmatrix} -(b_1 w^* + e_1) & 0 & -b_1 u^* \\ b_2 w^* & -e_2 & b_2 u^* \\ -b_3 w^* & 0 & -(b_3 u^* + e_3) \end{pmatrix}.$$

Multiplying the first three equations of system (4.2) by $\cos \frac{2k\pi x}{l}$ and integrating over $(0, l)$, we obtain

$$\begin{pmatrix} -(4d_1(\frac{k\pi}{l})^2 + b_1 w^* + e_1) & 4\chi_k^S u^* (1 - u^*) (\frac{k\pi}{l})^2 & -b_1 u^* \\ b_2 w^* & -(4d_2(\frac{k\pi}{l})^2 + e_2) & b_2 u^* \\ -b_3 w^* & 0 & -(4d_3(\frac{k\pi}{l})^2 + b_3 u^* + e_3) \end{pmatrix} \times \begin{pmatrix} \int_0^l \xi_1 \cos \frac{2k\pi x}{l} dx \\ \int_0^l \eta_1 \cos \frac{2k\pi x}{l} dx \\ \int_0^l \zeta_1 \cos \frac{2k\pi x}{l} dx \end{pmatrix} = \begin{pmatrix} (2u^* - 1) \chi_k^S M_k \frac{(k\pi)^2}{2l} + \frac{b_1 N_k l}{2} \\ -\frac{b_2 N_k l}{4} \\ \frac{b_3 N_k l}{4} \end{pmatrix}. \quad (4.17)$$

Applying Cramer's rule to system (4.17) gives

$$\int_0^l \xi_1 \cos \frac{2k\pi x}{l} dx = \frac{|\mathcal{R}_1|}{|\mathcal{R}_0|}, \quad \int_0^l \eta_1 \cos \frac{2k\pi x}{l} dx = \frac{|\mathcal{R}_2|}{|\mathcal{R}_0|}, \quad \int_0^l \zeta_1 \cos \frac{2k\pi x}{l} dx = \frac{|\mathcal{R}_3|}{|\mathcal{R}_0|}, \quad (4.18)$$

where

$$\begin{aligned} \mathcal{R}_1 &= \begin{pmatrix} (2u^* - 1)\chi_k^S M_k \frac{(k\pi)^2}{2l} + \frac{b_1 N_k l}{2} & 4\chi_k^S u^*(1 - u^*)\left(\frac{k\pi}{l}\right)^2 & -b_1 u^* \\ -\frac{b_2 N_k l}{4} & -(4d_2\left(\frac{k\pi}{l}\right)^2 + e_2) & b_2 u^* \\ \frac{b_3 N_k l}{4} & 0 & -(4d_3\left(\frac{k\pi}{l}\right)^2 + b_3 u^* + e_3) \end{pmatrix}, \\ \mathcal{R}_2 &= \begin{pmatrix} -(4d_1\left(\frac{k\pi}{l}\right)^2 + b_1 w^* + e_1) & (2u^* - 1)\chi_k^S M_k \frac{(k\pi)^2}{2l} + \frac{b_1 N_k l}{2} & -b_1 u^* \\ b_2 w^* & -\frac{b_2 N_k l}{4} & b_2 u^* \\ -b_3 w^* & \frac{b_3 N_k l}{4} & -(4d_3\left(\frac{k\pi}{l}\right)^2 + b_3 u^* + e_3) \end{pmatrix}, \\ \mathcal{R}_3 &= \begin{pmatrix} -(4d_1\left(\frac{k\pi}{l}\right)^2 + b_1 w^* + e_1) & 4\chi_k^S u^*(1 - u^*)\left(\frac{k\pi}{l}\right)^2 & (2u^* - 1)\chi_k^S M_k \frac{(k\pi)^2}{2l} + \frac{b_1 N_k l}{2} \\ b_2 w^* & -(4d_2\left(\frac{k\pi}{l}\right)^2 + e_2) & -\frac{b_2 N_k l}{4} \\ -b_3 w^* & 0 & \frac{b_3 N_k l}{4} \end{pmatrix}, \end{aligned}$$

