Supplymental Materials for "An attention economy model of co-evolution between content quality and audience selectivity" by M. Chujyo et al.

A Brief Description of the Proposed Model

We consider a simple evolutionary model of an attention economy composed of two types of agents: content providers and consumers. Each provider chooses between two strategies: producing high-quality content (strategy H) or low-quality content (strategy L). Producing high-quality content is costly, so providers adopting strategy H incur a cost $c_H > 0$, whereas those adopting strategy L bear no additional cost. Strategy H represents effort-intensive, high-quality content production.

Consumers also choose between two strategies: actively searching for and consuming high-quality content (active strategy A), or randomly consuming available content (passive strategy P). The active strategy entails a search cost $c_A > 0$, reflecting the time and cognitive effort required to identify high-quality content. Consumers receive higher benefits from high-quality content, earning $b_H > 0$ for high-quality and $b_L > 0$ for low-quality content, with $b_H > b_L$. Our main question is which strategy profiles become evolutionarily stable under the coupled dynamics of providers and consumers.

Attention is treated as a limited cognitive resource. Each consumer can allocate attention to only a fraction σ of all available content items, where $0 < \sigma < 1$, reflecting realistic constraints on time and cognitive capacity in online environments. This limitation shapes competition among providers for visibility and influences how consumers evaluate and select content.

We assume a population of mproviders and n consumers. At each time step, the following sequence occurs:

1. Each provider produces one content item according to their strategy: high-quality under H and low-quality under L.
2. Each consumer selects $m\sigma$ content items from those produced in the current step. Active consumers (A) preferentially choose high-quality items, while passive consumers (P) select items at random.
3. Each provider receives a reward $r > 0$ for each view of their content.
4. Each consumer obtains benefit b_H or b_L from the content they consume, depending on its quality.

These steps are iterated over time, and the population shares of the four strategies evolve according to replicator dynamics.

B Formulation of the Replicator Dynamics

Let $x \in [0, 1]$ be the proportion of providers adopting the high–effort strategy H (so $1 - x$ adopts L), and let $y \in [0, 1]$ be the proportion of consumers adopting the active strategy A (so $1 - y$ adopts P). We denote by π_H^{prov}, π_L^{prov}, π_A^{cons}, and π_P^{cons} the expected payoffs for the four strategies.

The average payoffs for each population are

$$\bar{\pi}_{\text{prov}} = x\,\pi_H^{\text{prov}} + (1-x)\,\pi_L^{\text{prov}}, \qquad \bar{\pi}_{\text{cons}} = y\,\pi_A^{\text{cons}} + (1-y)\,\pi_P^{\text{cons}}. \tag{S1}$$

The two–population evolutionary dynamics follow the standard replicator equations:

$$\dot{x} = x\left(\pi_H^{\text{prov}} - \bar{\pi}_{\text{prov}}\right) = x(1-x)\big(\pi_H^{\text{prov}} - \pi_L^{\text{prov}}\big), \tag{S2}$$

$$\dot{y} = y\left(\pi_A^{\text{cons}} - \bar{\pi}_{\text{cons}}\right) = y(1-y)\big(\pi_A^{\text{cons}} - \pi_P^{\text{cons}}\big). \tag{S3}$$

The expected payoffs can be expressed in closed form, but their expressions depend on whether the fraction of high-effort providers x is larger or smaller than the attention capacity σ.

First, we present the payoff expressions and then summarize the resulting piecewise replicator system. Each payoff is derived directly from the expected number of views received by each provider or the expected number of items consumed by each viewer, multiplied by the corresponding reward or benefit. Providers receive a reward r per view and incur a cost only when producing high-quality content. Consumers have $m\sigma$ attention slots, and whether these slots can be fully filled with high-quality items determines the structure of the payoffs.

A key observation is that the system naturally divides at the threshold $\sigma = x$. This threshold arises because each viewer has $m\sigma$ attention slots, while the total supply of high-quality items is mx; thus, the condition $m\sigma \le mx$ is equivalent to $\sigma \leq x$. When the available high-quality content (proportional to x) is sufficient to fill the attention capacity of all active consumers, they allocate their attention entirely to high-quality items. Conversely, when high-quality content is scarce ($\sigma > x$), active consumers must supplement their attention with low-quality content, leading to different mixing proportions across content types. This shift in attention allocation produces two distinct regimes, each with its own payoff expressions.

