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Nash Equilibrium Improvement in Quantum Extensions of Symmetric Games

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Abstract

Quantum strategies can alter the equilibrium structure of classical games, but apparent improvements may depend on arbitrary strategy labels or on the particular quantum operations admitted. Here, we characterize payoff improvement across all permissible three-strategy extensions of renaming-symmetric 2×2 games within the Eisert-Wilkens-Lewenstein framework. We preserve the classical game as a subgame and add one fixed unitary strategy, restricting attention to the three extension classes that remain invariant under isomorphic transformations of the classical game. Across six canonical payoff orderings, including coordination, anti-coordination, and dominant-strategy dilemmas, we derive the conditions under which the extended games admit pure Nash equilibria whose payoffs are no lower than those of the corresponding symmetric classical equilibrium. The classical-type A_0 extension produces no strict improvement, while B_0 does so only in restricted regions and generally asymmetrically. By contrast, the genuinely quantum C_0 extension supports non-inferior equilibria across all six classes and yields strict, mutually beneficial improvements in identifiable payoff regions. These results move beyond isolated game examples by showing precisely when representation-invariant quantum extensions can improve equilibrium payoffs and when they cannot. These conditions provide a design map for quantum-enabled

multi-agent protocols by identifying the payoff structures under which an added quantum strategy can improve equilibrium outcomes without altering the players' underlying preferences.

Keywords: Quantum games; Nash equilibrium; EWL scheme; symmetric games; quantum strategies; Pareto improvement; permissible extensions

1 Introduction

Quantum game theory studies strategic interactions implemented through quantum protocols, in which players apply local quantum operations to systems that may share entanglement. Its central question is whether and under what operational assumptions, quantum resources enlarge the set of attainable correlations or alter the equilibrium structure relative to classical implementations [1–3]. Since the seminal contributions of Meyer [1] and Eisert, Wilkens, and Lewenstein [2], quantum games have provided a natural framework for analyzing whether quantum correlations can alter the equilibrium structure of classical games without abandoning their original payoff interpretation.

The game-theoretic motivation stems from the well-known tension between equilibrium stability and efficiency. Nash equilibrium (NE) is the central solution concept of non-cooperative game theory, but it only excludes profitable unilateral deviations and therefore need not be Pareto efficient [4, 5]. In many strategic dilemmas, individually rational behaviour may lead to collectively inefficient or even Pareto-dominated outcomes. The Prisoner's Dilemma provides the canonical example, while Chicken or Hawk-Dove games and Battle-of-the-Sexes-type interactions exhibit related tensions, particularly for mixed or symmetric equilibria [6].

Classical game theory has explored several ways of enlarging the set of sustainable or implementable outcomes, including repeated interaction [7], mediated correlation and communication [8, 9], credible commitments [10], and bargaining or mechanism design [5, 11]. These approaches rely on additional temporal, informational, or institutional assumptions. Quantum game theory offers a different route: rather than modifying the players' preferences, it changes the physically admissible strategy and correlation structure available to them.

One of the most influential frameworks for this purpose is the Eisert–Wilkens–Lewenstein (EWL) quantization scheme [2]. In the EWL framework, a classical two-player game is embedded into a quantum protocol. An initial product state is entangled, players apply local unitary operations, a disentangling operation is performed, and a final measurement produces a probability distribution over classical outcomes. Expected payoffs are then computed from the original classical payoff table. The construction is designed so that the classical game is recovered when players restrict themselves to the operations representing classical strategies. At the same time, entanglement and the enlarged unitary strategy sets can generate joint outcome distributions that are unavailable under independent classical mixed strategies. Whether these distributions represent genuinely non-classical correlations or can be reproduced by a classical correlation device depends on the operational assumptions of the model.

Early work showed that quantum strategies can improve equilibrium outcomes in particular games. Eisert, Wilkens, and Lewenstein demonstrated that the prisoner's dilemma can

cease to pose a dilemma under a suitable quantum extension [2]. This conclusion, however, depends essentially on the admissible strategy space. Benjamin and Hayden showed that the Pareto-efficient pure equilibrium identified in the restricted EWL strategy set disappears when players are allowed to use arbitrary local unitary operations [12]. Subsequent work further demonstrated that the existence and structure of quantum equilibria depend not only on entanglement but also on the parametrization and scope of the players' strategy sets [13, 14]. Marinatto and Weber proposed an alternative quantization approach and applied it to static games of complete information, including the Battle of the Sexes [15]. Benjamin and Hayden extended the discussion to multiplayer quantum games [16]. Subsequent work analyzed, among other issues, the role of entanglement, equilibrium stability, and phase-transition-like changes in quantum games [17, 18]. These studies established that quantum resources can change strategic outcomes, but they also raised a deeper methodological issue: when does a quantum game remain a legitimate extension of the original classical game?

Because a quantum extension introduces additional structure including the embedding of classical strategies, the initial state, the admissible quantum operations, and the final measurement - its predictions should be invariant under strategically irrelevant changes in the representation of the input game. In particular, classically isomorphic games should generate isomorphic quantum extensions. Otherwise, equilibrium predictions may depend on the labelling of players or strategies rather than on the underlying game itself. This requirement was formulated explicitly as strong-isomorphism invariance for EWL-type quantum games [19] and subsequently used to construct restricted strategy spaces invariant under isomorphic transformations of the input game [20].

The present work tackles this issue by examining permissible finite quantum extensions of symmetric 2×2 games. We study a class of renaming-symmetric games, that is, classical games that are strategically equivalent to symmetric ones. This class encompasses a wide range of symmetric strategic dilemmas as well as coordination problems. The two classical strategies are represented by the local operations $\mathbb{1}$ and iX. We then add one fixed unitary strategy U to both players' strategy sets. This construction produces a finite 3×3 normal-form game in which the original 2×2 game appears as the upper-left subgame, while the remaining entries are determined by the EWL protocol.

The restriction to one additional unitary strategy is not merely a simplifying assumption. It allows the analysis to build directly on the classification of permissible three-strategy quantum extensions developed by Frąckiewicz and Szopa [21]. Their result identifies the conditions under which the added unitary operation preserves invariance with respect to isomorphic transformations of the input classical game. Under these conditions, only three types of permissible 3×3 extensions arise, denoted here by A_0, B_0, and C_0. By restricting attention to these extensions, the present paper ensures that the comparison between classical and quantum equilibria is not an artifact of strategy labeling.

The central research question is: under what conditions do permissible finite quantum extensions of symmetric strategic dilemmas generate NE whose payoffs are not inferior to the corresponding classical NE payoffs? We also identify the parameter regions in which the improvement is strict. The emphasis is therefore not on claiming that quantum strategies always improve strategic outcomes. Rather, the aim is to determine when such improvement is possible under a restricted, finite, and representation-invariant extension procedure.

The contribution of the paper is fourfold. First, we formulate a finite 3×3 EWL-type quantum extension of symmetric 2×2 games and explicitly preserve the classical game as a subgame. Second, we restrict the analysis to permissible extensions satisfying invariance with respect to isomorphic transformations of the classical game. Third, we classify the relevant classical equilibrium benchmarks across canonical payoff orderings, distinguishing cases in which the appropriate comparison is with pure or mixed classical equilibria. Fourth, we derive explicit conditions under which the permissible quantum extensions admit NE with payoffs that are non-inferior, and in some regions strictly superior, to the classical equilibrium payoffs.

The results show that Pareto-improved quantum equilibria arise only in specific regions of the payoff parameter space and depend on the type of permissible extension. In many cases, permissible extensions do not produce non-inferior quantum equilibria for the relevant strategy profiles. However, for selected canonical dilemma classes, the C_0 extension yields quantum NE that weakly or strictly improve upon the corresponding classical equilibrium payoff. The analysis therefore clarifies both the promise and the limitation of finite quantum extensions: quantum resources can create equilibrium improvements, but only under structurally identifiable conditions.

