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An Operations Research Tutorial on Fairness Metrics and Resource Allocation

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Abstract

Fairness is a pervasive yet multifaceted concept, with applications spanning economics, engineering, public policy, and computational systems. Yet there is no universal definition or metric of fairness, resulting in divergent and often incompatible approaches across disciplines. This paper addresses that gap by providing a structured tutorial that outlines, compares, and operationalizes established fairness notions in resource allocation and fair division. We center on homogeneous, disconnected resource settings under a centralized decision-maker, using the fair cake-cutting metaphor as a structural and pedagogical anchor. We outline utility-allocation schemes, fairness metrics, and inequality measures such as the Gini and Jain indices, and examine efficiency–fairness trade-offs through the Price of Fairness, the Price of Efficiency, the Price of Anarchy, and the Price of Choice, used here as a cross-scheme comparative lens. We revisit classical fair-division criteria—proportionality, super-proportionality, envy-freeness, strong envy-freeness, and equitability—alongside resource and population monotonicity principles, illustrating through a computational proof-of-concept where each scheme satisfies or violates these properties. The proof-of-concept, a stylized cloud computing credit-allocation scenario, further shows how strategic misreporting affects different fairness objectives unevenly. The result is intended as a structured, defensible reference that equips operations research scholars and practitioners to understand, compare, and apply fairness paradigms while remaining explicit about the assumptions, limitations, and trade-offs inherent in each.

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1 Introduction

A decision-maker charged with distributing a scarce, divisible resource—compute credits across projects, relief supplies across affected regions, or capacity across competing users—must choose not only how much each recipient receives, but which fairness principle governs that choice. The principle is rarely innocuous: egalitarian, utilitarian, proportional, and Rawlsian rules [1] can each be defended on normative grounds, yet they yield materially different allocations, and the “cost” of selecting one over another is seldom made explicit. Efficient allocation determines how opportunities, access to essential services, and well-being are distributed across individuals and institutions, while inefficiency can yield waste, inequality, and social unrest. These challenges span domains as diverse as inheritance distribution [2], investment planning [3], and the allocation of vital utilities such as natural gas, electricity, and water in interdependent networks [4], and equitable access to essential healthcare services [5]. They also extend to crisis contexts, such as the fair distribution of medical or survival resources during pandemics or disasters [6–10]. In each case, scarce resources must be distributed in ways that are not only efficient but also ethically and socially defensible. Fairness, however, is inherently difficult to define and operationalize [11–13]. It represents a form of distributive justice that balances competing moral, social, and economic objectives [14].

One of the earliest and most intuitive frameworks for thinking about fairness is the classical “divide and choose” rule, historically framed through the metaphor of dividing a cake [15]. In this procedure, one agent divides the cake and the other chooses first—forcing the divider to internalize fairness in the act of division. Despite its simplicity, this mechanism captures the strategic and moral essence of equitable allocation. The “cake-cutting” metaphor endures as both an analytical and pedagogical construct: it abstracts fairness from context, allowing moral, mathematical, and operational dimensions to coexist in a single image.

Fairness becomes especially salient under scarcity. When resources abound, efficiency dominates; when scarcity binds, fairness determines legitimacy. Competing principles lead to distinct and sometimes incompatible outcomes. Hence, fairness should not be treated as a static ideal but as a context-dependent construct shaped by problem structure, stakeholder priorities, and institutional rules. This recognition motivates the need for a structured tutorial on fairness-based resource allocation: one that synthesizes fairness definitions, metrics, and schemes into a single coherent reference that supports both comparison and application across disciplines. To this end, this paper systematizes insights from mathematics, economics, game theory, and operations research, with emphasis on the allocation of homogeneous resources under disconnected-allocation and divisibility properties.

In this context, a *homogeneous resource* is one whose units are indistinguishable—such as water, energy, or a uniformly flavored cake—so that its value does not depend on the specific portion received. A *disconnected allocation* property implies that receiving a resource in multiple separate parts is equivalent, in value, to receiving it as one contiguous piece of the same total size. Finally, the *divisibility* property assumes that resources can be partitioned continuously, allowing allocations of fractional quantities. Indivisible resources—such as discrete goods like televisions, jewelry, or vehicles—fall outside this scope, since exact fairness may become unattainable in such cases and could necessitate approximations or relaxations of fairness guarantees. Accordingly, this study restricts attention to divisible, homogeneous, and disconnected resources, for which fairness principles can be analyzed and compared under consistent mathematical assumptions

[16].