In both regimes, payoff terms reflect the distribution of attention across content types. Passive consumers sample content uniformly at random; therefore, the expected fraction of high-quality items they consume is simply x. Active consumers, in contrast, exhaust the available supply of high-quality items before allocating attention elsewhere.

In the regime $\sigma \leq x$, their attention can be fully concentrated on high-quality content. In the regime $\sigma > x$, only a fraction x/σ of attention can be allocated to high-quality items, and the remainder must be filled with low-quality items. For providers, the term $(1 - y)$ captures the contribution of passive consumers, who allocate their σm attention slots uniformly across all m items. The term y/x arises from active consumers: there are ny active viewers, each selecting σm items, so the total number of active views on high-quality content is $ny\sigma m$. Dividing this by the mx high–quality items yields an expected $ny\sigma/x$ views per high-quality item; after factoring out $rn\sigma$ in the payoff expression, this appears as the factor y/x. In regime $\sigma \leq x$, the resulting payoffs are

$$\pi_H^{\text{prov}}(x, y) = -c_H + rn\sigma \left((1 - y) + \frac{y}{x}\right), \tag{S4}$$
$$\pi_L^{\text{prov}}(x, y) = rn\sigma(1 - y), \tag{S5}$$
$$\pi_A^{\text{cons}}(x, y) = -c_A + m\sigma b_H, \tag{S6}$$
$$\pi_P^{\text{cons}}(x, y) = \sigma m \left(x b_H + (1 - x) b_L\right). \tag{S7}$$

In the regime $\sigma > x$, active consumers first exhaust all available high-quality items. Consequently, every high-quality item is viewed by all active consumers, and the total attention allocated to high-quality content is $nymx$. The remaining attention, $nym(\sigma - x)$, must be allocated to low-quality items. Since there are $m(1 - x)$ low-quality items, this residual attention is distributed uniformly across them, yielding an average of $(\sigma - x)/(1 - x)$ views per low-quality item for each active consumer. Therefore, in regime $\sigma > x$, the payoffs become

$$\pi_H^{\text{prov}}(x, y) = -c_H + rn \left(\sigma(1 - y) + y\right), \tag{S8}$$
$$\pi_L^{\text{prov}}(x, y) = rn \left[\sigma(1 - y) + y\frac{\sigma - x}{1 - x}\right], \tag{S9}$$
$$\pi_A^{\text{cons}}(x, y) = -c_A + m \left(x b_H + (\sigma - x) b_L\right), \tag{S10}$$
$$\pi_P^{\text{cons}}(x, y) = \sigma m \left(x b_H + (1 - x) b_L\right). \tag{S11}$$

For the subsequent stability analysis, we define the effective parameter $b := b_H - b_L > 0$ and incorporates b_L into the baseline payoff. With this substitution, the replicator dynamics can be expressed in the simplified piecewise system:

$$\frac{dx}{dt} = \begin{cases} (1 - x)\left(rn\sigma y - c_H x\right), & (\sigma \leq x), \\ x\left(rny(1 - \sigma) + c_H(x - 1)\right), & (\sigma > x), \end{cases} \tag{S12}$$

$$\frac{dy}{dt} = \begin{cases} y(1 - y)\left(bm\sigma(1 - x) - c_A\right), & (\sigma \leq x), \\ y(1 - y)\left(bmx(1 - \sigma) - c_A\right), & (\sigma > x), \end{cases} \tag{S13}$$

where $r, n, m, b, c_H, c_A > 0$ are constant parameters and $0 < \sigma < 1$ is the attention ratio.

C Derivation and Stability of Equilibria

The equilibria are determined by the simultaneous solutions of $\dot{x} = 0$ and $\dot{y} = 0$. Once these fixed points are identified, their stability can be assessed by examining the evolution of small perturbations under the dynamics. Since the replicator system is two-dimensional and nonlinear, local stability is evaluated by linearizing the system around each equilibrium.