This perspective contributes to quantum information processing by treating quantum strategies not merely as formal additions to classical games, but as controlled extensions of strategic interaction. The invariance requirement separates genuine quantum-enabled changes in equilibrium structure from changes induced by arbitrary representation choices. In this sense, the paper links quantum correlations, Nash stability, and Pareto improvement through a finite and admissible extension of the classical strategy space.

The remainder of the paper is organized as follows. Section 2 presents the EWL framework and shows how the classical 2×2 game is recovered as a special case. It then introduces finite 3×3 extensions obtained by adding one non-classical unitary strategy. Section 3 defines the class of symmetric classical games and classifies the relevant NE benchmarks. Section 4 introduces the admissible quantum extensions A_0, B_0, and C_0. Section 5 derives the conditions under which quantum NE yield non-inferior payoffs relative to classical equilibria. Sections 5.2 and 6 discuss the implications of the results and conclude.

2 The EWL scheme and finite quantum extensions

We briefly recall the EWL quantization scheme for a two-player, two-strategy game. The protocol embeds a classical bimatrix game into a quantum setting by allowing the players to act locally on an entangled two-qubit state, after which a measurement in the computational basis determines the probabilities of measurement outcomes and, consequently, the expected payoffs.

The protocol consists of three steps: first, an entangling operation is applied to an initial product state; next the players perform local unitary operations corresponding to their strategies; finally, a disentangling operation is applied, followed by a measurement.

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ and consider the initial state

$$\rho_0 = |00\rangle\langle 00|. \tag{1}$$

Let J be the entangling unitary operator acting on $\mathcal{H}$,

$$J = \frac{\mathbb{1} \otimes \mathbb{1} + i\sigma_x \otimes \sigma_x}{\sqrt{2}} \tag{2}$$

and denote the corresponding unitary channels by

$$\mathcal{J}(\rho) = J\rho J^\dagger, \qquad \mathcal{J}^\dagger(\rho) = J^\dagger \rho J. \tag{3}$$

The role of J is to introduce entanglement between the players' subsystems, while $J^\dagger$ ensures that classical strategies are faithfully embedded in the scheme.

The entangled input state is given by $\rho_E = \mathcal{J}(\rho_0)$. A direct computation yields

$$\rho_E = \mathcal{J}(\rho_0) = \frac{1}{2}\left(|00\rangle\langle 00| - i|00\rangle\langle 11| + i|11\rangle\langle 00| + |11\rangle\langle 11|\right). \tag{4}$$

Players 1 and 2 act independently on their respective subsystems by applying local unitary operations $U_1, U_2 \in \mathsf{SU}(2)$ to their respective subsystems. Nevertheless, since the state ρ_E is entangled, local actions can generate non-classical correlations in the resulting outcome distribution.

The joint action of the players is described by the unitary channel

$$\mathcal{U}_{U_1,U_2}(\rho) = (U_1 \otimes U_2)\rho(U_1^\dagger \otimes U_2^\dagger). \tag{5}$$

The state after the players' moves is therefore

$$\rho' = \mathcal{U}_{U_1,U_2}(\rho_E). \tag{6}$$

In the original EWL scheme, the disentangling operation $J^\dagger$ is applied before measurement. The resulting state is

$$\rho_f = \left(\mathcal{J}^\dagger \circ \mathcal{U}_{U_1,U_2} \circ \mathcal{J}\right)(\rho_0) \tag{7}$$

Measurement in the computational basis can be described by the quantum-to-classical channel [22]

$$\mathcal{M}(\rho) = \sum_{i,j\in\{0,1\}} \mathrm{Tr}(|ij\rangle\langle ij|\rho)|ij\rangle\langle ij|. \tag{8}$$

This channel maps the quantum state onto a classical diagonal state in the computational basis by removing all off-diagonal terms, so that the diagonal entries correspond to the probabilities of measurement outcomes. In particular, for the state ρ_f, we define

$$p_{ij} = \mathrm{Tr}(|ij\rangle\langle ij|\rho_f). \tag{9}$$

Then

$$\rho_C = \mathcal{M}(\rho_f) = \left(\mathcal{M} \circ \mathcal{J}^\dagger \circ \mathcal{U}_{U_1,U_2} \circ \mathcal{J}\right)(\rho_0) = \sum_{i,j\in\{0,1\}} p_{ij}|ij\rangle\langle ij|. \tag{10}$$

Thus, the EWL protocol induces a probability distribution over the four classical outcomes, and the players' expected payoffs are obtained by averaging the classical payoff table with respect to this distribution.

We define diagonal payoff observables

$$R_1 = \sum_{i,j\in\{0,1\}} a_{ij}|ij\rangle\langle ij|, \quad R_2 = \sum_{i,j\in\{0,1\}} b_{ij}|ij\rangle\langle ij|, \tag{11}$$

where a_{ij} and b_{ij} are the entries of the underlying classical bimatrix game corresponding to the outcome (i, j). Then the expected payoffs are given by

$$u_A = \mathrm{Tr}(R_1\rho_C), \quad u_B = \mathrm{Tr}(R_2\rho_C). \tag{12}$$

2.1 Relation to the classical game

Let us now show explicitly that the classical bimatrix game is recovered as a special case of the EWL protocol. We restrict the players' strategy sets to the two unitary operators

$$U_1^0 = U_2^0 = \mathbb{1}, \qquad U_1^1 = U_2^1 = i\sigma_x. \tag{13}$$

These two operations represent the two pure classical strategies. For $i, j \in \{0, 1\}$, we have

$$\mathcal{U}^{i,j}_{U_1,U_2}(\rho_E) = \left(U_1^i \otimes U_2^j\right)\rho_E\left(\left(U_1^i\right)^\dagger \otimes \left(U_2^j\right)^\dagger\right) \tag{14}$$

A direct computation gives

$$\mathcal{U}^{ij}_{U_1,U_2}(\rho_E) = \frac{1}{2}\left(|ij\rangle\langle ij| - i|ij\rangle\langle \bar{i}\bar{j}| + i|\bar{i}\bar{j}\rangle\langle ij| + |\bar{i}\bar{j}\rangle\langle \bar{i}\bar{j}|\right) = |\psi_{ij}\rangle\langle\psi_{ij}|, \tag{15}$$

where $|\psi_{ij}\rangle = \frac{1}{\sqrt{2}}\left(|ij\rangle + i|\bar{i}\bar{j}\rangle\right)$, $\bar{i} = 1 - i$ and $\bar{j} = 1 - j$. We now apply the channel $\mathcal{J}^\dagger$, that is,

$$\rho_f^{ij} = \mathcal{J}^\dagger\left(\mathcal{U}^{ij}_{U_1,U_2}(\rho_E)\right) = J^\dagger\mathcal{U}^{ij}_{U_1,U_2}(\rho_E)J = J^\dagger|\psi_{ij}\rangle\langle\psi_{ij}|J = |ij\rangle\langle ij|. \tag{16}$$

Hence the measurement probabilities are

$$p_{kl}^{ij} = \mathrm{Tr}\left(|kl\rangle\langle kl|\rho_f^{ij}\right) = \delta_{ki}\delta_{lj}. \tag{17}$$

Consequently,

$$u_1(U_1^i, U_2^j) = a_{ij}, \qquad u_2(U_1^i, U_2^j) = b_{ij}. \tag{18}$$

2.2 EWL with a fixed non-classical unitary extension

We now restrict the set of admissible EWL strategies to a finite set consisting of the two classical strategies and one additional non-classical unitary operation. Namely, we consider

$$\mathcal{S} = \{\mathbb{1}, iX, U\}, \tag{19}$$

where $U \in \mathsf{SU}(2)$ is fixed in advance and is available to both players. We write

$$U_1^0 = U_2^0 = \mathbb{1}, \qquad U_1^1 = U_2^1 = iX, \qquad U_1^2 = U_2^2 = U. \tag{20}$$

The operator iX is the Pauli matrix σ_x multiplied by the imaginary unit. Thus, the players do not optimize over the whole group $\mathsf{SU}(2)$. Instead, each player chooses one of three pure strategies. The first two correspond to the classical pure strategies of the underlying 2×2 game, while the third one represents an additional quantum strategy.