and

$$\mathcal{R}_0 = \begin{pmatrix} -(4d_1\left(\frac{k\pi}{l}\right)^2 + b_1 w^* + e_1) & 4\chi_k^S u^*(1 - u^*)\left(\frac{k\pi}{l}\right)^2 & -b_1 u^* \\ b_2 w^* & -(4d_2\left(\frac{k\pi}{l}\right)^2 + e_2) & b_2 u^* \\ -b_3 w^* & 0 & -(4d_3\left(\frac{k\pi}{l}\right)^2 + b_3 u^* + e_3) \end{pmatrix}.$$

Note that ρ_2 in (4.9) can be determined from (4.14), (4.16) and (4.18).

Summarizing the above analysis, we have the following result on the direction of the bifurcation branch.

Theorem 4.2. *For the bifurcation branch $\Gamma_k(s)$ of system (3.2) at $(\mathbf{z}^*, \chi_k^S)$, it turns to the right if $\rho_2 > 0$ and to the left if $\rho_2 < 0$.*

Next, we discuss the stability of the bifurcation branch. In some cases, the sign of ρ_2 determines the stability of the branch. Specifically, for $(u, v, w) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}$, consider the eigenvalue problem

$$D_{(u,v,w)} F(u_k(s, x), v_k(s, x), w_k(s, x), \chi_k(s))(u, v, w) = \sigma(u, v, w). \quad (4.19)$$

If the eigenvalue σ is real and negative, then the branch is locally stable. At $s = 0$, $\sigma(0) = 0$ and the kernel of $D_{(u,v,w)} F(\mathbf{z}^*, \chi_k^S)$ is spanned by $\{(1, M_k, N_k) \cos \frac{k\pi x}{l}\}$. The stability result is as follows.

Theorem 4.3. *Assume the conditions of Theorem 3.1 hold, and let*

$$\Gamma_k(s) = \{(u_k(s, x), v_k(s, x), w_k(s, x), \chi_k^S(s))\}$$

be the bifurcation branch (with expansions given by (3.13) and (3.14)). Then

(1) *If $\chi_0 = \chi_{k_0}^S := \min_{k \in \mathbb{N}^+} \chi_k^S < \min_{k \in \mathbb{N}^+} \chi_k^H$, then when $\rho_2 > 0$, the bifurcation branch $\Gamma_{k_0}(s)$ emanating from $(\mathbf{z}^*, \chi_{k_0}^S)$ is asymptotically stable, and when $\rho_2 < 0$ it is unstable; for every $k \neq k_0$, the bifurcation branch $\Gamma_k(s)$ near $(\mathbf{z}^*, \chi_k^S)$ is unstable.*

(2) *If $\chi_0 = \chi_{k_1}^H := \min_{k \in \mathbb{N}^+} \chi_k^H < \min_{k \in \mathbb{N}^+} \chi_k^S$, then for every $k \in \mathbb{N}^+$, the bifurcation branch $\Gamma_k(s)$ near $(\mathbf{z}^*, \chi_k^S)$ is unstable.*

Proof The proof idea follows from Theorem 1.13 of [8] and Theorems 5.5-5.6 of [7].

We study the stability of $(u_{k_0}(s, x), v_{k_0}(s, x), w_{k_0}(s, x), \chi_{k_0}^S(s))$. By Theorem 1.13 of [8], there exists an interval I containing $\chi_{k_0}^S(s) \in I$ and a C^1 smooth function $(\chi, s) : I \times (-\delta, \delta) \rightarrow (\sigma(\chi), \sigma(s))$ with $\sigma(0) = 0$ and $\sigma(\chi_{k_0}^S(s)) = 0$, such that $\sigma(s)$ is a simple eigenvalue of (4.19), and $\sigma(\chi)$ satisfies the same property for the linearized operator at the trivial branch:

$$\mathcal{D}_{(u,v,w)}\mathcal{F}(u^*, v^*, w^*, \chi)(u, v, w) = \sigma(s)(u, v, w), \quad (u, v, w) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X}. \quad (4.20)$$

Moreover, $\sigma(s)$ is the unique eigenvalue of the perturbed operator near the origin in the complex plane that bifurcates from zero (and similarly for $\sigma(\chi)$).