In this paper, we use—and intentionally extend—the concept of fair cake cutting as a unifying analytical lens. We adopt it not merely as a metaphor but as a structured device through which the mechanics of fairness, efficiency, and equity can be examined within the mathematical language of optimization and operations research. Under this lens, a centralized decision maker allocates resources fairly across participants. We assert that many real-world allocation problems can be represented as variations of a cake-cutting process, where the total resource represents the cake and fairness rules correspond to allocation criteria. While classical models emphasize the balance between fairness and efficiency, this tutorial extends this logic to incorporate topological, organizational, and systemic constraints—such as policy hierarchies or technological capacities—that shape fairness in applied contexts. Defining fairness under such conditions is necessarily pluralistic, requiring reconciliation between competing axioms and fairness metrics that may point toward different “just” outcomes.

This paper contributes a structured Operations Research perspective on fairness in resource allocation by: (i) organizing utility-based allocation schemes, inequality metrics, valuation-based fair-division criteria, and sensitivity axioms under a common notation; (ii) illustrating, through worked examples and numerical scenarios, conditions under which these notions coincide, diverge, or become incompatible under homogeneous divisible resources; and (iii) demonstrating, through a reproducible stylized resource allocation model, how alternative fairness choices affect efficiency, inequality, envy, monotonicity, and strategic vulnerability. As the principal lens for comparing schemes, we employ the Price of Choice (PoC) [17], which quantifies the fairness-specific performance loss incurred when one allocation scheme is adopted in place of another, evaluated under each scheme’s own normative logic rather than a fixed ideological baseline.

Intended as a structured tutorial and reference, this paper consolidates representative fairness concepts that are widely used in operations research, economics, game theory, and engineering resource allocation. We focus on concepts that can be operationalized as optimization objectives, constraints, diagnostic metrics, or sensitivity properties under homogeneous, divisible, and disconnected resource assumptions. Unlike existing surveys of fair division and fairness measurement, which typically catalogue definitions and metrics in isolation, our tutorial unifies allocation schemes, inequality indices, valuation-based criteria, and sensitivity axioms under a single notation and a common cake-cutting anchor, and uses the Price of Choice as a scheme-neutral instrument for making the trade-offs among them explicit and comparable.

The remainder of this tutorial is organized as follows. Section 2 reviews the conceptual foundations, including utility allocation schemes, fairness metrics, and their axiomatic properties, along with measures such as the Price of Fairness, the Price of Efficiency, the Price of Anarchy, and the Price of Choice; it closes by casting these schemes as instances of a single generic optimization formulation (Section 3), which anchors the applied material that follows. We also discuss valuation-based fairness criteria and monotonicity principles related to changes in resources, participants, and technology. Section 4, without claiming empirical generality, presents a proof-of-concept case study that instantiates this formulation in a stylized allocation scenario. Finally, Section 5 concludes with a discussion of key insights, limitations, and directions for future research.

How to use this tutorial. Readers seeking a specific entry point may navigate selectively. Those new to fair allocation can begin with the cake-cutting preliminaries and allocation

schemes in Section 2, which establish the common notation used throughout. Readers focused on *measuring* fairness can turn directly to the fairness metrics and inequality indices, while those concerned with *comparing* schemes will find the efficiency–fairness trade-off discussion and the Price of Choice most relevant. Practitioners who prefer to learn by example may start with the proof-of-concept scenario in Section 4 and refer back to the conceptual material as needed. Throughout, the cake-cutting metaphor serves as a consistent anchor, so concepts introduced in one section carry their notation and intuition into the next.

2 Conceptual Foundations

This section introduces and systematizes the core ideas that underpin fairness in resource division and allocation, integrating insights from fair division in game theory, utility theory in economics, and axiomatic analyses from engineering and operations research. Throughout, we adopt the *cake-cutting* lens as an organizing device—one that allows intuitive notions of fairness to be systematically translated into formal, operationalizable constructs.