For each pair of pure strategies (U_1^k, U_2^l), where $k, l \in \{0, 1, 2\}$, we define

$$\mathcal{U}_{U_1^k, U_2^l}(\rho) = \left(U_1^k \otimes U_2^l\right) \rho \left(\left(U_1^k\right)^\dagger \otimes \left(U_2^l\right)^\dagger\right). \tag{21}$$

The resulting classical state after the EWL protocol is

$$\rho_C^{kl} = \left(\mathcal{M} \circ \mathcal{J}^\dagger \circ \mathcal{U}_{U_1^k, U_2^l} \circ \mathcal{J}\right)(\rho_0). \tag{22}$$

The corresponding payoff pair is

$$u(U_1^k, U_2^l) = \left(u_A(U_1^k, U_2^l), u_B(U_1^k, U_2^l)\right), \tag{23}$$

where

$$u_A(U_1^k, U_2^l) = \mathrm{Tr}(R_1 \rho_C^{kl}), \qquad u_B(U_1^k, U_2^l) = \mathrm{Tr}(R_2 \rho_C^{kl}). \tag{24}$$

From the previous subsection, when both players use only the classical strategies in the EWL scheme, the original classical game is recovered. Hence, for $k, l \in \{0, 1\}$,

$$u(U_1^k, U_2^l) = (a_{kl}, b_{kl}). \tag{25}$$

The entries involving U are obtained from the same EWL payoff rule and may depend on the quantum features of the protocol. Then, each player's quantum strategy is represented by a unitary operator

$$U(\theta, \alpha, \beta) = \begin{pmatrix} e^{i\alpha}\cos(\theta/2) & ie^{i\beta}\sin(\theta/2) \\ ie^{-i\beta}\sin(\theta/2) & e^{-i\alpha}\cos(\theta/2) \end{pmatrix}, \tag{26}$$

where $\theta \in [0, \pi]$ and $\alpha, \beta \in [0, 2\pi)$. The classical strategies are embedded as

$$\mathbb{1} = U(0, 0, 0), \qquad iX = U(\pi, 0, 0).$$

Therefore, the restricted EWL scheme induces the following finite 3×3 game :

$$\begin{array}{c} \\ \mathbb{1} \\ iX \\ U \end{array} \begin{array}{c} \begin{array}{ccc} \mathbb{1} & iX & U \end{array} \\ \begin{pmatrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) & u(\mathbb{1}, U) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) & u(iX, U) \\ u(U, \mathbb{1}) & u(U, iX) & u(U, U) \end{pmatrix} \end{array}. \tag{27}$$

The original classical 2×2 game appears as the upper-left subgame, while the additional unitary U creates the third row and the third column.

Now let the players use mixed strategies. A mixed strategy of player A is a probability vector

$$p = (p_0, p_1, p_2), \qquad p_k \geq 0, \qquad \sum_{k=0}^{2} p_k = 1, \tag{28}$$

and similarly a mixed strategy of player B is

$$q = (q_0, q_1, q_2), \qquad q_l \geq 0, \qquad \sum_{l=0}^{2} q_l = 1. \tag{29}$$

In the quantum description, these mixed strategies are represented by local quantum channels:

$$\mathcal{E}_1(\rho) = \sum_{k=0}^{2} p_k U_1^k \rho \left(U_1^k\right)^\dagger, \quad \mathcal{E}_2(\rho) = \sum_{l=0}^{2} q_l U_2^l \rho \left(U_2^l\right)^\dagger. \tag{30}$$

Thus, the joint channel corresponding to the mixed strategy profile (p, q) is

$$\mathcal{E}(\rho) = \sum_{k,l=0}^{2} p_k q_l \left(U_1^k \otimes U_2^l\right) \rho \left(\left(U_1^k\right)^\dagger \otimes \left(U_2^l\right)^\dagger\right). \tag{31}$$

This means that the pair of pure strategies $\left(U_1^k, U_2^l\right)$ is implemented with probability $p_k q_l$. After the EWL protocol, the resulting classical state is

$$\rho_C(p, q) = \left(\mathcal{M} \circ \mathcal{J}^\dagger \circ \mathcal{E} \circ \mathcal{J}\right)(\rho_0). \tag{32}$$

By linearity,

$$\rho_C(p, q) = \sum_{k,l=0}^{2} p_k q_l \rho_C^{kl}. \tag{33}$$

Therefore,

$$u_1(p, q) = \sum_{k,l=0}^{2} p_k q_l u_1(U_1^k, U_2^l), \tag{34}$$

and

$$u_2(p, q) = \sum_{k,l=0}^{2} p_k q_l u_2(U_1^k, U_2^l). \tag{35}$$

Hence, the random-unitary channels $\mathcal{E}_1$ and $\mathcal{E}_2$ are exactly the quantum representation of classical mixed strategies over the finite set $\{\mathbb{1}, iX, U\}$. Consequently, once the admissible EWL strategies are restricted to this finite set, the model becomes an ordinary finite game in normal form with three pure strategies for each player, although the entries of the payoff matrix are still determined by the quantum EWL protocol.

2.3 Permissible three-strategy extensions

A finite quantum extension obtained by adding a unitary strategy to the two classical strategies should not depend on an arbitrary representation of the initial classical game. In particular, if two 2×2 classical games differ only by a relabeling of the players' strategies, i.e., they are strongly isomorphic, then their quantum extensions should also be strongly isomorphic. Otherwise, the quantum procedure would not extend the underlying game-theoretic problem itself, but rather a particular ordering of its strategies. As shown in [21], this requirement is nontrivial: for a generic unitary strategy U, isomorphic forms of the same classical game may lead to non-equivalent quantum games, possibly with different NE. Therefore, a three-strategy quantum extension is called permissible if the additional unitary strategy preserves invariance with respect to isomorphic transformations of the initial classical game.

The permissibility condition restricts the admissible unitary operators to those for which the extended 3×3 game is independent, up to strong isomorphism, of row and column permutations in the original 2×2 payoff matrix. In the EWL scheme with maximal entanglement, these restrictions imply that the additional strategy must satisfy $\theta = \pi/2$ together with appropriate phase conditions on α and β. Consequently, all permissible extensions fall into three and only three classes. The first one (A_0) corresponds to an extension that can be interpreted as adding a classical mixed strategy, while the remaining two (B_0, C_0) generate genuinely new quantum extensions that still preserve the strategic identity of the initial game.

Then a finite 3×3 quantum extension of the game is

$$\begin{array}{c} \\ \mathbb{1} \\ iX \\ U \end{array} \begin{array}{c} \begin{array}{ccc} \mathbb{1} & iX & U \end{array} \\ \begin{pmatrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) & u(I, U) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) & u(iX, U) \\ u(U, I) & u(U, iX) & u(U, U) \end{pmatrix} \end{array}. \tag{36}$$

where $U(\theta, \alpha, \beta) \equiv U$ is an arbitrary but fixed unitary strategy (26).

