Supplementary material for the manuscript '*Free in-plane vibration analysis of a rectangular plate with an edge V-notch*'

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1 Derivation of V-notch corner functions for the in-plane extension problem

Since a V-notch can be treated locally as an angular sector with traction-free radial boundaries, the local singular functions near the notch vertex can be derived from Williams's asymptotic solution for plates in extension [1].

1.1 Williams's asymptotic representation

For the in-plane elasticity problem, Williams expressed the radial and circumferential displacements, U_r and U_θ, in terms of the Airy stress function Υ and an auxiliary harmonic function ψ_1, namely

$$2\mu U_r = -\frac{\partial \Upsilon}{\partial r} + \frac{r}{1+\nu}\frac{\partial \psi_1}{\partial \theta} \tag{1.1}$$

$$2\mu U_\theta = -\frac{1}{r}\frac{\partial \Upsilon}{\partial \theta} + \frac{r^2}{1+\nu}\frac{\partial \psi_1}{\partial r} \tag{1.2}$$

where μ is the shear modulus. The corresponding stresses are

$$\sigma_r = \frac{1}{r}\frac{\partial \Upsilon}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Upsilon}{\partial \theta^2} \tag{1.3}$$

$$\sigma_\theta = \frac{\partial^2 \Upsilon}{\partial r^2} \tag{1.4}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Upsilon}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Upsilon}{\partial r\,\partial \theta} \tag{1.5}$$

Williams assumed separated asymptotic forms of the type

$$\Upsilon = r^{\lambda+1} F(\theta) \tag{1.6}$$

$$\psi_1 = r^m G(\theta) \tag{1.7}$$

with

$$F(\theta) = b_1 sin[(\lambda+1)\theta] + b_2 cos[(\lambda+1)\theta] + b_3 sin[(\lambda-1)\theta] + b_4 cos[(\lambda-1)\theta] \tag{1.8}$$

Compatibility requires

$$\lambda = m + 1 \tag{1.9}$$

and Williams further obtained the relations between the coefficients in G and those in F as

$$a_1 = -\frac{4\lambda}{\lambda-1} b_3, \qquad a_2 = \frac{4\lambda}{\lambda-1} b_4 \tag{1.10}$$

where

$$G(\theta) = a_1 cos[(\lambda-1)\theta] + a_2 sin[(\lambda-1)\theta] \tag{1.11}$$

The following derivations assume $\lambda \neq 1$ since Eq. (1.10) contains the factor $(\lambda-1)^{-1}$. The case $\lambda = 1$ corresponds to a bounded-stress term and is not retained as a singular corner-enrichment function in the present formulation.

1.2 Stress field

Substituting Eq. (1.6) into Eqs. (1.3)-(1.5) yields

$$\sigma_r = r^{\lambda-1}[F''(\theta) + (\lambda+1)F(\theta)] \tag{1.12}$$

$$\sigma_\theta = r^{\lambda-1}\lambda(\lambda+1)F(\theta) \tag{1.13}$$

$$\tau_{r\theta} = -r^{\lambda-1}\lambda F'(\theta) \tag{1.14}$$

Hence, the stress singularity order is governed by $r^{\lambda-1}$. Therefore, roots satisfying $0 < \lambda < 1$ correspond to singular stresses at the notch vertex.

1.3 Boundary conditions

The two V-notch faces are traction-free. Therefore,

$$\sigma_\theta = 0, \qquad \tau_{r\theta} = 0 \tag{1.15}$$

on $\theta = \pm\kappa$. From Eqs. (1.13) and (1.14), this implies

$$F(\pm\kappa) = 0, \qquad F'(\pm\kappa) = 0 \tag{1.16}$$

Because the V-notch is symmetric with respect to its bisector, the angular solution can be decomposed into antisymmetric and symmetric displacement families. The antisymmetric family corresponds to an odd Airy angular function, whereas the symmetric family corresponds to an even Airy angular function.