In a *vanilla* fair cake-cutting problem, the stakeholders (participants) are each entitled to a portion of the total resource pool—the whole cake—represented as C . C is typically modeled as a finite 1D segment, a 2D polygon, or a finite subset of $\mathbb{R}^d$. Let $\mathcal{N}$ denote the set of participants, with $|\mathcal{N}| = n$. Feasible allocations belong to $\mathcal{X} \subseteq \mathbb{R}_+^n$, where $x_i \in \mathcal{X}$ denotes the portion of C assigned to participant i . Following the *Subjective Theory of Value*, Nash [18] argues that the value of an item reflects individual preferences and perceptions. Consequently, no single objective valuation of C can be universally agreed upon by all participants: each $v_i(C)$ may differ across $i \in \mathcal{N}$, as may the pairwise comparisons $v_i(x_i)$ versus $v_i(x_j)$. To capture this heterogeneity, subjective utility (or value) functions $u_i(x_i)$ or $v_i(x_i)$ map allocations to $[0, 1]$ —for normalized values—or, more generally, to $[\underline{u}, \bar{u}]$, where $\underline{u}, \bar{u} \in \mathbb{R}$ and $\underline{u} < \bar{u}$ for each $i \in \mathcal{N}$. In what follows, we use “item” and “resource” interchangeably, as well as “allocation” and “division”.

Classical cake-cutting models evaluate valuations on specific *fractions* of C (e.g., $[0, \frac{1}{2}]$ or $[\frac{1}{3}, \frac{2}{3}]$). Our focus differs: participants are *indifferent to location* and care only about the *size* of the fraction. For example, any interval of length $\frac{1}{3}$ of C is equivalent from the agent’s perspective. This corresponds precisely to the homogeneous, disconnected-allocation setting: units are indistinguishable and receiving multiple disjoint pieces totaling a given size is equivalent, in value, to receiving a single contiguous piece of that size.

2.1 Fairness Metrics

Central to the analysis of fair allocation is the notion of a *fairness metric*: a scalar function $h(\mathbf{u})$ that maps an allocation’s induced utility vector $\mathbf{u} = (u_1, \dots, u_n)$ to a real number representing the degree of fairness of that allocation. A fairness metric can be put to two distinct uses. It may serve as an *objective*—a quantity that an allocation rule optimizes in order to *produce* an allocation in the first place—or as a *diagnostic*, applied *after the fact* to an already-determined allocation in order to assess its equality on a common scale and compare allocations produced by different rules. The same functional form can serve either purpose, and the direction in which an objective is optimized—maximizing or minimizing h —is a feature of the rule that uses

it, not of the metric itself. We introduce fairness metrics first, as a self-contained concept, so that the allocation rules developed in the remainder of this section can each be characterized economically as the optimization of a particular fairness objective.

Regardless of role, a well-behaved fairness metric is expected to satisfy certain properties, including continuity, scale-invariance (so that proportional rescaling of all allocations does not change the fairness assessment), and Schur-concavity (so that any transfer from a better-off to a worse-off participant does not decrease fairness) [19]. Lan et al. [12] formalize these intuitions axiomatically and show that they generate a family of fairness measures unifying Jain’s index, Shannon entropy, and the α -fair utility function as special cases; we refer the reader there for the full construction.

2.2 Allocation Schemes: Axiomatic Analysis

We review representative utility-allocation schemes—*Social Welfare*, *Kalai–Smorodinsky*, *Max-min*, *Nash/Proportional Fairness*, and *Entropy*—and the axioms often used to evaluate them: Pareto optimality, symmetry, (weak) affine invariance, independence of irrelevant alternatives, and restricted monotonicity. Sampat and Zavala [11] synthesize these axioms (rooted in Nash [20] and Roth [21]) and summarize which schemes satisfy which properties.

Let $f_i : \mathcal{X} \rightarrow \mathbb{R}_+$ be the utility induced for participant $i \in \mathcal{N}$ by an allocation, and define the induced feasible utility set

$$\mathcal{U} := \{u : \exists x \in \mathcal{X}, f_i(x_i) = u_i \ \forall i \in \mathcal{N}\}. \quad (1)$$

Let $\bar{u}$ denote the vector of utopia (component-wise maximal) utilities and $\underline{u}$ the vector of worst (component-wise minimal) utilities; typically $\bar{u}, \underline{u} \notin \mathcal{U}$. A scheme returns $\varphi(\mathcal{U}) \in \mathcal{U}$ by optimizing a fairness metric $h : \mathcal{U} \rightarrow \mathbb{R}$; for the schemes presented here this takes the form $\varphi(\mathcal{U}) := \arg \max_{u \in \mathcal{U}} h(u)$, though minimizing formulations also arise in practice.