It was proven that there are only three permissible types of 3×3 extensions [21], namely:

$$A_0 = \begin{pmatrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) & \left(\frac{a_{00}+a_{01}}{2}, \frac{b_{00}+b_{01}}{2}\right) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) & \left(\frac{a_{10}+a_{11}}{2}, \frac{b_{10}+b_{11}}{2}\right) \\ \left(\frac{a_{00}+a_{10}}{2}, \frac{b_{00}+b_{10}}{2}\right) & \left(\frac{a_{01}+a_{11}}{2}, \frac{b_{01}+b_{11}}{2}\right) & \left(\sum\limits_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum\limits_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) \end{pmatrix}, \tag{37}$$

where $U = U\left(\frac{\pi}{2}, \alpha, \beta\right),\ \alpha, \beta \in \{0, \pi\}$. The second class is

$$B_0 = \begin{pmatrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) & \left(\frac{a_{10}+a_{11}}{2}, \frac{b_{10}+b_{11}}{2}\right) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) & \left(\frac{a_{00}+a_{01}}{2}, \frac{b_{00}+b_{01}}{2}\right) \\ \left(\frac{a_{01}+a_{11}}{2}, \frac{b_{01}+b_{11}}{2}\right) & \left(\frac{a_{00}+a_{10}}{2}, \frac{b_{00}+b_{10}}{2}\right) & \left(\sum\limits_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum\limits_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) \end{pmatrix}, \tag{38}$$

where $U = U\left(\frac{\pi}{2}, \alpha, \beta\right),\ \alpha, \beta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. And the third class is

$$C_0 = \begin{pmatrix} (a_{00}, b_{00}) & (a_{01}, b_{01}) & \left(\sum_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) \\ (a_{10}, b_{10}) & (a_{11}, b_{11}) & \left(\sum_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) \\ \left(\sum_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) & \left(\sum_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) & \left(\sum_{i,j\in\{0,1\}} \frac{a_{ij}}{4}, \sum_{i,j\in\{0,1\}} \frac{b_{ij}}{4}\right) \end{pmatrix}. \tag{39}$$

where $U = U\left(\frac{\pi}{2}, \alpha, \beta\right)$, and $\alpha, \beta \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$.

Adding an additional unitary strategy does not by itself guarantee that equilibrium payoffs will improve or even remain unchanged. The following example illustrates that an unrestricted unitary extension may eliminate a classical NE and replace it with an equilibrium yielding strictly lower payoffs.

Example 1. *Consider the symmetric classical game*

$$\begin{pmatrix} (3,3) & (0,5) \\ (5,0) & (4,4) \end{pmatrix} \tag{40}$$

The second strategy strictly dominates the first strategy for both players. Consequently, the game has a unique NE, given by the second-strategy profile, with payoff profile $(4,4)$*. Substituting operator* $U(\pi/2, \pi/2, \pi) = \frac{iZ-iX}{\sqrt{2}}$ *into the EWL protocol gives the following extended game:*

$$\begin{array}{c} \\ \mathbb{1} \\ iX \\ U \end{array} \begin{array}{c} \begin{array}{ccc} \mathbb{1} & iX & U \end{array} \\ \begin{pmatrix} (3,3) & (0,5) & (2,9/2) \\ (5,0) & (4,4) & (2,9/2) \\ (9/2,2) & (9/2,2) & (3,3) \end{pmatrix} \end{array}. \tag{41}$$

The classical equilibrium (iX, iX) *is no longer a NE of the extended game. Indeed, when the other player chooses* iX*, a unilateral deviation to* U *increases the deviating player's payoff from* 4 *to* $9/2$*. By contrast,* (U, U) *is a NE, since a unilateral deviation from* U *to either* $\mathbb{1}$ *or* iX *reduces the deviating player's payoff from* 3 *to* 2*. In fact,* (U, U) *is the unique NE of the extended game. To see this, consider a mixed strategy* $(0, p, 1-p)$ *that assigns probability* $p \in (0,1)$ *to* iX *and probability* $1-p$ *to* U*. Its payoffs against* $\mathbb{1}$*,* iX*, and* U *are, respectively,* $\frac{9}{2} + \frac{p}{2}, \frac{9}{2} - \frac{p}{2}$ *and* $3-p$*. These payoffs are strictly greater than the corresponding payoffs 3, 0, and 2 obtained from* $\mathbb{1}$*. Thus,* $\mathbb{1}$ *is strictly dominated by a mixed strategy and cannot be used with positive probability in any NE. After eliminating* $\mathbb{1}$*, the strategy* U *strictly dominates* iX*. Hence,* (U, U) *is the unique NE of the extended game, and the equilibrium payoff decreases from 4 in the classical game to 3 in its quantum extension.*

The unitary operation considered in this example does not generate a permissible extension, because the parameter pair $(\alpha, \beta) = \left(\frac{\pi}{2}, \pi\right)$ satisfies none of the conditions defining the permissible classes A_0, B_0, and C_0. Consequently, extension (41) violates the game-isomorphism-invariance requirement adopted in this paper.

The example shows that merely enlarging the strategy space does not guarantee an equilibrium payoff at least as high as that of the corresponding classical equilibrium. An additional unitary strategy may introduce profitable unilateral deviations and thereby eliminate equilibria that are stable in the classical game. This motivates restricting the analysis to permissible, game-isomorphism-invariant extensions. Within this restricted family, our results identify extensions admitting an equilibrium whose payoff is no lower than that of the corresponding classical equilibrium.

3 Classical games

We consider a class of 2×2 games that are strategically equivalent to symmetric games. This class includes symmetric games

$$G = \begin{pmatrix} (s_1, s_1) & (s_2, s_3) \\ (s_3, s_2) & (s_4, s_4) \end{pmatrix}, \tag{42}$$

or games that can be represented as symmetric games by reversing the order of one player's strategies, as in the case of a game

$$G' = \begin{pmatrix} (s_2, s_3) & (s_1, s_1) \\ (s_4, s_4) & (s_3, s_2) \end{pmatrix}, \tag{43}$$

which can be transformed into game G if the order of the Column player's strategies is reversed. These games, named renaming-symmetric games, constitute a fairly large class of games, such as the Prisoners' Dilemma, the Hawk-Dove game, the Battle of the Sexes, and the Chicken game.

3.1 Classification of 2×2 games

In our analysis, we examine the general form of the game (42), which includes all 24 possible linear orderings of the payoff values s_1, s_2, s_3 and s_4. Note that by permuting the rows and columns in the game (42), we obtain an isomorphic game

$$G'' = \begin{pmatrix} (s_4, s_4) & (s_3, s_2) \\ (s_2, s_3) & (s_1, s_1) \end{pmatrix} \tag{44}$$

in which the positions of the payoffs s_2 and s_3 are interchanged. Without loss of generality, we may therefore assume that $s_3 \geq s_2$. As a result, the number of cases to be examined is reduced to 12, which, for the purpose of our discussion, can be divided into 12 orderings. We now rule out those orderings for which the highest payoff occurs at one of the diagonal outcomes. If

$$s_1 = \max\{s_1, s_2, s_3, s_4\},$$

then in particular we have $s_1 \geq s_3$. Hence, the profile $((1, 0), (1, 0))$ constitutes a symmetric NE of the classical game. Its payoff is $v_C^* = s_1$, which is Pareto-optimal. Likewise, if

$$s_4 = \max\{s_1, s_2, s_3, s_4\},$$

then $s_4 \geq s_2$, so $((0,1),(0,1))$ forms a symmetric NE of the classical game with payoff $v_C^* = s_4$, which is again Pareto-optimal.