The eigenfunction corresponding to (4.20) can be expressed as $(u(\chi, x), v(\chi, x), w(\chi, x))$, which depends smoothly on χ and satisfies

$$(u(\chi_{k_0}^S(s), x), v(\chi_{k_0}^S(s), x), w(\chi_{k_0}^S(s), x)) = \left(\cos \frac{k\pi x}{l}, M_k \cos \frac{k\pi x}{l}, N_k \cos \frac{k\pi x}{l} \right)$$

and $(u(\chi, x), v(\chi, x), w(\chi, x)) - (\cos \frac{k\pi x}{l}, M_k \cos \frac{k\pi x}{l}, N_k \cos \frac{k\pi x}{l}) \in \mathcal{Z}$, uniquely determined, where M_k, N_k are given in (3.10) and $\mathcal{Z}$ is defined in (3.15).

From system (3.5), problem (4.20) is equivalent to

$$\begin{cases} d_1 u'' - \chi u^*(1 - u^*)v'' - (b_1 w^* + e_1)u - b_1 u^* w = \sigma u, & x \in (0, l), \\ d_2 v'' + b_2 w^* u - e_2 v + b_2 u^* w = \sigma v, & x \in (0, l), \\ d_3 w'' - b_3 w^* u - (b_3 u^* + e_3)w = \sigma w, & x \in (0, l), \\ u'(x) = v'(x) = w'(x) = 0, & x = 0, l. \end{cases} \quad (4.21)$$

Differentiating system (4.21) with respect to χ and then setting $\chi = \chi_{k_0}^S$, we obtain

$$\begin{cases} d_1 \dot{u}'' - \chi_{k_0}^S u^*(1 - u^*)\dot{v}'' - u^*(1 - u^*) \cos'' \frac{k_0 \pi x}{l} - (b_1 w^* + e_1)\dot{u} - b_1 u^* \dot{w} = \dot{\sigma}(\chi_{k_0}^S) \cos \frac{k_0 \pi x}{l}, & x \in (0, l), \\ d_2 \dot{v}'' + b_2 w^* \dot{u} - e_2 \dot{v} + b_2 u^* \dot{w} = \dot{\sigma}(\chi_{k_0}^S) M_k \cos \frac{k_0 \pi x}{l}, & x \in (0, l), \\ d_3 \dot{w}'' - b_3 w^* \dot{u} - (b_3 u^* + e_3)\dot{w} = \dot{\sigma}(\chi_{k_0}^S) N_k \cos \frac{k_0 \pi x}{l}, & x \in (0, l), \\ \dot{u}'(x) = \dot{v}'(x) = \dot{w}'(x) = 0, & x = 0, l, \end{cases} \quad (4.22)$$

where the dot denotes differentiation with respect to χ evaluated at $\chi = \chi_{k_0}^S$, i.e., $\dot{u} = \frac{\partial u(\chi, x)}{\partial \chi}|_{\chi=\chi_{k_0}^S}$, etc.

Multiplying each equation of system (4.22) by $\cos \frac{k_0 \pi x}{l}$ and integrating over $(0, l)$, we obtain

$$\begin{pmatrix} -d_1 \left(\frac{k_0 \pi}{l} \right)^2 - (b_1 w^* + e_1) & \chi_{k_0}^S u^*(1 - u^*) \left(\frac{k_0 \pi}{l} \right)^2 & -b_1 u^* \\ b_2 w^* & -d_2 \left(\frac{k_0 \pi}{l} \right)^2 - e_2 & b_2 u^* \\ -b_3 w^* & 0 & -d_3 \left(\frac{k_0 \pi}{l} \right)^2 - (b_3 u^* + e_3) \end{pmatrix} \times \begin{pmatrix} \int_0^l \dot{u} \cos \frac{k_0 \pi x}{l} dx \\ \int_0^l \dot{v} \cos \frac{k_0 \pi x}{l} dx \\ \int_0^l \dot{w} \cos \frac{k_0 \pi x}{l} dx \end{pmatrix} = \begin{pmatrix} \frac{l}{2} \dot{\sigma}(\chi_{k_0}^S) - \frac{l}{2} u^*(1 - u^*) \left(\frac{k_0 \pi}{l} \right)^2 \\ -\frac{M_k l}{2} \dot{\sigma}(\chi_{k_0}^S) \\ \frac{N_k l}{2} \dot{\sigma}(\chi_{k_0}^S) \end{pmatrix}.$$