Axioms

1. **Pareto optimality:** $\varphi(\mathcal{U})$ is Pareto optimal if no $u \in \mathcal{U}$ satisfies $u_i \geq \varphi(\mathcal{U})_i$ for all i with strict inequality for some i .
2. **Symmetry:** For any permutation operator $\mathcal{P}$ on indices, $\mathcal{P}(\varphi(\mathcal{U})) = \varphi(\mathcal{P}(\mathcal{U}))$.
3. **(Weak) affine invariance:** For scaling operators $\mathcal{A}(u)_i = r_i \cdot u_i$ with $r_i > 0$, $\mathcal{A}(\varphi(\mathcal{U})) = \varphi(\mathcal{A}(\mathcal{U}))$. The weak version uses a common $r_i = r$.
4. **Independence of irrelevant alternatives:** If $\mathcal{U} \subset \mathcal{U}'$ and $\varphi(\mathcal{U}') \in \mathcal{U}$, then $\varphi(\mathcal{U}) = \varphi(\mathcal{U}')$.
5. **Restricted monotonicity:** If $\mathcal{U} \subset \mathcal{U}'$, then $\varphi(\mathcal{U}')$ dominates $\varphi(\mathcal{U})$.

Social Welfare

The utilitarian scheme maximizes total (shifted) utility:

$$\varphi_{SW}(\mathcal{U}) = \arg \max_{u \in \mathcal{U}} \left(h_{SW}(u) := \sum_{i \in \mathcal{N}} (u_i - \underline{u}_i) \right). \quad (2)$$

It may be non-unique and satisfies axioms 1, 3 (weak), and 4.

Kalai–Smorodinsky

For two participants, the KS solution selects the point on the line segment joining $\underline{u}$ and $\bar{u}$ that lies on the Pareto frontier. The *Pareto frontier* (or *Pareto-efficient set*) represents the set of allocations for which no participant's utility can be increased without decreasing that of another. In other words, an allocation $u \in \mathcal{U}$ is Pareto-optimal if there exists no other $u' \in \mathcal{U}$ such that $u'_i \geq u_i$ for all $i \in \mathcal{N}$ and $u'_j > u_j$ for at least one j [22].

$$\frac{u_1 - \underline{u}_1}{\bar{u}_1 - \underline{u}_1} = \frac{u_2 - \underline{u}_2}{\bar{u}_2 - \underline{u}_2}. \quad (3)$$

It satisfies axioms 1, 2, 3, and 5 for $n = 2$ and does not derive from an explicit $h(u)$; generalization to $n \geq 3$ is nontrivial, as discussed by Sampat and Zavala [11].

Max-min Scheme

Rooted in Rawlsian fairness [1], it maximizes the minimum shifted utility:

$$\varphi_{MM}(\mathcal{U}) = \arg \max_{u \in \mathcal{U}} \left(h_{MM}(u) := \min_{i \in \mathcal{N}} (u_i - \underline{u}_i) \right). \quad (4)$$

Solutions are often non-unique; it satisfies axioms 3 (weak) and 4.

Nash Scheme and Proportional Fairness

$$\varphi_{NS}(\mathcal{U}) = \arg \max_{u \in \mathcal{U}} \left(h_{NS}(u) := \prod_{i \in \mathcal{N}} (u_i - \underline{u}_i) \right). \quad (5)$$

If $\mathcal{U}$ is convex, applying a logarithm yields proportional fairness with the optimality condition

$$\sum_{i \in \mathcal{N}} \frac{u_i - \varphi_{PF}(\mathcal{U})_i}{\varphi_{PF}(\mathcal{U})_i} \leq 0 \quad \forall u \in \mathcal{U}, \quad (6)$$

and objective

$$\varphi_{PF}(\mathcal{U}) = \arg \max_{u \in \mathcal{U}} \left(h_{PF}(u) := \sum_{i \in \mathcal{N}} \log(u_i - \underline{u}_i) \right). \quad (7)$$

Nash and proportional fairness emphasize balance via the product (or log-sum) of utilities, giving relative weight to lower utilities; under standard conditions the PF optimizer is unique.