1.4 Antisymmetric family

For the antisymmetric family,

$$F_A(\theta) = b_1\sin[(\lambda+1)\theta] + b_3\sin[(\lambda-1)\theta] \tag{1.17}$$

where $b_2 = b_4 = 0$. Since $F_A(\theta)$ is odd, the condition $F_A(-\kappa) = 0$ is automatically satisfied if $F_A(\kappa) = 0$. Similarly, $F_A'(\theta)$ is even, so $F_A'(-\kappa) = 0$ is automatically satisfied if $F_A'(\kappa) = 0$. Thus, it is sufficient to impose

$$F_A(\kappa) = 0, \qquad F_A'(\kappa) = 0 \tag{1.18}$$

Substituting Eq. (1.17) into Eq. (1.18) gives

$$b_1\sin[(\lambda+1)\kappa] + b_3\sin[(\lambda-1)\kappa] = 0 \tag{1.19}$$

$$b_1(\lambda+1)\cos[(\lambda+1)\kappa] + b_3(\lambda-1)\cos[(\lambda-1)\kappa] = 0 \tag{1.20}$$

For a non-trivial solution $(b_1, b_3) \neq (0,0)$, the determinant of Eqs. (1.19)-(1.20) must vanish, therefore

$$(\lambda-1)\sin[(\lambda+1)\kappa]\cos[(\lambda-1)\kappa]-(\lambda+1)\cos[(\lambda+1)\kappa]\sin[(\lambda-1)\kappa]=0 \tag{1.21}$$

Let

$$\mathbb{A}=(\lambda+1)\kappa, \qquad \mathbb{B}=(\lambda-1)\kappa \tag{1.22}$$

Then

$$\mathbb{A}-\mathbb{B}=2\kappa=\varpi, \qquad \mathbb{A}+\mathbb{B}=2\lambda\kappa=\lambda\varpi$$

Using trigonometric sum and difference formulas, Eq. (1.21) becomes

$$\lambda\sin(\mathbb{A}-\mathbb{B})-\sin(\mathbb{A}+\mathbb{B})=0 \tag{1.23}$$

Hence,

$$\sin(\lambda\varpi)=\lambda\sin\varpi \tag{1.24}$$

Equation (1.24) is the characteristic equation for the antisymmetric displacement family. Since Eq. (1.19)-(1.20) are homogeneous, they determine only the ratio between b_1 and b_3. A convenient choice of the coefficient ratio can be obtained from Eq. (1.19) (when this choice degenerates for a special parameter value, an equivalent non-zero ratio may instead be obtained from Eq. (1.20); for $\lambda \neq 1$, the two choices cannot vanish simultaneously) as follows

$$b_1=\sin[(\lambda-1)\kappa], \qquad b_3=-\sin[(\lambda+1)\kappa] \tag{1.25}$$

Thus,

$$F_A(\theta)=\sin[(\lambda-1)\kappa]\sin[(\lambda+1)\theta]-\sin[(\lambda+1)\kappa]\sin[(\lambda-1)\theta] \tag{1.26}$$

1.5 Symmetric family

For the symmetric family,

$$F_S(\theta)=b_2\cos[(\lambda+1)\theta]+b_4\cos[(\lambda-1)\theta] \tag{1.27}$$

where $b_1=b_3=0$. Since $F_S(\theta)$ is even, the condition $F_S(-\kappa)=0$ is automatically satisfied if $F_S(\kappa)=0$. Similarly, $F_S'(\theta)$ is odd, so $F_S'(-\kappa)=0$ is automatically satisfied if $F_S'(\kappa)=0$. Thus, it is sufficient to impose

$$F_S(\kappa) = 0, \qquad F_S'(\kappa) = 0 \tag{1.28}$$

Substituting Eq. (1.27) into Eq. (1.28) gives

$$b_2\cos[(\lambda+1)\kappa] + b_4\cos[(\lambda-1)\kappa] = 0 \tag{1.29}$$

$$-b_2(\lambda+1)\sin[(\lambda+1)\kappa] - b_4(\lambda-1)\sin[(\lambda-1)\kappa] = 0 \tag{1.30}$$

For a non-trivial solution $(b_2, b_4) \neq (0,0)$, the determinant of Eqs. (1.29)-(1.30) must vanish, therefore,

$$(\lambda+1)\sin[(\lambda+1)\kappa]\cos[(\lambda-1)\kappa] - (\lambda-1)\cos[(\lambda+1)\kappa]\sin[(\lambda-1)\kappa] = 0 \tag{1.31}$$