From (3.8) the coefficient matrix on the left-hand side is invertible. Denoting its cofactors by A_{j1} ($j = 1, 2, 3$), we have

$$A_{11} = \left(d_2 \left(\frac{k_0 \pi}{l} \right)^2 + e_2 \right) \left(d_3 \left(\frac{k_0 \pi}{l} \right)^2 + b_3 u^* + e_3 \right) > 0,$$

$$A_{12} = b_2 w^* \left(d_3 \left(\frac{k_0 \pi}{l} \right)^2 + e_3 \right) > 0,$$

$$A_{13} = -b_3 w^* \left(d_2 \left(\frac{k_0 \pi}{l} \right)^2 + e_2 \right) < 0,$$

and since $M_k > 0$, $N_k < 0$, we obtain

$$\dot{\sigma}(\chi_{k_0}^S) = \frac{A_{11} u^* (1 - u^*) \left(\frac{k_0 \pi}{l} \right)^2}{A_{11} + A_{12} M_k + A_{13} N_k} > 0.$$

By Theorem 1.16 of [8], for small $s \neq 0$, $\sigma(s)$ has the same sign as $-s \chi_{k_0}^{S'}(s) \dot{\sigma}(\chi_{k_0}^S)$, and for $\sigma(s) \neq 0$,

$$\lim_{s \rightarrow 0} \frac{-s \chi_{k_0}^{S'}(s) \dot{\sigma}(\chi_{k_0}^S)}{\sigma(s)} = 1.$$

Since $\rho_1 = 0$, we have $\text{sgn}(\sigma(0)) = -\text{sgn}(\rho_2)$. Furthermore, by standard perturbation theory for eigenvalues [16], $\text{sgn}(\sigma(s)) = -\text{sgn}(\rho_2)$ for $s \in (-\delta, \delta)$.

Thus, if $\rho_2 > 0$, then $\sigma(s) < 0$, meaning that all eigenvalues of (4.19) have negative real parts; hence the bifurcation branch $\Gamma_{k_0}(s)$ near $(\mathbf{z}^*, \chi_{k_0}^S)$ is asymptotically stable. Conversely, if $\rho_2 < 0$, then $\Gamma_{k_0}(s)$ is unstable.

For the case $k \in \mathbb{N}^+$ with $\chi_0 = \min_{k \in \mathbb{N}^+} \chi_k^H < \chi_k^S$, or for $k \neq k_0$ when $\chi_0 = \min_{k \in \mathbb{N}^+} \chi_k^S < \chi_k^H$, it follows from the proof of Theorem 2.1 that the linearized operator (4.20) has an eigenvalue with positive real part. By standard eigenvalue perturbation theory [16], for small s , the perturbed operator (4.19) also has an eigenvalue $\sigma(s)$ with positive real part, so $(u_k(s, x), v_k(s, x), w_k(s, x), \chi_k(s))$ is unstable for $s \in (-\theta, \theta)$. This completes the proof. $\square$

A schematic bifurcation diagram for case (1) is shown in Figure 7.