Both satisfy axioms 1, 2, 3, and 4.

Entropy (Theil Index)

The Theil index measures dispersion relative to equality and decomposes into within- and between-group terms [23]:

$$\varphi_T(\mathcal{U}) = \arg \max_{u \in \mathcal{U}} \left(h_E(u) := - \sum_{i \in \mathcal{N}} p_i(u) \log p_i(u) \right), \quad (8)$$

where $\sum_i p_i(u) = 1$ and $p_i(u) = \frac{u_i - \underline{u}_i}{\sum_{j \in \mathcal{N}} (u_j - \underline{u}_j)}$. With fixed aggregate utility, entropy is maximized by equal gains, i.e., $u_i - \underline{u}_i$ equal across i . Entropy satisfies axioms 1, 2, 3 (weak), and 4. Other generalized-entropy and superquantile schemes are surveyed by Sampat and Zavala [11], who also note that no single scheme satisfies all axioms 1–5 simultaneously.

2.3 Max-min Fairness

Max-min fairness lexicographically maximizes the ordered utility vector, aligning with Rawlsian egalitarianism [24]. When the utopia utilities are equal ($\bar{u}_i = r \in \mathbb{R}$ for all i), max-min and Rawlsian outcomes coincide; with heterogeneous $\bar{u}_i = r_i$, max-min fairness seeks the lexicographically largest utility vector under a sorting operator $\mathcal{T}$:

$$\mathcal{T}(u) = (u_{(1)}, u_{(2)}, \dots, u_{(n)}), \quad u_{(1)} \leq \dots \leq u_{(n)}. \quad (9)$$

An allocation u' is lexicographically larger than u (denoted $u' >_{\text{lex}} u$) if there exists k such that $u'_i = u_i$ for all $i < k$ and $u'_k > u_k$ ($u' \succeq_{\text{lex}} u$ for weak inequality). Then

$$\mathcal{T}(\varphi_{MMF}(\mathcal{U})) \succeq_{\text{lex}} \mathcal{T}(u) \quad \forall u \in \mathcal{U}. \quad (10)$$

As in Kalai-Smorodinsky, max-min fairness does not arise from a single explicit $h(u)$ but from the lexicographic order [24–26]. When utilities are normalized to a common scale (e.g., $[0, 1]$), the resulting allocation mirrors the Kalai-Smorodinsky proportional-gain spirit.

Remark. Max-min fairness should not be conflated with the Max-min *scheme* of (4), despite their shared name and common emphasis on the worst-off participant. The Max-min scheme maximizes only the single smallest shifted utility, $\min_i (u_i - \underline{u}_i)$, and is indifferent to how utility is distributed among the remaining participants; consequently its optimum is frequently non-unique. Max-min fairness refines this criterion lexicographically: having maximized the smallest component, it then maximizes the second-smallest among the resulting optima, and so on through the entire ordered vector via (10). The two therefore coincide only on the worst-off participant; beyond that, max-min fairness resolves the indeterminacy that the Max-min scheme leaves open, yielding a unique allocation under standard conditions.

2.4 The α -fair Utility Function

The schemes presented above can be unified through the α -fair utility function, a tractable parametric family [12]:

$$\sum_i U_\alpha(u_i), \quad U_\alpha(u) = \begin{cases} \frac{u^{1-\alpha}}{1-\alpha}, & \alpha \geq 0, \alpha \neq 1, \\ \log(u), & \alpha = 1, \end{cases} \quad (11)$$

where α controls the emphasis placed on fairness relative to efficiency. The family recovers several of the schemes already introduced as special cases: $\alpha = 1$ yields Proportional Fairness (Section 2.2) and $\alpha \rightarrow \infty$ approaches Max-min fairness (Section 2.3), with larger α encoding stricter fairness requirements while preserving Pareto optimality. The α -fair function thus acts as a single dial interpolating between efficiency-oriented and equality-oriented objectives, and provides a convenient reference family for the comparative price metrics introduced in Section 2.6.

2.5 Diagnostic Metrics

We now present two widely used *diagnostic* metrics—Jain’s index and the Gini coefficient—which quantify how equal a given allocation is, independently of the scheme that produced it.

