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Solution of the Advection-Diffusion Equation in Two Dimensions using the Deposition Velocity

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Abstract

In this study, we utilize the crosswind integrated normalized forecasted concentrations at centerline taking the power law of vertical eddy diffusivity and wind speed using the separation method. The observed SF₆ concentrations on Copenhagen, Denmark, are compared with the crosswind integrated normalized forecasted concentrations at centerline. We find that the crosswind integrated normalized forecasted concentrations at centerline achieved 100% of its observed concentrations.

Keywords: Crosswind Integrated Normalized Concentrations; The Separation Method; The Observed SF₆ Concentrations on Copenhagen, Denmark.

Introduction

Recent studies have demonstrated the applicability of these techniques in unstable conditions. For example, Essa and Mosallem (2025) applied separation methods combined with Laplace and Fourier transforms to investigate pollutant dispersion under weak wind regimes, showing strong agreement with the Prairie-Grass field experiment. Similarly, Essa and Elsaid (2023) extended the advection diffusion equation (ADE) to three dimensions under unstable conditions, employing Laplace transforms to model radioactive iodine dispersion in Cairo, with turbulence parameterization based on Monin–Obukhov similarity theory. These works highlight the robustness of transform-based solutions in capturing the essential physics of pollutant transport in unstable boundary layers.

Essa and Taha (2023) developed two analytical model solutions of the steady-state ADE using variable separation, Fourier transform, Laplace transform, and multilayer methods. Their work incorporated wind speed and eddy diffusivity as power-law functions of vertical distance, enabling predictions under neutral, stable, and unstable conditions. Malik et al. (2024) introduced a subdivision collocation method for solving ADE numerically, validating results against analytical solutions. While primarily numerical, their work emphasizes the importance of analytical benchmarks in unstable regimes. Fenton (2023) explored analytical solutions of ADE in sediment transport, reinforcing the role of diffusion in unstable hydrodynamic environments.

This work presents an analytical treatment of the two-dimensional advection-diffusion equation (ADE) under unstable conditions, using the separation of variables technique. The forms of the vertical eddy diffusivity and the wind speed are taken as function as power law in the vertical height. The observed SF₆ concentrations on Copenhagen, Denmark, are compared with the crosswind integrated normalized forecasted concentrations at centerline.

Method technique

The advection-diffusion equation in two dimensions is as follows:

$$u(z) \frac{\partial C_y(x,z)}{\partial x} = \frac{\partial}{\partial z} \left(k_z(z) \frac{\partial C_y(x,z)}{\partial z} \right) \quad (1)$$

where, $u(z)$ is the wind speed (m/s), $C_y(x,z)$ is the crosswind concentration in two dimensions (g/m^2), (Bq/m^2) and $k_z(z)$ is the vertical eddy diffusivity (m^2/s).

Eq. (1) is solved under the boundary conditions are as follows:

(a) The flux is equal zero at the vertical height equals the mixing height is ($z=h$).

$$k_z(z) \frac{\partial C_y(x,z)}{\partial z} = 0 \quad \text{at } z = h \quad (2a)$$

(b) The flux equals the multiply the deposition velocity and crosswind concentration at the ground ($z=0$)

$$k_z(z) \frac{\partial C_y(x,z)}{\partial z} = -V_d C_y(x,z) \quad \text{at } z = 0 \quad (2b)$$

where, V_d is the deposition velocity

© The source term is as follows:

$$u C_y(0,z) = Q \delta(z - h_s) \quad \text{at } x = 0 \quad (2c)$$

where, δ is the Dirac delta function, Q is the emission rate (g/s) or Bq/s and h_s is the stack height (m).

Also, the vertical eddy diffusivity and the wind speed are as follows:

$$k_z(z) = 0.31 \left(\frac{w_*}{u} \right)^2 z^q \quad (3)$$

$$u(z) = 0.16 \left(\frac{w_*}{u} \right)^2 z^p \quad (4)$$

where, w_* is the vertical convective velocity (m/s) and p, q are constant values depend on the stability classes.