Let the system be expressed as $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. The Jacobian matrix

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

captures the local behavior of trajectories near (x, y). Evaluating $J(x, y)$ at equilibrium (x^*, y^*) provides a linear approximation of the dynamics. The equilibrium is

- locally asymptotically stable if both eigenvalues of $J(x^*, y^*)$ have negative real parts,
- unstable if at least one eigenvalue has a positive real part.

Because the payoff functions that determine f and g differ depending on whether $\sigma \leq x$ or $\sigma > x$, Partial derivatives must be computed separately for these two regions. Accordingly, the correct branch of the Jacobian must be used to evaluate the stability of each equilibrium. If a candidate equilibrium lies at the switching boundary $x = \sigma$, the Jacobian should be evaluated using the appropriate one-sided limit.

For reference, we summarize the Jacobian matrix in these two regions. From Eqs. (S12)–(S13), we obtain the Jacobian matrix

$$J(x, y) = \begin{bmatrix} -rn\sigma y + c_H(2x-1) & (1-x)rn\sigma \\ -y(1-y)bm\sigma & (1-2y)(bm\sigma(1-x) - c_A) \end{bmatrix} \quad (\sigma \leq x), \tag{S14}$$

and

$$J(x, y) = \begin{bmatrix} rny(1-\sigma) + c_H(2x-1) & x\, rn(1-\sigma) \\ y(1-y)\, bm(1-\sigma) & (1-2y)(bmx(1-\sigma) - c_A) \end{bmatrix} \quad (\sigma > x). \tag{S15}$$

C.1 Corner Equilibria

The state space $[0, 1]^2$ has the vertexes $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Each represents a homogeneous population of providers and consumers. Because the replicator equations– Eqs. (S2)-(S3)– have the factors of $x(1-x)$ and $y(1-y)$, all four corners automatically satisfy $\dot{x} = \dot{y} = 0$.

Stability is determined by evaluating the Jacobian matrix $J(x, y)$ introduced above. Since the Jacobian is piecewise, an appropriate branch must be selected based on whether $\sigma \leq x$ or $\sigma > x$.

(i) $(x, y) = (0, 0)$**.** Here, $x < \sigma$. Thus, we use the Jacobian for $\sigma > x$. Substituting $(0, 0)$ yields

$$J(0, 0) = \begin{bmatrix} -c_H & 0 \\ 0 & -c_A \end{bmatrix}, \tag{S16}$$

whose eigenvalues $(-c_H, -c_A)$ are both negative. Therefore, $(0, 0)$ is always asymptotically stable.

(ii) $(x, y) = (0, 1)$**.** Again, $x < \sigma$; hence, the branch $\sigma > x$ applies:

$$J(0, 1) = \begin{bmatrix} rn(1-\sigma) - c_H & 0 \\ 0 & c_A \end{bmatrix}. \tag{S17}$$

Because $c_A > 0$, this point is unstable.

(iii) $(x, y) = (1, 0)$**.** Here, $x = 1 \geq \sigma$. Thus, we use the branch $\sigma \leq x$:

$$J(1, 0) = \begin{bmatrix} c_H & 0 \\ 0 & -c_A \end{bmatrix}. \tag{S18}$$

One eigenvalue is positive and one is negative; thus, $(1, 0)$ is unstable.

(iv) $(x, y) = (1, 1)$**.** Again, $\sigma \leq x$:

$$J(1, 1) = \begin{bmatrix} c_H - rn\sigma & 0 \\ 0 & c_A \end{bmatrix}. \tag{S19}$$

Because $c_A > 0$, the point is unstable, regardless of the sign of $c_H - rn\sigma$.

Hence, among the corner equilibria, only $(0, 0)$ is asymptotically stable and the remaining three corners are unstable.

C.2 Boundary Behavior on $x = 0, 1$ and $y = 0$

Next, we examined the equilibria that lie on the edges of the state space. Each boundary of $[0, 1]^2$ is invariant under the dynamics, so the flow along an edge reduces to a one-dimensional equation. Before analyzing the nontrivial case $y = 1$, we first summarize the behavior along the other boundaries $x = 0$, $x = 1$, and $y = 0$.

Boundary $x = 0$. For $x = 0$ we have $x < \sigma$; therefore, the case $\sigma > x$ in Eqs. (S12)–(S13) applies:

$$\dot{x} = 0, \qquad \dot{y} = y(1-y)\big(bm\sigma - c_A\big). \tag{S20}$$

Thus, along $x = 0$ the system reduces to one-dimensional dynamics in y, and the only equilibria on this edge are the corners $(0,0)$ and $(0,1)$.