Using the definitions in Eq. (1.22), Eq. (1.31) becomes

$$\lambda\sin(\mathbb{A}-\mathbb{B}) + \sin(\mathbb{A}+\mathbb{B}) = 0 \tag{1.32}$$

Thus,

$$\sin(\lambda\varpi) = -\lambda\sin\varpi \tag{1.33}$$

Equation (1.33) is the characteristic equation for the symmetric displacement family. For this family, a convenient choice of the coefficient ratio, for example based on Eq. (1.29), is

$$b_2 = \cos[(\lambda-1)\kappa], \qquad b_4 = -\cos[(\lambda+1)\kappa] \tag{1.34}$$

Thus,

$$F_S(\theta) = \cos[(\lambda-1)\kappa]\cos[(\lambda+1)\theta] - \cos[(\lambda+1)\kappa]\cos[(\lambda-1)\theta] \tag{1.35}$$

Similar to the antisymmetric case, the coefficient ratio can be chosen from either Eq. (1.29) or Eq. (1.30) as long as the choice gives a non-zero pair.

1.6 Displacement fields

The displacement components are now obtained by substituting Eqs. (1.6)-(1.11) into Eqs. (1.1)-(1.2). For the antisymmetric family, using the coefficient choice in Eq. (1.25), the radial and circumferential displacement functions are

$$U_r^{(A)} = \frac{r^\lambda}{2\mu}\left\{-(\lambda+1)\sin[(\lambda-1)\kappa]\sin[(\lambda+1)\theta] + \left(\lambda+1-\frac{4\lambda}{1+\nu}\right)\sin[(\lambda+1)\kappa]\sin[(\lambda-1)\theta]\right\} \tag{1.36}$$

$$U_\theta^{(A)} = \frac{r^\lambda}{2\mu}\left\{-(\lambda+1)\sin[(\lambda-1)\kappa]\cos[(\lambda+1)\theta] + (\lambda-1+\frac{4\lambda}{1+\nu})\sin[(\lambda+1)\kappa]\cos[(\lambda-1)\theta]\right\} \tag{1.37}$$

The roots $\lambda = \lambda_n^{(A)}$ used in Eqs. (1.36)-(1.37) are obtained from Eq. (1.24). For the symmetric family, using the coefficient choice in Eq. (1.34), the radial and circumferential displacement functions are

$$U_r^{(S)} = \frac{r^\lambda}{2\mu}\left\{-(\lambda+1)\cos[(\lambda-1)\kappa]\cos[(\lambda+1)\theta] + (\lambda+1-\frac{4\lambda}{1+\nu})\cos[(\lambda+1)\kappa]\cos[(\lambda-1)\theta]\right\} \tag{1.38}$$

$$U_\theta^{(S)} = \frac{r^\lambda}{2\mu}\left\{(\lambda+1)\cos[(\lambda-1)\kappa]\sin[(\lambda+1)\theta] - (\lambda-1+\frac{4\lambda}{1+\nu})\cos[(\lambda+1)\kappa]\sin[(\lambda-1)\theta]\right\} \tag{1.39}$$

The roots $\lambda = \lambda_n^{(S)}$ used in Eqs. (1.38)-(1.39) are obtained from Eq. (1.33). The displacement functions derived above correspond to individual Williams-type eigenfunctions associated with the characteristic roots $\lambda_n^{(A)}$ and $\lambda_n^{(S)}$. The complete local asymptotic representation consists of a linear combination of these functions over all admissible roots in both families. In a Ritz formulation, however, only a finite number of roots are selected, and the corresponding corner functions are included as enrichment functions in the displacement approximation. The most important roots are those satisfying $0 < \lambda < 1$ because they generate singular stresses proportional to $r^{\lambda-1}$.