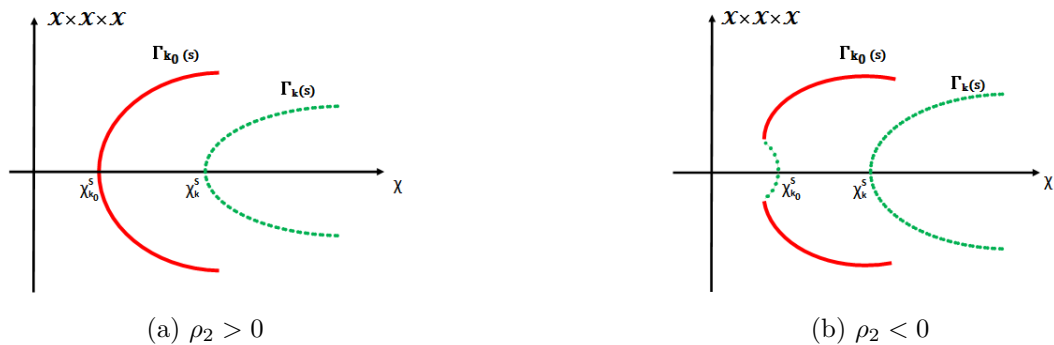


Figure 7. Bifurcation diagram for Theorem 4.3(1): red solid line-stable, green dashed-unstable. As can be seen from the figure, if $\Gamma_{k_0}(s)$ turns right, it is stable near $(\mathbf{z}^*, \chi_{k_0}^S)$, see (a); if $\Gamma_{k_0}(s)$ turns left, it is unstable near $(\mathbf{z}^*, \chi_{k_0}^S)$, see (b). If $k \neq k_0$, $\Gamma_k(s)$ is always unstable near $(\mathbf{z}^*, \chi_k^S)$, see (a) and (b).

5. Discussions and Conclusion Remarks

This work establishes that the the volume-filling mechanism fundamentally reshapes the bifurcation landscape of immune chemotaxis. The minimal chemotactic threshold emerges as a quantifiable biomarker for the transition to stable chronic inflammation. Future work will incorporate dynamic drug interventions

to explore whether lowering χ below this critical value can force the regression of established inflammatory aggregates.

(i) Volume-filling as an intrinsic anti-inflammatory constraint. The saturation factor $u(1-u)$ in the chemotactic flux rigorously enforces $0 \leq u \leq 1$, independent of the magnitude of χ . Biologically, this term represents contact inhibition and excluded volume effects among immune cells. Consequently, even when χ far exceeds the instability threshold, the peak density of immune cells is capped near 1 (see Figure 5), preventing uncontrolled inflammatory expansion. Mathematically, this accounts for the confinement of granulomas rather than their unlimited expansion. Common pathological examples for example include tuberculosis and foreign body reactions, where physical clustering of macrophages and T cells generates a self-restricting feedback loop to contain lesion spread.

(ii) Threshold $\chi_{k_0}^S$ distinguishes immune surveillance from chronic foci. Theorem 4.3 establishes that only the minimal bifurcation point $\chi_{k_0}^S = \min_k \chi_k^S$ generates stable spatially heterogeneous steady states. This defines a sharp immunological switch: below this threshold, the immune system maintains a uniform, non-aggressive surveillance state; exceeding it triggers irreversible self-organization into stable, localized inflammatory plaques. Therapeutically, this suggests that partial suppression of chemotactic sensitivity (e.g., via CCR5/CXCR3 antagonists) below $\chi_{k_0}^S$ may suffice to destabilize established chronic lesions, without abolishing beneficial chemotaxis entirely.

(iii) Tissue size dictates lesion multiplicity. Table 2 and Figure 6 demonstrate that the admissible spatial mode k_0 scales linearly with the tissue length l through $\mu_k = (\frac{k\pi}{l})^2$. Small microenvironments support only a single stable aggregate ($k_0 = 1$), whereas larger organs accommodate multiple periodic foci ($k_0 \geq 1$). This provides a mathematical rationale for the clinical observation that large organs (e.g., liver) frequently exhibit multi-focal necrosis, while small lymphoid tissues typically develop solitary germinal centers. The specific mode number k_0 selected by the system is precisely the integer that minimizes the critical chemotactic threshold χ_k^S , linking the lesion count directly to the diffusion-to-response balance of immune cells.

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