Boundary $x = 1$. For $x = 1$, $\sigma \leq x$. Hence,

$$\dot{x} = 0, \qquad \dot{y} = y(1-y)\big(bm\sigma(1-1) - c_A\big) = -c_A\, y(1-y). \tag{S21}$$

Again, the dynamics are one-dimensional in y, and the only equilibria on $x = 1$ are the corners $(1,0)$ and $(1,1)$.

Boundary $y = 0$. For $y = 0$, we have $\dot{y} = 0$ and

$$\dot{x} = \begin{cases} (1-x)(0 - c_H x) = -c_H x(1-x), & (\sigma \leq x), \\ x\big(c_H(x-1)\big), & (\sigma > x), \end{cases} \tag{S22}$$

such that $\dot{x} < 0$ holds for all $0 < x < 1$. Along $y = 0$ the flow runs from $(1,0)$ to $(0,0)$, and the only equilibria on this edge are the corners $(0,0)$ and $(1,0)$.

In summary, there are no nontrivial equilibria along the boundaries $x = 0$, $x = 1$, or $y = 0$, and all boundary equilibria other than the four corners must lie on $y = 1$. Therefore, we next focus on the line $y = 1$ to analyze the existence and stability of interior points on this boundary.

C.3 Boundary Equilibria on $y = 1$

Finally, we examine the boundary $y = 1$, which is the only edge that may contain nontrivial equilibria. By setting $y = 1$ in Eq. (S12), for $\sigma \leq x$:

$$\dot{x} = (1-x)(rn\sigma - c_H x), \tag{S23}$$

such that the nontrivial equilibria satisfy

$$x^* = \frac{rn\sigma}{c_H}. \tag{S24}$$

For $\sigma > x$:

$$\dot{x} = x\big(rn(1-\sigma) + c_H(x-1)\big), \tag{S25}$$

such that the nontrivial equilibria satisfy

$$x^* = 1 - \frac{rn(1-\sigma)}{c_H}. \tag{S26}$$

Thus, $y = 1$ may host two candidate boundary equilibria

$$(x^*, y^*) = \left(\frac{rn\sigma}{c_H}, 1\right), \qquad (x^*, y^*) = \left(1 - \frac{rn(1-\sigma)}{c_H}, 1\right), \tag{S27}$$

provided they fall within $(0,1)$ and satisfy the corresponding regional conditions for x.

Stability of $(x^*, y^*) = (\frac{rn\sigma}{c_H}, 1)$ in $\sigma \leq x$

For $\sigma \leq x$, we use the following Jacobian for $\sigma \leq x$ evaluated at $(x^*, 1)$:

$$J(x^*, 1) = \begin{bmatrix} -rn\sigma + c_H(2x^* - 1) & (1-x^*)rn\sigma \\ 0 & -(bm\sigma(1-x^*) - c_A) \end{bmatrix}. \tag{S28}$$

This matrix is upper-triangular; therefore, the eigenvalues are diagonal. Substituting $x^* = rn\sigma/c_H$ yields

$$\begin{aligned} \lambda_1 &= -rn\sigma + c_H\left(2\frac{rn\sigma}{c_H} - 1\right) = rn\sigma - c_H, \\ \lambda_2 &= -\left(bm\sigma\left(1 - \frac{rn\sigma}{c_H}\right) - c_A\right) = c_A - bm\sigma\left(1 - \frac{rn\sigma}{c_H}\right). \end{aligned}$$

Thus, $(x^*, 1)$ is asymptotically stable if and only if

$$c_H > rn\sigma, \tag{S29}$$

$$c_A < bm\sigma \left(1 - \frac{rn\sigma}{c_H}\right). \tag{S30}$$

Note that the existence of this point as a nontrivial equilibrium with $\sigma < x^* < 1$ requires

$$rn\sigma < c_H < rn. \tag{S31}$$

Combined with (S30), a convenient form of the stability condition is as follows:

$$rn\sigma < c_H < rn, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} < 1. \tag{S32}$$