2 Matrix entries

Since $D_U(\xi,\eta) = H_\xi^U(\xi)\, H_\eta^U(\eta)$ and $D_V(\xi,\eta) = H_\xi^V(\xi)\, H_\eta^V(\eta)$ (i.e. Eqs. (7)-(8) and (13) in the manuscript), define combined 1D functions

$$\begin{aligned} \Psi_m^U(\xi) &= H_\xi^U(\xi)\,\chi_m(\xi), \qquad & \Psi_n^U(\eta) &= H_\eta^U(\eta)\,\chi_n(\eta) \\ \Psi_m^V(\xi) &= H_\xi^V(\xi)\,\chi_m(\xi), \qquad & \Psi_n^V(\eta) &= H_\eta^V(\eta)\,\chi_n(\eta) \end{aligned} \tag{2.1}$$

so that the effective regular basis functions take tensor-product form: $\Phi_{mn}^U = \Psi_m^U(\xi)\Psi_n^U(\eta)$ and $\Phi_{mn}^V = \Psi_m^V(\xi)\Psi_n^V(\eta)$. Define also the effective corner basis functions

$$\begin{aligned} \Pi_n^{AU}(\xi,\eta) &= D_U(\xi,\eta)U_{x,n}^{(A)}(r,\theta), \qquad & \Pi_n^{AV}(\xi,\eta) &= D_V(\xi,\eta)U_{y,n}^{(A)}(r,\theta) \\ \Pi_n^{SU}(\xi,\eta) &= D_U(\xi,\eta)U_{x,n}^{(S)}(r,\theta), \qquad & \Pi_n^{SV}(\xi,\eta) &= D_V(\xi,\eta)U_{y,n}^{(S)}(r,\theta) \end{aligned} \tag{2.2}$$

2.1 Regular-regular stiffness blocks $\boldsymbol{K}_{pp}$

All domain integrals are taken over the physical notched domain Ω. The regular-regular stiffness entries are

$$K_{mn,rs}^{pp,uu} = A_{11}\frac{b}{a}\iint_{\Omega}\frac{\partial\Phi_{mn}^{U}}{\partial\xi}\frac{\partial\Phi_{rs}^{U}}{\partial\xi}d\xi d\eta + A_{66}\frac{a}{b}\iint_{\Omega}\frac{\partial\Phi_{mn}^{U}}{\partial\eta}\frac{\partial\Phi_{rs}^{U}}{\partial\eta}d\xi d\eta \tag{2.3}$$

$$K_{mn,rs}^{pp,uv} = A_{12}\iint_{\Omega}\frac{\partial\Phi_{mn}^{U}}{\partial\xi}\frac{\partial\Phi_{rs}^{V}}{\partial\eta}d\xi d\eta + A_{66}\iint_{\Omega}\frac{\partial\Phi_{mn}^{U}}{\partial\eta}\frac{\partial\Phi_{rs}^{V}}{\partial\xi}d\xi d\eta \tag{2.4}$$

$$K_{mn,rs}^{pp,vv} = A_{22}\frac{a}{b}\iint_{\Omega}\frac{\partial\Phi_{mn}^{V}}{\partial\eta}\frac{\partial\Phi_{rs}^{V}}{\partial\eta}d\xi d\eta + A_{66}\frac{b}{a}\iint_{\Omega}\frac{\partial\Phi_{mn}^{V}}{\partial\xi}\frac{\partial\Phi_{rs}^{V}}{\partial\xi}d\xi d\eta \tag{2.5}$$

and $K_{mn,rs}^{pp,vu} = K_{rs,mn}^{pp,uv}$ by symmetry. The tensor-product form of Φ^{U} and Φ^{V} permits the full-rectangle contributions to be evaluated as products of 1D integrals. Define

$$\begin{cases} I_{mr}^{\varrho\varsigma,W} = \int_{-1}^{1}(\Psi_{m}^{W})^{(\varrho)}(\Psi_{r}^{W})^{(\varsigma)}\,d\xi \\ J_{ns}^{\varrho\varsigma,W} = \int_{-1}^{1}(\Psi_{n}^{W})^{(\varrho)}(\Psi_{s}^{W})^{(\varsigma)}\,d\eta \end{cases} \qquad (W = U \text{ or } V) \tag{2.6}$$

where superscript (ϱ) denotes the ϱ-th derivative. The cross integrals are $I_{mr}^{\varrho\varsigma,UV} = \int_{-1}^{1}(\Psi_{m}^{U})^{(\varrho)}(\Psi_{r}^{V})^{(\varsigma)}\,d\xi$ and $J_{ns}^{\varrho\varsigma,UV} = \int_{-1}^{1}(\Psi_{n}^{U})^{(\varrho)}(\Psi_{s}^{V})^{(\varsigma)}\,d\eta$. The full-rectangle contributions to Eqs. (2.3)-(2.5) then take the closed forms