When the classical game already admits a symmetric Pareto-optimal Nash equilibrium, no payoff improvement is possible within a payoff-preserving quantum extension. The role of the extension is therefore to preserve this optimal equilibrium. This is achieved by the A_0 extension: a classical Pareto-optimal NE attaining the maximal payoff remains an equilibrium of the extended game, because the additional strategy generates only averages of the original payoffs and thus creates no profitable unilateral deviation. Hence, the A_0 extension also admits a symmetric Pareto-optimal NE with the same payoff as its classical counterpart. Consequently,

$$v_Q^* = v_C^*,$$

where v_Q^* denotes the equal payoff attained at the corresponding NE of the quantum extension. It remains to examine those payoff orderings in which the maximal payoff is s_3. Under the convention $s_2 \leq s_3$, these reduce to the following six canonical cases. In game theory, we impose strict inequalities to guarantee that the game is generic and free of payoff ties.

$1a$: $s_4 < s_1 < s_2 < s_3$ (Battle of the Sexes)

$1b$: $s_1 < s_4 < s_2 < s_3$ (Battle of the Sexes)

$2a$: $s_4 < s_2 < s_1 < s_3$ (Chicken game, Hawk-Dove game)

$2b$: $s_1 < s_2 < s_4 < s_3$ (Route Choice Game)

$3a$: $s_2 < s_4 < s_1 < s_3$ (Prisoners' Dilemma)

$3b$: $s_2 < s_1 < s_4 < s_3$ (Deadlock game)

Without loss of generality, in cases $1a$ - $3b$, we may normalize the payoffs by setting $s_3 = 1$ and shifting s_1, s_2, s_4 so that their minimum is equal to zero, i.e.,

$$\min_{k\in\{1,2,4\}} s_k = 0.$$

3.2 Types of Nash equilibria in classical games

For the symmetric game (42), the pure-strategy NE are obtained directly from the best-response conditions. The strategy profile $(1,1)$ is a NE whenever neither player has an incentive to deviate unilaterally to strategy 2, which is equivalent to

$$s_1 \geq s_3.$$

Similarly, the profile $(2,2)$ is a NE whenever

$$s_4 \geq s_2.$$

Finally, the asymmetric profiles $(1,2)$ and $(2,1)$ are NE precisely when strategy 1 is a best response to strategy 2 and, simultaneously, strategy 2 is a best response to strategy 1. This is

equivalent to the conditions

$$s_2 \geq s_4 \quad \text{and} \quad s_3 \geq s_1.$$

Besides pure equilibria, the game may also admit a symmetric mixed-strategy NE. Suppose each player adopts the mixed strategy

$$\sigma = (p, 1-p),$$

where $p \in [0,1]$ denotes the probability of playing strategy 1. If the opponent uses the mixed strategy σ, then the expected payoff from choosing strategy 1 is

$$u(1,\sigma) = ps_1 + (1-p)s_2,$$

whereas the expected payoff from choosing strategy 2 is

$$u(2,\sigma) = ps_3 + (1-p)s_4.$$

In a mixed-strategy NE both pure strategies must yield the same expected payoff. Hence,

$$ps_1 + (1-p)s_2 = ps_3 + (1-p)s_4.$$

Solving for p gives

$$p^* = \frac{s_4 - s_2}{s_1 - s_2 - s_3 + s_4},$$

provided that

$$s_1 - s_2 - s_3 + s_4 \neq 0.$$

Therefore, whenever

$$0 \leq p^* \leq 1,$$

the game admits the symmetric mixed-strategy NE

$$(\sigma^*, \sigma^*), \qquad \sigma^* = (p^*, 1-p^*).$$

The above discussion remains valid for renaming-symmetric games. Since NE are invariant under relabelling of strategies, a renaming-symmetric game can first be transformed into its canonical symmetric form. The equilibrium is then computed for this representation and mapped back to the original game by applying the inverse relabelling of strategies.

3.3 Classification of Nash equilibria in classical games

Consider a game (42). Let us now define the following parameters:

$$\Delta_1 := s_1 - s_3, \qquad \Delta_2 := s_4 - s_2 \tag{45}$$

Depending on the signs of Δ_1 and Δ_2, the game can be classified into the following cases.

- **Case A: Coordination-type game.** Suppose $\Delta_1 > 0, \Delta_2 > 0$, or equivalently, $s_1 > s_3$ and $s_4 > s_2$. Then the game has two pure symmetric NE, defined by the following pairs of strategies, i.e. (1,1) and (2,2), and one symmetric mixed equilibrium.

- **Case B: Anti-coordination game.** Suppose $\Delta_1 < 0, \Delta_2 < 0$, or equivalently, $s_1 < s_3$ and $s_4 < s_2$. Then the game has two pure asymmetric NE, defined by the following pairs of strategies, i.e. (1,2) and (2,1), and one symmetric mixed equilibrium. This corresponds to anti-coordination games such as Hawk–Dove or Chicken.
- **Case C: Strategy 1 dominant.** Suppose $\Delta_1 > 0, \Delta_2 < 0$, or equivalently, $s_1 > s_3$ and $s_4 < s_2$. The unique NE is given by a pair of strategies (1,1). The mixed equilibrium lies outside the interval [0,1], so no interior mixed equilibrium exists. Hence the game possesses exactly one pure NE.
- **Case D: Strategy 2 dominant.** Suppose $\Delta_1 < 0, \Delta_2 > 0$, or equivalently, $s_1 < s_3$ and $s_4 > s_2$. The unique NE is given by a pair of strategies (2,2). Again, no interior mixed equilibrium exists. Hence the game possesses exactly one pure NE.

We do not discuss cases in which $\Delta_i = 0$, given the conditions in Section 3.1. In these boundary situations, at least one player is indifferent between the two pure strategies, giving rise to degenerate equilibrium structures. Accordingly, four general equilibrium structures can be distinguished as Cases A-D. These cases classify the game according to the number and symmetry of its NE, independently of the specific ordering of the payoff parameters.

The classical game classes defined in the Section 3.1 are defined by additional restrictions on the relative ordering of s_1, s_2, s_3, s_4. Therefore, each class can be assigned to one of the general Nash-equilibrium categories by evaluating the corresponding best-response conditions. The resulting correspondence is given in Table 1.

Class	Game type	NE case
$1a$: $s_4 < s_1 < s_2 < s_3$	Battle of the Sexes	Case B
$1b$: $s_1 < s_4 < s_2 < s_3$	Battle of the Sexes	Case B
$2a$: $s_4 < s_2 < s_1 < s_3$	Chicken / Hawk-Dove	Case B
$2b$: $s_1 < s_2 < s_4 < s_3$	Route Choice Game	Case D
$3a$: $s_2 < s_4 < s_1 < s_3$	Prisoner's Dilemma	Case D
$3b$: $s_2 < s_1 < s_4 < s_3$	Deadlock Game	Case D

Table 1: Mapping of payoff-ordering (canonical) classes to NE structures

4 Existence and payoff structure of Nash Equilibria in 3 × 3 quantum extensions

4.1 Research Objectives

This study considers the extension of classical games into the quantum domain by augmenting each player's two classical strategies with an additional unitary strategy, resulting in a three-strategy framework [21]. To ensure theoretical consistency, such extensions must remain invariant under isomorphic transformations of the underlying classical game.

Previous research has shown that only three classes of extensions satisfy these conditions, with two being inherently quantum [21]. Using non-permissible unitary operators may lead to inequivalent quantum games, for example differing in their Nash equilibria despite originating from the same classical structure.

Within this setting, the study investigates whether the incorporation of permissible unitary strategies can yield outcomes that are at least as favorable as, or potentially superior to, their classical counterparts.

The detailed objectives of this study include the following:

- Investigating NE in quantum-extended classical games: The analysis focuses on a general, parametrized classical game to examine how NE emerge within the expanded strategic space.
- Evaluating quantum payoffs: A primary objective is to establish conditions ensuring that both players' payoffs associated with quantum NE are not lower than those arising in the classical framework.
- Assessing the strategic advantages of quantum extensions: The study aims to determine how quantum strategies can resolve classical decision-making dilemmas and improve the equilibrium solutions.