Stability of $(x^*, y^*) = (1 - \frac{rn(1-\sigma)}{c_H}, 1)$ **in** $\sigma > x$

For $\sigma > x$, we used the Jacobian with $\sigma > x$:

$$J(x^*, 1) = \begin{bmatrix} rn(1-\sigma) + c_H(2x^* - 1) & x^* rn(1-\sigma) \\ 0 & -(bmx^*(1-\sigma) - c_A) \end{bmatrix}. \tag{S33}$$

Substituting $x^* = 1 - \frac{rn(1-\sigma)}{c_H}$ yields

$$\lambda_1 = c_H - rn(1-\sigma),$$
$$\lambda_2 = c_A - bm(1-\sigma)\left(1 - \frac{rn(1-\sigma)}{c_H}\right).$$

For $(x^*, 1)$ to be in the region of $\sigma > x$, we require $0 < x^* < \sigma$, which is equivalent to

$$0 < 1 - \frac{rn(1-\sigma)}{c_H} < \sigma \quad \Rightarrow \quad 0 < c_H - rn(1-\sigma) < c_H\sigma. \tag{S34}$$

However, this implies that $\lambda_1 = c_H - rn(1-\sigma) > 0$. Thus, $(x^*, 1)$ cannot be stable in its defined region. Therefore, the equilibrium $(1 - \frac{rn(1-\sigma)}{c_H}, 1)$ is always unstable.

Thus, the only possible stable equilibrium for $y = 1$ is

$$(x^*, y^*) = \left(\frac{rn\sigma}{c_H}, 1\right), \tag{S35}$$

and it is stable for

$$rn\sigma < c_H < rn, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} < 1. \tag{S36}$$

C.4 Interior Equilibria

We now consider the equilibria for $0 < x < 1$ and $0 < y < 1$.

Case $\sigma \leq x$

From Eqs. (S12)–(S13), interior equilibria in $\sigma \leq x$ satisfy

$$(1-x)(rn\sigma y - c_H x) = 0 \quad \Rightarrow \quad y = \frac{c_H x}{rn\sigma},$$
$$y(1-y)\big(bm\sigma(1-x) - c_A\big) = 0 \quad \Rightarrow \quad x = 1 - \frac{c_A}{bm\sigma}.$$

Thus, the interior equilibrium is

$$(x^*, y^*) = \left(1 - \frac{c_A}{bm\sigma},\ \frac{c_H}{rn\sigma}\left(1 - \frac{c_A}{bm\sigma}\right)\right). \tag{S37}$$

For $0 < \sigma < 1$ and suitable choices for c_A and c_H, the resulting point (x^*, y^*) satisfies $\sigma < x^* < 1$ and $0 < y^* < 1$. The precise existence conditions are summarized below:

$$bm\sigma(1-\sigma) > c_A, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} > 1. \tag{S38}$$

Evaluating the Jacobian for $\sigma \leq x$ at (x^*, y^*) and simplifying it, we obtain

$$J = \begin{bmatrix} -c_H(1-x^*) & rn\sigma(1-x^*) \\ -bm\sigma y^*(1-y^*) & 0 \end{bmatrix}. \tag{S39}$$

The trace and determinant of the matrix are as follows:

$$\mathrm{tr}(J) = -c_H(1-x^*) < 0, \qquad \det(J) = -J_{12}J_{21} > 0 \tag{S40}$$

because $J_{12} > 0$ and $J_{21} < 0$ for $0 < y^* < 1$. Therefore, both eigenvalues have negative real parts, and we conclude that the interior equilibrium (x^*, y^*) in region $\sigma \leq x$ is always asymptotically stable if it exists. The existence conditions $x^* \in (\sigma, 1)$ and $y^* \in (0, 1)$ can be summarized as follows:

$$bm\sigma(1-\sigma) > c_A, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} > 1. \tag{S41}$$

Case $\sigma > x$

In this region, the interior equilibrium satisfies

$$x\big(rny(1-\sigma) + c_H(x-1)\big) = 0 \quad \Rightarrow \quad y = \frac{c_H(1-x)}{rn(1-\sigma)},$$
$$y(1-y)\big(bmx(1-\sigma) - c_A\big) = 0 \quad \Rightarrow \quad x = \frac{c_A}{bm(1-\sigma)}.$$

Thus, the interior equilibrium is

$$(x^*, y^*) = \left(\frac{c_A}{bm(1-\sigma)}, \frac{c_H}{rn(1-\sigma)}\left(1 - \frac{c_A}{bm(1-\sigma)}\right)\right), \tag{S42}$$

with $0 < x^*, y^* < 1$ and $\sigma > x^*$ assumed.