Assuming that the solution of Eq. (1) in the form of separation technique as follows:

$$C_y(x,z) = X(x)Z(z) \quad (5)$$

Substitute from Eq. (5) in Eq. (1) one gets as follows:

$$u(z) \dot{X}(x) Z(z) = \frac{d}{dz} \left(k_z(z) \dot{Z}(z) \right) X(x) \quad (6)$$

Dividing on $u(z)X(x)Z(z)$, one gets as follows:

$$\frac{\dot{X}(x)}{X(x)} = \frac{1}{Z(z)} \frac{1}{u(z)} \frac{d}{dz} \left(k_z(z) \dot{Z}(z) \right) = -\lambda \quad (7)$$

where, λ is a constant value.

From Eq. (7) one can get for x value as follows:

$$\frac{\dot{X}(x)}{X(x)} = -\lambda \rightarrow X(x) = e^{-\lambda x} \quad (8)$$

But for respect to $Z(z)$, one gets:

$$\frac{d}{dz} \left(k_z(z) \dot{Z}(z) \right) + \lambda Z(z) u(z) = 0 \quad (9)$$

Substitute from Eqns. (3) and (4) in Eq. (9) one can get as follows:

$$\frac{d}{dz} \left(0.31 \left(\frac{w_*}{u} \right)^2 z^q \dot{Z}(z) \right) + 0.16 \lambda \left(\frac{w_*}{u} \right)^2 z^p Z(z) = 0 \quad (10)$$

Eq. (10) becomes as follows:

$$0.31 \left(\frac{w_*}{u} \right)^2 \frac{d}{dz} \left(z^q \dot{Z}(z) \right) = -0.16 \lambda \left(\frac{w_*}{u} \right)^2 z^p Z(z) \quad (11)$$

Divide by $0.31 \left(\frac{w_*}{u}\right)^2$ one gets as follows:

$$\frac{d}{dz} \left(z^q \dot{Z}(z) \right) + \frac{0.16}{0.31} \lambda z^p Z(z) = 0 \quad (12)$$

Let $\alpha = 0.16/0.31$

Eq. (12) becomes as follows:

$$\frac{d}{dz} \left(z^q \dot{Z}(z) \right) + \alpha \lambda z^p Z(z) = 0 \quad (13)$$

The general solution is as follows:

$$C_y(x, z) = \sum_n A_n Z_n(z) e^{-\lambda_n x} \quad (14)$$

Now we calculate A_n as follows:

From the condition (2c) one gets as follows:

$$u C_y(0, z) = Q \delta(z - h_s) \quad \text{at } x = 0 \quad (2c)$$

Substitute from Eq. (14) in Eq. (2c) one gets:

$$u(z) \sum_{n=1}^{\infty} A_n Z_n(z) = Q \delta(z - h_s) \quad (15)$$

Thus the orthogonality condition is:

$$\int_0^h u(z) Z_m(z) Z_n(z) dz = 0 \quad \text{for } m \neq n \quad (16)$$

Multiply both side of Eq. (15) by $Z_m(z)$ and integrate with respect to z from 0 to h , one gets as follows:

$$\int_0^h u(z) \sum_{n=1}^{\infty} A_n Z_n(z) Z_m(z) dz = Q \int_0^h \delta(z - h_s) Z_m(z) dz$$

Using the orthogonality condition one gets as follows:

$$\begin{aligned} A_m \int_0^h u(z) Z_m^2(z) dz &= Q Z_m(h_s) \\ A_m &= \frac{Q Z_m(h_s)}{\int_0^h u(z) Z_m^2(z) dz} \end{aligned} \quad (17)$$

Also, we calculate $Z_n(z)$ as follows:

Eq. (13) is as follows:

$$\frac{d}{dz} \left(z^q \dot{Z}(z) \right) + \alpha \lambda z^p Z(z) = 0$$

where, $\alpha = \frac{0.16}{0.31}$, since $p+q=1$, then Eq. (13) becomes:

$$\frac{d}{dz} \left(z^q \dot{Z}(z) \right) + \alpha \lambda z^{1-q} Z(z) = 0 \quad (18)$$

Expand Eq. (18) as follows:

$$z^q \ddot{Z}(z) + q z^{q-1} \dot{Z}(z) + \alpha \lambda z^{1-q} Z(z) = 0$$

Divide on z^q , one can get as follows:

$$\ddot{Z}(z) + \frac{q}{z} \dot{Z}(z) + \alpha \lambda z^{1-2q} Z(z) = 0$$

If $q=0.85$

$$\ddot{Z}(z) + \frac{0.85}{z} \dot{Z}(z) + \alpha \lambda z^{-0.7} Z(z) = 0 \quad (19)$$

This is a Sturm-Liouville type equation with variable coefficient $z^{-0.7}$. It resemble a generalized Bessel equation but with a fractional power.