4.2 Definitions and notation

Following the construction recalled in Section 2.3 and originally derived in [21], we now apply the three permissible quantum extensions to the symmetric game (42). In Section 2.3 the extensions were written as full bimatrix games, with each entry containing the payoffs of both players. Here, for the sake of further analysis, we use a reduced notation and display only the payoff matrix of the row player.

This reduction is justified by the symmetry of the underlying game. If the payoff matrix of the row player is known, then the payoff matrix of the column player is obtained by its transposition. Thus, the full bimatrix extension is completely determined by the scalar payoff matrix of one player.

Accordingly, we denote by A_0, B_0, and C_0 the row-player payoff matrices corresponding to the three permissible extensions introduced in Section 2.3. These matrices are obtained by substituting

$$a_{00} = b_{00} = s_1, \qquad a_{01} = b_{10} = s_2, \qquad a_{10} = b_{01} = s_3, \qquad a_{11} = b_{11} = s_4,$$

into the first-player components of the general formulas for the permissible extensions. The resulting scalar matrices will be used below as the payoff matrices of the extended games:

$$A_0 = \begin{pmatrix} s_1 & s_2 & \frac{s_1+s_2}{2} \\ s_3 & s_4 & \frac{s_3+s_4}{2} \\ \frac{s_1+s_3}{2} & \frac{s_2+s_4}{2} & \frac{s_1+s_2+s_3+s_4}{4} \end{pmatrix}, \tag{46}$$

$$B_0 = \begin{pmatrix} s_1 & s_2 & \frac{s_3+s_4}{2} \\ s_3 & s_4 & \frac{s_1+s_2}{2} \\ \frac{s_2+s_4}{2} & \frac{s_1+s_3}{2} & \frac{s_1+s_2+s_3+s_4}{4} \end{pmatrix}, \tag{47}$$

$$C_0 = \begin{pmatrix} s_1 & s_2 & \frac{s_1+s_2+s_3+s_4}{4} \\ s_3 & s_4 & \frac{s_1+s_2+s_3+s_4}{4} \\ \frac{s_1+s_2+s_3+s_4}{4} & \frac{s_1+s_2+s_3+s_4}{4} & \frac{s_1+s_2+s_3+s_4}{4} \end{pmatrix}. \tag{48}$$

Consequently, in the remainder of this section, the symbols A_0, B_0, and C_0 refer to the payoff matrices of the row player, not to the full bimatrix representations of the corresponding extended games.

Let $c_k \in S_k$ denote a strategy available to player k, where $S_k = \{\mathbb{1}, iX, U\}$ is the player's strategy set. We compare the payoffs corresponding to classical NE u_C (the best among pure and mixed strategies) with those obtained for quantum equilibria u_Q. The quantity referred to as the *Gain* is defined as the difference between these payoffs:

$$g^k = u_Q^k(c_i, c_j) - u_C^k(c_i, c_j), \tag{49}$$

where $k \in \{1, 2\}$ denotes the player, and $i, j \in \{1, 2, 3\}$ denote the indices of the corresponding strategy profiles c. The maximum gain is reported in the tables in Section 5.

As shown in the results presented in Section 5, whenever both players benefit from the quantum extension, their gains coincide. Consequently, the notation $g^{1,2}$ is used to denote this common positive gain, whereas g^1 (respectively, g^2) indicates that only player 1 (respectively, player 2) benefits.

5 Quantum Nash equilibria with non-inferior payoffs

5.1 Main results

This section presents the results on the existence of quantum NE with non-inferior payoffs in 3×3 quantum-extended games. The analysis encompasses equilibria evaluated against both pure- and mixed-strategy classical payoffs, following the classification in Section 3.3, with particular attention given to identifying conditions under which the quantum outcomes are at least as favorable as those obtained in the classical case. By systematically examining these equilibria, the study highlights how the introduction of permissible unitary strategies influences both their existence and the associated payoff structure.

Tables 2-4 provide a strategy-oriented characterization of the quantum NE obtained for the extensions A_0, B_0, and C_0. For each strategy profile, the tables identify the canonical classes in which the profile constitutes a NE and yields payoffs at least as high as those of the corresponding classical equilibrium. The associated payoff constraints and the resulting upper bounds on the quantum payoff gains are also reported.

The obtained results reveal a clear hierarchy among the three considered quantum extensions. Although all of them preserve the structure of the underlying symmetric game, they differ substantially in their ability to generate quantum NE with non- inferior payoffs.

The analytical conditions summarized in Tables 2-4 are illustrated geometrically in Figs. 1-3. The figures represent the regions of the payoff-parameter space for which a given quantum strategy profile satisfies the NE conditions and provides non-inferior payoffs with respect to the corresponding classical equilibrium. Therefore, the regions shown in the figures illustrate the extent to which each quantum extension expands the set of admissible payoff parameters.

For the A_0 extension, the results presented in Table 2 indicate that only the strategy profile $(3, 3)$ can generate non-inferior quantum NE, and only for the condition $s_2 = 1 - s_1$

Extension A_0			
Strategy	Canonical class (NE case)	Conditions for NE existence with non-inferior payoffs	Gain
(3,3) Fig. 1	$1a$ (B)	$0 < s_1 < \frac{1}{2} \wedge s_2 = 1 - s_1$	0
	$2a$ (B)	$\frac{1}{2} < s_1 < 1 \wedge s_2 = 1 - s_1$	0

Table 2: Conditions for the existence of quantum NE with non-inferior payoffs in the A_0 extension.

Extension B_0			
Strategies	Canonical class (NE case)	Conditions for NE existence with non-inferior payoffs	Gain
(3,2) Fig. 2	$1a$ (B)	$\frac{1}{3} < s_1 < \frac{1}{2} \wedge 1 - s_1 \leq s_2 \leq \frac{s_1+1}{2}$ $\frac{1}{2} \leq s_1 < 1 \wedge s_1 < s_2 \leq \frac{s_1+1}{2}$ $s_1 = \frac{1}{3} \wedge s_2 = \frac{2}{3}$	$g^1 \to \frac{1}{4}$
	$2a$ (B)	$\frac{1}{2} < s_1 < 1 \wedge s_2 = 1 - s_1$	$g^1 \to \frac{1}{2}$
(3,3) Fig. 2	$1a$ (B)	$0 < s_1 < \frac{1}{2} \wedge s_2 = 1 - s_1$	0
	$2a$ (B)	$\frac{1}{2} < s_1 < 1 \wedge s_2 = 1 - s_1$	0

Table 3: Conditions for the existence of quantum NE with non-inferior payoffs in the B_0 extension.

corresponding to canonical classes $1a$ and $2a$. This restriction is reflected in Fig. 1, where the admissible payoff parameter region reduces to a single line in the (s_1, s_2) plane. Moreover, the absence of color variation in the gain distribution confirms the analytical result that no payoff improvement is obtained, i.e., the gain remains equal to zero. There are no additional NE with non-inferior quantum payoffs in the remaining parameter planes.

The extension B_0 introduces a qualitative change with respect to A_0. While the symmetric strategy profile $(3, 3)$ reproduces the equilibrium structure observed for the A_0 extension, as confirmed by Fig. 2, the additional strategy profile $(3, 2)$ gives rise to new quantum NE. As shown in Table 3, canonical class $1a$ reveals equilibria over a two-dimensional region of the payoff parameter space, rather than being restricted only to the boundary $s_2 = 1 - s_1$.