In this case, the Jacobian is simplified to

$$J = \begin{bmatrix} c_H x^* & J_{12} \\ J_{21} & 0 \end{bmatrix}, \tag{S43}$$

where

$$J_{12} = x^* rn(1-\sigma) = \frac{rnc_A}{bm} > 0,$$
$$J_{21} = y^*(1-y^*)bm(1-\sigma) > 0.$$

Thus

$$\mathrm{tr}(J) = c_H x^* > 0, \qquad \det(J) = -J_{12}J_{21} < 0. \tag{S44}$$

Therefore, the eigenvalues have opposite signs, and we conclude that the interior equilibrium in the region $\sigma > x$ is always unstable.

C.5 Summary of Stable Equilibria

In summary, the system admits the following stable equilibria:

(1) The collapse equilibrium

$$(x, y) = (0, 0), \tag{S45}$$

which is always asymptotically stable ($\lambda_1 = -c_H < 0$, $\lambda_2 = -c_A < 0$).

(2) A boundary equilibrium on $y = 1$:

$$(x, y) = \left(\frac{rn\sigma}{c_H}, 1\right) \tag{S46}$$

with $x \in (\sigma, 1)$, which is stable if and only if

$$rn\sigma < c_H < rn, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} < 1. \tag{S47}$$

(3) An interior equilibrium in the region $\sigma \leq x$:

$$(x, y) = \left(1 - \frac{c_A}{bm\sigma}, \frac{c_H}{rn\sigma}\left(1 - \frac{c_A}{bm\sigma}\right)\right), \tag{S48}$$

with $\sigma < x < 1$ and $0 < y < 1$, which is always asymptotically stable whenever it exists. The existence conditions can be written as

$$bm\sigma(1-\sigma) > c_A, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} > 1. \tag{S49}$$

D Boundary Between Collapse and Non–Collapse Regimes

Finally, we clarify the conditions for a purely collapsing system, in which the only stable equilibrium is $(x, y) = (0, 0)$. In the classification above, the collapse equilibrium $(0, 0)$ is always locally stable, whereas additional stable equilibria may exist in certain regions of parameter space. In particular, nontrivial stable equilibria (on $y = 1$ or in the interior) require

$$bm\sigma(1 - \sigma) > c_A \tag{S50}$$

along with the inequalities involving c_H.

For a stable boundary equilibrium at $y = 1$, we require

$$rn\sigma < c_H < rn, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} < 1. \tag{S51}$$

From $rn\sigma < c_H < rn$ we obtain:

$$1 - \frac{rn\sigma}{c_H} > 1 - \frac{rn\sigma}{rn} = 1 - \sigma, \tag{S52}$$

and hence

$$c_A < bm\sigma\left(1 - \frac{rn\sigma}{c_H}\right) < bm\sigma(1 - \sigma). \tag{S53}$$

Thus, the condition $bm\sigma(1 - \sigma) > c_A$ is automatically satisfied when the boundary equilibrium is stable.

For a stable interior equilibrium in $\sigma \leq x$, we explicitly require

$$bm\sigma(1 - \sigma) > c_A, \qquad \frac{c_A}{bm\sigma} + \frac{rn\sigma}{c_H} > 1. \tag{S54}$$

The parameter region in which the system admits nontrivial stable equilibria (either on $y = 1$ or in the interior) is contained within the domain $bm\sigma(1 - \sigma) > c_A$. Conversely, the complementary inequality $bm\sigma(1 - \sigma) < c_A$ implies that no additional stable equilibria exist, and the only stable equilibrium is the collapse equilibrium $(x, y) = (0, 0)$. Thus, the equation

$$bm\sigma(1 - \sigma) = c_A \tag{S55}$$

defines the boundary that separates the collapse regime (where only $(0, 0)$ is stable) from the coexistence regime, where high-quality strategies can persist.