Let set:

$$t = \beta z^p, \quad \beta = \sqrt{\frac{\alpha \lambda}{(p)^2}} \quad (20)$$

This substitution is chosen because the exponent $1-2q=1-1.7=-0.7$ matches with z^{p-1} , where, $p=0.15$. With this substitution, the ordinary diffusion equation (ODE) reduces to a Bessel equation of order $\nu = \frac{q-1}{2p}$.

So the solution is:-

$$Z(z) = c_1 J_{\frac{q-1}{2p}}(\beta z^p) + c_2 Y_{\frac{q-1}{2p}}(\beta z^p) \quad (21)$$

The Half order Bessel functions are as follows:

$$J_{\frac{q-1}{2p}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x), \quad Y_{\frac{q-1}{2p}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \quad (22)$$

Then the form of $Z(z)$ is as follows:

$$Z(z) = \sqrt{\frac{2}{\pi \beta z^p}} (c_1 \cos(\beta z^p) + c_2 \sin(\beta z^p)) \quad (23)$$

Then the final expression for $Z_n(z)$, where $q=0.85$, one can get as follows:

$$Z_n(z) = \sum_{n=1} \sqrt{\frac{2}{\pi \beta_n z^p}} (c_1 \cos(\beta_n z^p) + c_2 \sin(\beta_n z^p)) \quad (24)$$

where,

$$\beta_n = \sqrt{\frac{\alpha \lambda_n}{p^2}}, \quad \alpha = \frac{0.16}{0.31} \quad (24a)$$

At $z = h \rightarrow \dot{Z}(h) = 0$, differentiate $Z(z)$ at $z=h$, the derivative condition leads to as follows:

$$\tan(\beta h^p) = -\frac{c_1}{c_2} \quad (25)$$

For nontrivial solutions, the ratio c_1/c_2 must be finite so, the eigenvalue condition reduces to:

$$\beta h^p = n\pi, \quad n = 1, 2, \dots$$

Since, $\beta_n = \sqrt{\frac{\alpha \lambda_n}{p^2}}$, then $\sqrt{\frac{\alpha \lambda_n}{p^2}} h^p = n\pi \rightarrow \sqrt{\alpha \lambda_n} = \frac{n\pi p}{h^p}$

$$\lambda_n = \sum_{n=1} \frac{(n\pi p)^2}{\alpha h^{2p}}, \quad n = 1, 2, 3, \dots, \alpha = \frac{0.16}{0.31} \quad (26)$$

From Eq. (25), one gets as follows:

$$c_1 \sin(\beta h^p) = c_2 \cos(\beta h^p) = 0$$

So

$$\frac{c_1}{c_2} = \cot(\beta h^p) \quad (27)$$

From condition (2b), one can get:

$$k_z(0) \dot{Z}(0) = -V_d Z(0)$$

Since, $k_z(z) = 0.31(w_*/u)^2 z^q$ at $z = 0 \rightarrow k_z(z) = \text{zero}$

The Robin condition effectively enforces:

$$Z(0) \neq 0 \rightarrow c_1 \cos(0) + c_2 \sin(0) \rightarrow c_1 = 0$$

So, the Eigen function simplified to:

$$Z_n(z) = c_2 \sqrt{\frac{2}{\pi \beta_n z^p}} \sin(\beta_n z^p) \quad (28)$$

Now we calculate c_2 as follows:

Since $\int_0^h u(z) Z_n^2(z) dz = 1$, $u(z) = 0.16(w_*/u)^2 z^p$

$$\int_0^h 0.16(w_*/u)^2 z^p \left[c_2^2 \frac{2}{\pi \beta_n z^p} \sin^2(\beta_n z^p) \right] dz = 1$$

$$c_2^2 \frac{0.32}{\pi \beta_n} \left(\frac{w_*}{u} \right)^2 \int_0^h \sin^2(\beta_n z^p) dz = 1 \quad (29)$$

Let $t = \beta_n z^p$, then, $z^p = t/\beta_n$, $z = \left(\frac{t}{\beta_n} \right)^{1/p} \rightarrow dz = (1/p) \left(\frac{t}{\beta_n} \right)^{\frac{1}{p}-1} \frac{dt}{\beta_n} \rightarrow dz = \frac{1}{p} \beta_n^{-\frac{1}{p}} t^{\frac{1}{p}-1} dt$

Then,

$$\int_0^h \sin^2(\beta_n z^p) dz = \frac{1}{p} \beta_n^{-\frac{1}{p}} \int_0^{\beta_n h^{0.15}} t^{\frac{1}{p}-1} \sin^2(t) dt$$

148 Then,

$$149 \quad c_2 = \sqrt{\frac{\pi p \beta_n^{\frac{1}{p+1}}}{0.32 \left(\frac{w_*}{u}\right)^2 \int_0^{\beta_n h^p} \frac{1}{t^{p-1}} \sin^2(t) dt}} \quad (30)$$

150 Substituting from Eqns. (30), (28), (26), (24a),(17) and (8) in Eq. (14) one can get on the final solution in
151 the form:

$$152 \quad C_y(x, z)/Q = \sum_{n=1}^{10} A_n Z_n(z) e^{-\lambda_n x} \quad (31)$$

$$153 \quad \text{where, } A_n = \frac{Q Z_n(h_s)}{\int_0^h u(z) Z_n^2(z) dz}, \lambda_n = \frac{(n\pi p)^2}{\alpha h^{2p}}, \quad n = 1, 2, 3, \dots, \alpha = \frac{0.16}{0.31}, Z_n(z) = c_2 \sqrt{\frac{2}{\pi \beta_n z^p}} \sin(\beta_n z^p),$$

$$154 \quad \beta_n = \sqrt{\frac{\alpha \lambda_n}{p^2}}, \text{ and } c_2 = \sqrt{\frac{\pi p \beta_n^{\frac{1}{p+1}}}{0.32 \left(\frac{w_*}{u}\right)^2 \int_0^{\beta_n h^p} \frac{1}{t^{p-1}} \sin^2(t) dt}}$$

155

156 Results and Discussion:

157

158 The quantities of sulfur Hexafluoride (SF₆) from Copenhagen, Denmark (Gryning and Lyck,
159 1984) are compared to the levels predicted by Eq. (31). The stack height " h_s " is 115 meters. The values
160 of p and q are shown in Table (1) and the results are shown in Table (2) as follows:

161

162 Table (1): includes eddy diffusivity and wind speed exponents p and q for urban air stability.

	A	B	C	D	E	F
p	0.15	0.15	0.20	0.25	0.40	0.60
q	0.85	0.85	0.80	0.75	0.60	0.40

163

164 Table (2): The observed concentrations are compared with crosswind integrated normalized predicted
165 concentrations for multiple runs at different downwind distances, wind speeds, and distances.

Run No.	Date	PG Stabili ty	h(m) mixing height	w* (ms ⁻¹)	u ₁₀ m/s	u ₁₁₅ m/s	distance (m)	observed 10 ⁻⁴ sm ⁻²	Pred. Conc. 10 ⁻⁴ sm ⁻²
1	20-9-78	A	1980	0.83	2.1	3.4	1900	6.48	6.2
1	20-9-78	A	1980	0.83	2.1	3.4	3700	2.31	4.2
2	26-9-78	C	1920	1.07	4.9	10.6	2100	5.38	6.7
2	26-9-78	C	1920	1.07	4.9	10.6	4200	2.95	2.8
3	19-10-78	B	1120	0.68	2.4	5.0	1900	8.20	7.2
3	19-10-78	B	1120	0.68	2.4	5.0	3700	6.22	5.4
3	19-10-78	B	1120	0.68	2.4	5.0	5400	4.30	3.7
5	9-11-78	C	820	0.71	3.1	6.7	2100	6.72	5.4
5	9-11-78	C	820	0.71	3.1	6.7	4200	5.84	3.7
5	9-11-78	C	820	0.71	3.1	6.7	6100	4.97	2.7
6	30-4-78	C	1300	1.33	7.2	13.2	2000	3.96	5.0