This enlargement is illustrated in Fig. 2, where the permissible region corresponds to the inequalities obtained for classes $1a$ and $2a$, as indicated in the Table 3. Furthermore, the heat map reflects the positive payoff improvements predicted analytically, with the gain for player 1 approaching $g^1 \to \frac{1}{4}$ for class $1a$ and $g^1 \to \frac{1}{2}$ for class $2a$, while the corresponding gain for player 2 is not negative yet smaller. Hence, all non-trivial advantages of the B_0 extension

Extension C_0			
Strategies	Canonical class (NE case)	Conditions for NE existence with non-inferior payoffs	Gain
(3,2) Fig. 3	$1a$ (B)	$0 < s_1 < \frac{1}{2} \wedge s_1 < s_2 \leq \frac{1+s_1}{3}$	$g^{1,2} \to \frac{1}{4}$
	$1b$ (B)	$0 < s_2 \leq \frac{1}{3} \wedge\ 0 < s_4 < s_2$ $\frac{1}{3} < s_2 < \frac{1}{2} \wedge\ 3s_2 - 1 \leq s_4 < s_2$	$g^{1,2} \to \frac{1}{4}$
	$2a$ (B)	$0 < s_1 \leq \frac{1}{2} \wedge 0 < s_2 < s_1$ $\frac{1}{2} < s_1 < 1 \wedge 0 < s_2 \leq 1 - s_1$	$g^{1,2} \to \frac{1}{2}$
	$2b$ (D)	$0 < s_2 < \frac{1}{2} \wedge s_2 < s_4 \leq \frac{1+s_2}{3}$	$g^{1,2} \to \frac{1}{4}$
	$3a$ (D)	$0 < s_1 \leq \frac{1}{2} \wedge\ 0 < s_4 < s_1$ $\frac{1}{2} < s_1 < 1 \wedge\ 0 < s_4 \leq \frac{1+s_1}{3}$	$g^{1,2} \to \frac{1}{2}$
	$3b$ (D)	$0 < s_1 < \frac{1}{2} \wedge s_1 < s_4 \leq \frac{1+s_1}{3}$	$g^{1,2} \to \frac{1}{4}$
(3,3) Fig. 3	$1a$ (B)	$0 < s_1 < \frac{1}{2} \wedge s_1 < s_2 \leq 1 - s_1$	$g^{1,2} \to \frac{1}{4}$
	$1b$ (B)	$0 < s_2 \leq \frac{1}{2} \wedge\ 0 < s_4 < s_2$ $\frac{1}{2} < s_2 < 1 \wedge\ 0 < s_4 \leq 1 - s_2$	$g^{1,2} \to \frac{1}{4}$
	$2a$ (B)	$0 < s_1 \leq \frac{1}{2} \wedge 0 < s_2 < s_1$ $\frac{1}{2} < s_1 < 1 \wedge 0 < s_2 \leq 1 - s_1$	$g^{1,2} \to \frac{1}{2}$
	$2b$ (D)	$0 < s_2 < \frac{1}{2} \wedge s_2 < s_4 \leq \frac{1+s_2}{3}$	$g^{1,2} \to \frac{1}{4}$
	$3a$ (D)	$0 < s_1 \leq \frac{1}{2} \wedge\ 0 < s_4 < s_1$ $\frac{1}{2} < s_1 < 1 \wedge\ 0 < s_4 \leq \frac{1+s_1}{3}$	$g^{1,2} \to \frac{1}{2}$
	$3b$ (D)	$0 < s_1 < \frac{1}{2} \wedge s_1 < s_4 \leq \frac{1+s_1}{3}$	$g^{1,2} \to \frac{1}{4}$

Table 4: Conditions for the existence of quantum NE with non-inferior payoffs in the C_0 extension.

originate from the asymmetric strategy profile $(3, 2)$, whereas the symmetric profile $(3, 3)$ introduces no additional quantum benefit, similarly to the A_0 extension.

The most substantial extension of the permissible quantum equilibrium regions is obtained for C_0. The analytical classification in Table 4 shows that both strategy profiles $(3, 2)$ and $(3, 3)$ admit non-inferior quantum NE for a larger number of NE cases, including canonical classes $1a$, $1b$, $2a$, $2b$, $3a$, and $3b$. This result is reflected in Fig. 3, where the admissible regions occupy a substantially larger volume of the payoff parameter space. In particular, the three-dimensional representation demonstrates that the existence conditions depend not only on s_1 and s_2, but also on the payoff parameter s_4. The gain distribution shown by the heat maps is consistent with the values reported in Table 4, where the achievable improvements reach $g^{1,2} \to \frac{1}{4}$ for several canonical classes and $g^{1,2} \to \frac{1}{2}$ for classes $2a$ and $3a$.

Among the considered extensions, C_0 exhibits the richest equilibrium structure. Unlike the previous extensions, it supports quantum NE for all canonical classes considered in this study. In addition to the equilibrium classes inherited from the A_0 and B_0 extensions, new equilibria arise in classes $1b$, $2b$, $3a$, and $3b$, enlarging the regions of the parameter space in which non-inferior quantum equilibria exist.

Moreover, the quantum advantage is no longer restricted to a single category of the classical NE classification. Whereas the A_0 and B_0 extensions produce advantageous quantum equilibria only for cases classified as B, the C_0 extension additionally covers cases classified as D. Consequently, the C_0 extension applies to a broader range of classical equilibrium configurations.

The progression from A_0 through B_0 to C_0 is clearly visible both analytically and geometrically. These findings suggest that the choice of the quantum extension is at least as important as the choice of the quantum strategy itself, and that certain extensions may fundamentally modify the equilibrium landscape of symmetric games rather than perturb existing classical solutions.

Moreover, it is worth noticing that the three-dimensional representation of the C_0 extension also reveals the geometric structure of the canonical classes classification. Each pair of payoff parameters determines a different subset of canonical classes. Specifically, classes $1a$ and $2a$ are described by inequalities involving (s_1, s_2), classes $1b$ and $2b$ depend on (s_2, s_4), whereas classes $3a$ and $3b$ are determined by (s_1, s_4). Consequently, the coordinate planes of Fig. 3 provide natural projections of the analytical conditions summarized in Table 4, illustrating how the various canonical classes are embedded within the complete three-dimensional payoff parameter space.

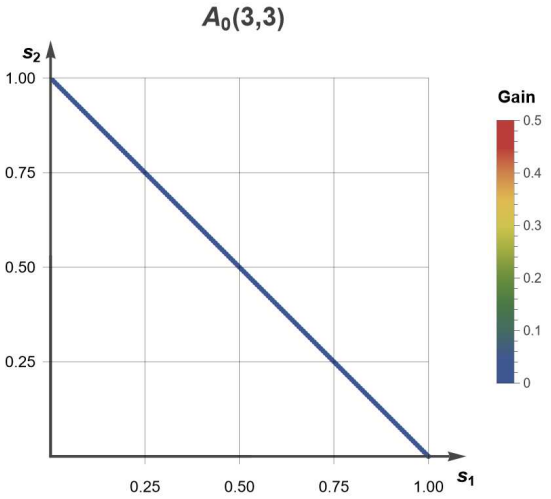


Figure 1: The plot shows the regions of payoff parameters for which the strategy profile (3,3) in the A_0 extension constitutes NE with quantum payoffs exceeding or equal to their classical counterparts. The heat map color displays the "Gain".