$$A_{11}\frac{b}{a}I_{mr}^{11,U}J_{ns}^{00,U} + A_{66}\frac{a}{b}I_{mr}^{00,U}J_{ns}^{11,U} \tag{2.7}$$

$$A_{12}I_{mr}^{10,UV}J_{ns}^{01,UV} + A_{66}I_{mr}^{01,UV}J_{ns}^{10,UV} \tag{2.8}$$

$$A_{22}\frac{a}{b}I_{mr}^{00,V}J_{ns}^{11,V} + A_{66}\frac{b}{a}I_{mr}^{11,V}J_{ns}^{00,V} \tag{2.9}$$

The complete $\boldsymbol{K}_{pp}$ entries are obtained by subtracting the contribution of the triangular notch region from each full-rectangle result. These notch corrections involve the same integrands evaluated over the triangular notch domain and are computed numerically.

2.2 Corner-corner stiffness blocks $\boldsymbol{K}_{cc}$

Because $C_n^{(A)}$ contributes to both U (via Π_n^{AU}) and V (via Π_n^{AV}), the antisymmetric-antisymmetric sub-block combines contributions from both displacement components:

$$\begin{aligned} K_{n,k}^{cc,AA} = \iint_\Omega \Bigg[& A_{11}\frac{b}{a}\frac{\partial \Pi_n^{AU}}{\partial \xi}\frac{\partial \Pi_k^{AU}}{\partial \xi} + A_{22}\frac{a}{b}\frac{\partial \Pi_n^{AV}}{\partial \eta}\frac{\partial \Pi_k^{AV}}{\partial \eta} + A_{12}\left(\frac{\partial \Pi_n^{AU}}{\partial \xi}\frac{\partial \Pi_k^{AV}}{\partial \eta} + \frac{\partial \Pi_k^{AU}}{\partial \xi}\frac{\partial \Pi_n^{AV}}{\partial \eta}\right) \\ & + A_{66}\frac{a}{b}\frac{\partial \Pi_n^{AU}}{\partial \eta}\frac{\partial \Pi_k^{AU}}{\partial \eta} + A_{66}\frac{b}{a}\frac{\partial \Pi_n^{AV}}{\partial \xi}\frac{\partial \Pi_k^{AV}}{\partial \xi} + A_{66}\left(\frac{\partial \Pi_n^{AU}}{\partial \eta}\frac{\partial \Pi_k^{AV}}{\partial \xi} + \frac{\partial \Pi_k^{AU}}{\partial \eta}\frac{\partial \Pi_n^{AV}}{\partial \xi}\right)\Bigg] d\xi d\eta \end{aligned} \tag{2.10}$$

The sub-block $K_{n,k}^{cc,SS}$ follows by replacing A with S throughout, and $K_{n,k}^{cc,AS} = K_{k,n}^{cc,SA}$ are obtained by mixing the two families accordingly. The matrix is symmetric, i.e., $K_{n,k}^{cc,AA} = K_{k,n}^{cc,AA}$, $K_{n,k}^{cc,SS} = K_{k,n}^{cc,SS}$, and $K_{n,k}^{cc,SA} = K_{k,n}^{cc,AS}$.

2.3 Coupling stiffness blocks $\boldsymbol{K}_{pc}$ and $\boldsymbol{K}_{cp}$

The coupling entry between the mn-th U-regular function and the k-th antisymmetric corner coefficient is

$$K_{mn,k}^{pc,u^A} = \iint_\Omega \left[A_{11}\frac{b}{a}\frac{\partial \Phi_{mn}^U}{\partial \xi}\frac{\partial \Pi_k^{AU}}{\partial \xi} + A_{12}\frac{\partial \Phi_{mn}^U}{\partial \xi}\frac{\partial \Pi_k^{AV}}{\partial \eta} + A_{66}\frac{a}{b}\frac{\partial \Phi_{mn}^U}{\partial \eta}\frac{\partial \Pi_k^{AU}}{\partial \eta} + A_{66}\frac{\partial \Phi_{mn}^U}{\partial \eta}\frac{\partial \Pi_k^{AV}}{\partial \xi}\right] d\xi d\eta \tag{2.11}$$