6	30-4-78	C	1300	1.33	7.2	13.2	4200	2.22	3.5
6	30-4-78	C	1300	1.33	7.2	13.2	5900	1.83	2.6
7	27-6-78	B	1850	0.87	4.1	7.6	2000	6.70	6.5
7	27-6-78	B	1850	0.87	4.1	7.6	4100	3.25	2.7
7	27-6-78	B	1850	0.87	4.1	7.6	5300	2.23	2.0
8	6-7-78	D	810	0.72	4.2	9.4	1900	4.16	5.3
8	6-7-78	D	810	0.72	4.2	9.4	3600	2.02	1.5
8	6-7-78	D	810	0.72	4.2	9.4	5300	1.52	2.1
9	19-7-78	C	2090	0.98	5.1	10.5	2100	4.58	5.9
9	19-7-78	C	2090	0.98	5.1	10.5	4200	3.11	3.7
9	19-7-78	C	2090	0.98	5.1	10.5	6000	2.59	2.5

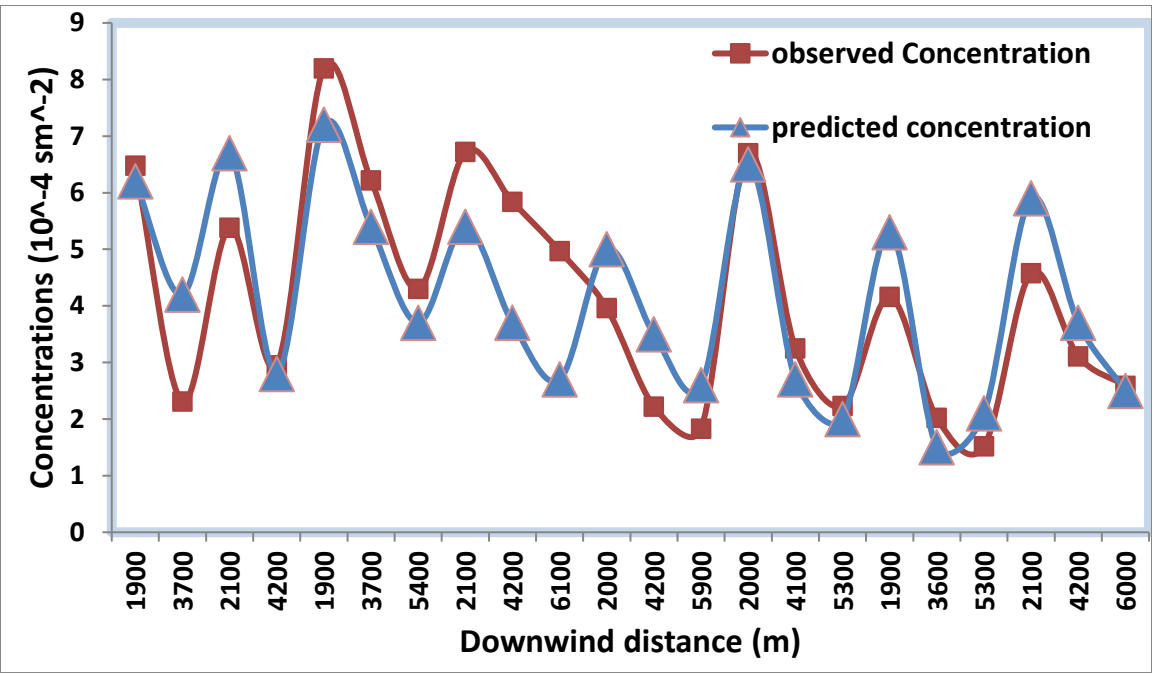


Fig. (1) shows that the downwind distance (m) at Copenhagen, Denmark, affects the crosswind normalized forecast, and observation concentrations (10^{-4}sm^{-2}).