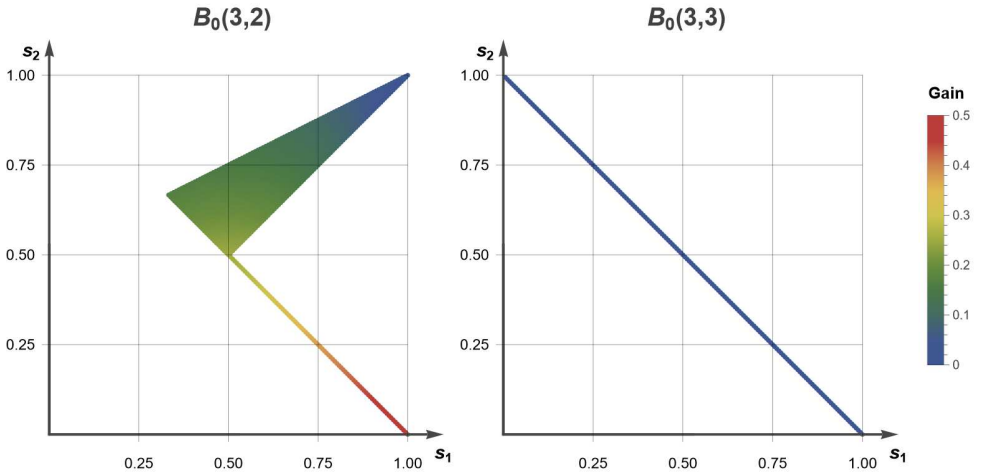


Figure 2: The plot shows the regions of payoff parameters for which the strategy profiles (3,2) and (3,3) in the B_0 extension constitute NE with quantum payoffs exceeding or equal to their classical counterparts. The heat map colour displays the "Gain".

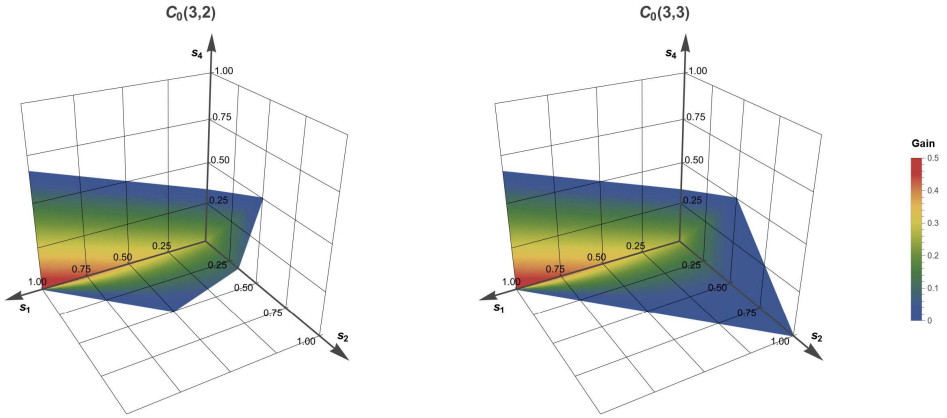


Figure 3: The plot shows the regions of payoff parameters for which the strategy profiles (3,2) and (3,3) in the C_0 extension constitute NE with quantum payoffs exceeding or equal to their classical counterparts. The heat map colour displays the "Gain"

5.2 Comparison and discussion

In the proposed framework, each player is allowed to select either a classical or a quantum strategy, enabling both paradigms to coexist within the same decision space and be evaluated

on equal footing. This hybrid setting makes it possible to assess the strategic value introduced by quantum options without constraining the players' rational choices.

The obtained results consistently indicate that the C_0 extension is the most effective mechanism for generating quantum NE with non-inferior payoffs. This behavior can be attributed to the structural properties of the corresponding payoff transformation resulting from the EWL scheme. While the A_0 and B_0 extensions construct the additional strategy from partial averages of selected classical payoffs, the C_0 extension defines the quantum-induced additional strategy using the global average payoff, $(s_1+s_2+s_3+s_4)/4$. As a consequence, the third strategy becomes independent of local payoff asymmetries and depends only on the aggregated characteristics of the underlying game.

This global averaging mechanism effectively smooths the payoff landscape and reduces the influence of single payoff values. Consequently, the Nash equilibrium conditions become less sensitive to the particular ordering of the payoff parameters, increasing the likelihood that both equilibrium existence and payoff non-inferiority are satisfied simultaneously. This trend is clearly reflected in the results, where C_0 is the only extension yielding advantageous pure-strategy equilibria and producing the largest number of admissible regions across the canonical classes.

In contrast, the A_0 and B_0 extensions preserve a stronger dependence on the original payoff structure because the additional strategy is generated from local and crossed averages of selected payoffs. As a result, the Nash inequalities remain more restrictive and the admissible regions are either absent or substantially smaller. Therefore, the results suggest that the global averaging mechanism inherent in C_0 provides a more favorable balance between equilibrium feasibility and payoff enhancement than the more localized payoff transformations represented by the extensions A_0 and B_0.

An additional perspective is provided by the evolution of the geometry of the admissible payoff parameter regions shown in Figs. 1-3. The progression from the one-dimensional region obtained for the A_0 extension, through the two-dimensional polygonal regions appearing in B_0, to the substantially larger three-dimensional regions associated with C_0 reflects a gradual relaxation of the joint NE and payoff non-inferiority conditions. In the A_0 extension, the admissible set is confined to the line $s_2 = 1 - s_1$, indicating that the existence of quantum equilibria requires a very specific relationship between the payoff parameters. In contrast, the B_0 extension admits two-dimensional regions in the (s_1, s_2) plane, while the C_0 extension further enlarges these regions into three-dimensional subsets of the (s_1, s_2, s_4) parameter space. Consequently, the quantum equilibria become feasible for progressively wider ranges of payoff configurations, following the rationale on averaging provided above.

The geometric enlargement of the admissible regions of payoff parameters is accompanied by quantum payoff improvements.

Although no general monotonic relationship between the size of the admissible region of payoff parameters and the gain is observed, Figs. 1-3 show that larger feasible regions correspond to a broader range of payoff parameters for which positive quantum advantages are achievable. This observation is consistent with the construction of the C_0 extension, in which the additional quantum strategy is defined by the global average payoff, $(s_1 + s_2 + s_3 + s_4)/4$, rather than by partial averages of selected classical payoffs as in A_0 and B_0. Consequently, the existence conditions become less dependent on local payoff asymmetries, leading to broader admissible regions in the payoff parameter space.

6 Conclusions

The results presented in this paper show that the introduction of quantum strategies can fundamentally reshape the landscape of equilibria in symmetric games. However, this conclusion is not valid for arbitrary quantization procedures or unrestricted extensions of the strategy space. As illustrated by the Example 1, an unrestricted choice of additional unitary strategies may introduce new unilateral deviations and eliminate classical equilibria that were stable in the original game.

The main result of this paper therefore is derived within the quantization framework satisfying the proposed permissibility and isomorphism invariance requirements, which is chosen to ensure that the properties of the underlying classical game are preserved. Under such procedure it is possible to construct a quantum version of any finite symmetric 2×2 game such that additional equilibria with non-inferior payoffs may emerge. This finding generalizes earlier insights obtained for specific cases such as the Prisoner's Dilemma and Battle of the Sexes [2, 23], and establishes a minimal condition for the existence of quantum advantage in symmetric games.

An important implication from this study is that the existence of quantum NE with non-inferior payoffs is not confined to isolated payoff configurations. Across the three quantum extensions considered, admissible parameter regions emerge systematically and expand from the highly restrictive conditions of the A_0 extension to the substantially larger regions obtained for C_0. This progression indicates that the construction of the quantum extension, particularly the definition of the additional quantum strategy, plays a decisive role in determining both the existence of quantum equilibria and the extent of the payoff improvements.

From a design perspective, these results provide a screening framework for quantum-enabled multi-agent protocols. They show that adding a quantum strategy is not inherently beneficial: its value depends jointly on the structure of the classical payoffs and the form of the quantum extension. The derived conditions identify when a representation-invariant extension can improve equilibrium outcomes without changing the players' underlying preferences or the classical payoff table. They can therefore guide protocol designers in selecting suitable quantum operations and, equally importantly, in recognizing games for which no advantage should be expected. Translating this framework into operational systems will require examining the robustness of the identified equilibria to noise, implementation errors, and small payoff perturbations, as well as comparing them with equally resourced classical coordination mechanisms.

Declarations

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