Note that both Π_k^{AU} and Π_k^{AV} appear in Eq. (2.11) because $C_k^{(A)}$ contributes to both U and V. The coupling entry from the V-equation is

$$K_{mn,k}^{pc,v^A} = \iint_\Omega \left[A_{22}\frac{a}{b}\frac{\partial \Phi_{mn}^V}{\partial \eta}\frac{\partial \Pi_k^{AV}}{\partial \eta} + A_{12}\frac{\partial \Phi_{mn}^V}{\partial \eta}\frac{\partial \Pi_k^{AU}}{\partial \xi} + A_{66}\frac{b}{a}\frac{\partial \Phi_{mn}^V}{\partial \xi}\frac{\partial \Pi_k^{AV}}{\partial \xi} + A_{66}\frac{\partial \Phi_{mn}^V}{\partial \xi}\frac{\partial \Pi_k^{AU}}{\partial \eta}\right] d\xi d\eta \tag{2.12}$$

The entries $K_{mn,k}^{pc,u^S}$ and $K_{mn,k}^{pc,v^S}$ follow by substituting S for A. The blocks $\boldsymbol{K}_{cp}$ are the transposes of the corresponding $\boldsymbol{K}_{pc}$ blocks.

To evaluate Eqs. (2.10)-(2.12), the partial derivatives of Π_n^{AU} and Π_n^{AV} are required. Applying the product rule to Eq. (2.2):

$$\frac{\partial \Pi_n^{AU}}{\partial \xi} = (H_\xi^U)' H_\eta^U \cdot U_{x,n}^{(A)} + H_\xi^U H_\eta^U \cdot \frac{\partial U_{x,n}^{(A)}}{\partial \xi} \tag{2.13}$$

where $(H_\xi^U)' = dH_\xi^U/d\xi$ is evaluated analytically from Eq. (8) in the manuscript, and $\partial U_{x,n}^{(A)}/\partial\xi$ is obtained via the chain rule through the local polar coordinates (r,θ):

$$\frac{\partial U_{x,n}^{(A)}}{\partial \xi} = \frac{\partial U_{x,n}^{(A)}}{\partial r}\frac{\partial r}{\partial \xi} + \frac{\partial U_{x,n}^{(A)}}{\partial \theta}\frac{\partial \theta}{\partial \xi} \tag{2.14}$$

Differentiating Eqs. (27)-(28) in the manuscript gives

$$\frac{\partial r}{\partial \xi} = \frac{a^2 \Delta\xi}{4r}, \qquad \frac{\partial r}{\partial \eta} = \frac{b^2 \Delta\eta}{4r} \tag{2.15}$$

$$\frac{\partial \theta}{\partial \xi} = \frac{ab\,\Delta\eta}{4r^2}, \qquad \frac{\partial \theta}{\partial \eta} = -\frac{ab\,\Delta\xi}{4r^2} \tag{2.16}$$

where $\Delta\xi = \xi - \xi_0$ and $\Delta\eta = \eta - \eta_0$. Writing $U_{x,n}^{(A)} = r^{\lambda_n^{(A)}}/(2\mu)\cdot\mathcal{F}_x^{(A)}(\theta)$ and $U_{y,n}^{(A)} = r^{\lambda_n^{(A)}}/(2\mu)\cdot\mathcal{F}_y^{(A)}(\theta)$, where the angular functions are determined from Eqs. (15)-(16) and (21) in the manuscript as

$$\begin{aligned} \mathcal{F}_x^{(A)}(\theta) &= -\mathcal{F}_r^{(A)}(\theta)\cos\theta + \mathcal{F}_\theta^{(A)}(\theta)\sin\theta \\ \mathcal{F}_y^{(A)}(\theta) &= \mathcal{F}_r^{(A)}(\theta)\sin\theta + \mathcal{F}_\theta^{(A)}(\theta)\cos\theta \end{aligned} \tag{2.17}$$

with $\mathcal{F}_r^{(A)}$ and $\mathcal{F}_\theta^{(A)}$ read off from Eqs. (15) and (16) in the manuscript respectively, the chain-rule derivatives become