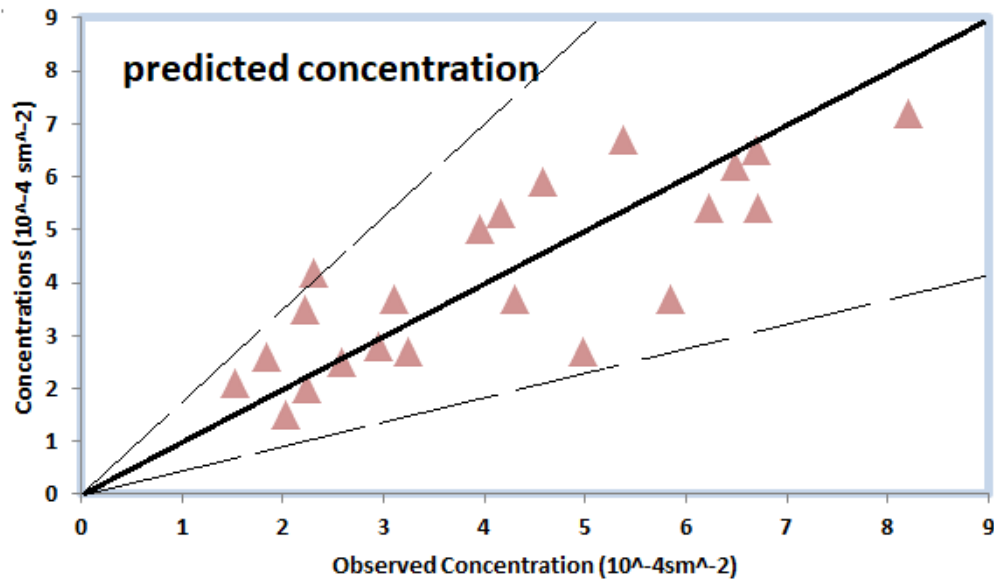


Fig. (2) Shows that the observed concentrations affect the crosswind normalized concentration models.

In comparison between the observed concentrations of SF_6 and crosswind normalized integrated predicted using separation method taking power law of eddy diffusivity and wind velocity in two dimensions. We get on high degree of closed agreement between observed and predicted concentrations, as shown in Fig. (1). Also we find that the predicted concentrations are inside a factor of two, as seen in Fig. (2).

Statistics Techniques

The observed and predicted outcomes are compared using the statistical method (Hanna 1989). The typical statistical performance between prior observations ($C_O = C_{Obs}$) and predicted ($C_p = C_{pred}$) are explained. A flawless model needs to execute as follows: $COR = FAC2 = 1.0$ and $NMSE = FB = 0$.

Table (3): crosswind integrated normalized concentrations for SF_6 are compared to observed concentrations using separation method in two dimensions.

Statistical functions	SF_6			
	NMSE	FB	COR	Fac2
Predicted Concentration	0.07	0.003	0.82	0.997

Compared to the predicted concentrations using the power law of the vertical eddy diffusivity and wind speed, using separation method, we obtain on a good better agreement with their observed concentrations at centerline using the statistical method in Table (3). The projected achieved 100% with the observed concentrations. Also the Normalized Mean Square Error and the Fraction Bias are near to zero and the Correlation is near to one.

Conclusions

In this study, the crosswind normalized integrated projected concentrations at centerline with the power law of vertical eddy diffusivity and wind speed using separation method. The two models are contrasted with the SF₆ concentrations found in Copenhagen, Denmark.

We get on a high degree of closed agreement between observed and predicted concentrations. Also we find that the predicted concentrations are inside a factor of two, with observed concentrations.

We obtain on a good better agreement with their observed concentrations at centerline using the statistical method. The projected achieved 100% with the observed concentrations. Also the Normalized Mean Square Error and the Fraction Bias are near to zero and the Correlation is near from one.

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Khaled S. M. Essa conceived the research idea and participated in writing the research and creating the illustrations.

H. M. Taha wrote and participated in creating the illustrations.

Ahmed M. Mosallem wrote the research and participated in creating the illustrations.

Ethics and Consent to Participate declarations

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