$$\frac{\partial U_{x,n}^{(A)}}{\partial \xi} = \frac{r^{\lambda_n^{(A)}-2}}{8\mu}\left[\lambda_n^{(A)}\, a^2\Delta\xi\, \mathcal{F}_x^{(A)} + ab\,\Delta\eta\, \frac{d\mathcal{F}_x^{(A)}}{d\theta}\right] \tag{2.18}$$

$$\frac{\partial U_{x,n}^{(A)}}{\partial \eta} = \frac{r^{\lambda_n^{(A)}-2}}{8\mu}\left[\lambda_n^{(A)}\, b^2\Delta\eta\, \mathcal{F}_x^{(A)} - ab\,\Delta\xi\, \frac{d\mathcal{F}_x^{(A)}}{d\theta}\right] \tag{2.19}$$

and analogously for $\partial U_{y,n}^{(A)}/\partial\xi$ and $\partial U_{y,n}^{(A)}/\partial\eta$ with $F_x^{(A)}$ replaced by $F_y^{(A)}$. The angular derivatives $dF_x^{(A)}/d\theta$ and $dF_y^{(A)}/d\theta$ are computed analytically from Eqs. (15)-(16) and (21) in the manuscript. Expressions for the symmetric family follow by substituting S for A and using the eigenvalues $\lambda_n^{(S)}$ from Eq. (20) in the manuscript. The full derivatives $\partial\Pi_n^{AU}/\partial\eta$, $\partial\Pi_n^{AV}/\partial\xi$, and $\partial\Pi_n^{AV}/\partial\eta$ follow the same pattern.

2.4 Regular-regular mass blocks $\boldsymbol{M}_{pp}$

The regular–regular mass entries are

$$M_{mn,rs}^{pp,uu} = \frac{\rho hab}{4} \iint_{\Omega} \Phi_{mn}^{U} \, \Phi_{rs}^{U} \, d\xi d\eta \tag{2.20}$$

$$M_{mn,rs}^{pp,vv} = \frac{\rho hab}{4} \iint_{\Omega} \Phi_{mn}^{V} \, \Phi_{rs}^{V} \, d\xi d\eta \tag{2.21}$$

As with the stiffness blocks, these are computed as the full-rectangle factored products (i.e., $(\rho hab/4)\, I_{mr}^{00,U} J_{ns}^{00,U}$ and $(\rho hab/4)\, I_{mr}^{00,V} J_{ns}^{00,V}$, respectively) minus the numerically evaluated notch-region correction.

2.5 Corner-corner mass blocks $\boldsymbol{M}_{cc}$

The corner-corner mass sub-block receives contributions from both displacement components, because $C_n^{(A)}$ appears in both U and V:

$$M_{n,k}^{cc,AA} = \frac{\rho hab}{4} \iint_{\Omega} [\Pi_n^{AU} \, \Pi_k^{AU} + \Pi_n^{AV} \, \Pi_k^{AV}] d\xi d\eta \tag{2.22}$$

The entries $M_{n,k}^{cc,SS}$ and $M_{n,k}^{cc,AS} = M_{k,n}^{cc,SA}$ follow analogously.

2.6 Coupling mass blocks $\boldsymbol{M}_{pc}$ and $\boldsymbol{M}_{cp}$

The coupling mass entries between the U-regular functions and the antisymmetric corner coefficients arise from the U contribution to the kinetic energy:

$$M_{mn,k}^{pc,u^A} = \frac{\rho hab}{4} \iint_{\Omega} \Phi_{mn}^{U} \, \Pi_k^{AU} \, d\xi d\eta \tag{2.23}$$

The corresponding coupling from the V contribution is:

$$M_{mn,k}^{pc,v^A} = \frac{\rho hab}{4} \iint_{\Omega} \Phi_{mn}^{V} \, \Pi_k^{AV} \, d\xi d\eta \tag{2.24}$$

Entries for the symmetric family follow by replacing A with S. The blocks $\boldsymbol{M}_{cp}$ are the transposes of the corresponding $\boldsymbol{M}_{pc}$ blocks.

References

1. Williams M (1952) Stress singularities resulting from various boundary conditions in angular corners of plates in extension. J. Appl. Mech. 19(4): 526-528. https://doi.org/10.1115/1.4010553