Jain’s index

Jain’s index measures equality of allocation [27]:

$$J = \frac{\left(\sum_{i=1}^n u_i\right)^2}{n \cdot \sum_{i=1}^n u_i^2}. \quad (12)$$

The index ranges from $\frac{1}{n}$ to 1, where $\frac{1}{n}$ corresponds to maximal disparity (one participant holds all resources) and 1 to perfect equality. More generally, if k of the n participants share the resource equally while the remaining $n - k$ receive none, Jain’s index takes the value $J = \frac{k}{n}$, quantifying partial equality among subsets of participants.

Gini coefficient

The Gini coefficient measures inequality on $[0, 1]$, where 0 denotes perfect equality and 1 represents maximal concentration of resources [28]. Let $\tilde{u} = \frac{1}{n} \sum_i u_i$ denote the mean utility across participants. Then

$$G = \frac{\sum_{i=1}^n \sum_{j=1}^n |u_i - u_j|}{2n^2 \tilde{u}}. \quad (13)$$

A key advantage of Gini is its compliance with the *Dalton–Pigou transfer principle*: redistributing resources from better-off to worse-off participants—without reversing their relative

rankings—reduces measured inequality [29]. This property can also help break ties when schemes such as Social Welfare yield multiple maximizers of their objective. While Gini and Theil differ in functional form, both can serve as diagnostic measures of distributional inequality (the Theil measure also appearing as a scheme objective in (8)); Gini relates closely to Jain’s index in that while Gini increases with inequality, Jain’s index decreases correspondingly, providing complementary views of the same fairness–inequality continuum.

2.6 Price-Based Trade-off Metrics

The Price of Fairness

The *Price of Fairness* (PoF) quantifies efficiency loss relative to a baseline efficiency objective (typically Social Welfare) when adopting a fairness-oriented scheme [24]. With φ_{SW} , φ_{PF} , and φ_{MMF} as in (2), (7), and (10), define

$$PoF_{SW,PF} = 1 - \frac{h_{SW}(\varphi_{PF}(\mathcal{U}))}{h_{SW}(\varphi_{SW}(\mathcal{U}))}, \quad PoF_{SW,MMF} = 1 - \frac{h_{SW}(\varphi_{MMF}(\mathcal{U}))}{h_{SW}(\varphi_{SW}(\mathcal{U}))}. \quad (14)$$

Because φ_{SW} maximizes h_{SW} , each ratio is at most 1. Moreover, since $\underline{u}$ is the vector of worst-case (component-wise minimal) utilities, every feasible allocation satisfies $u_i \geq \underline{u}_i$, so each shifted term $u_i - \underline{u}_i$ is nonnegative and $h_{SW} \geq 0$; hence each PoF lies in $[0, 1]$.

The Price of Efficiency

To capture fairness loss when prioritizing efficiency, Bertsimas, Farias, and Trichakis [30] introduce the *Price of Efficiency* (PoE). Taking the minimum utility as the fairness metric—that is, h_{MM} of (4)—the fairest allocation is the one maximizing this minimum, namely the Max-min scheme φ_{MM} , which attains the largest possible minimum-utility guarantee $\max_{u \in \mathcal{U}} h_{MM}(u)$. The PoE measures the fractional loss in this guarantee incurred by adopting an efficiency-tilted allocation $\varphi_{\alpha\text{-fair}}$ (an α -fair allocation with finite α) instead:

$$PoE = \frac{h_{MM}(\varphi_{MM}(\mathcal{U})) - h_{MM}(\varphi_{\alpha\text{-fair}}(\mathcal{U}))}{h_{MM}(\varphi_{MM}(\mathcal{U}))} = 1 - \frac{h_{MM}(\varphi_{\alpha\text{-fair}}(\mathcal{U}))}{h_{MM}(\varphi_{MM}(\mathcal{U}))}. \quad (15)$$

Because φ_{MM} maximizes h_{MM} by construction, the ratio is at most 1; and since $h_{MM} \geq 0$ on shifted nonnegative utilities, $PoE \in [0, 1]$, interpretable as the percentage loss in the minimum-utility guarantee when efficiency is prioritized over maximal fairness. Although the Max-min scheme may admit multiple optimal allocations, all of them attain the same value of h_{MM} , so the PoE is well-defined regardless of which is selected.

The Price of Anarchy

In game-theoretic settings, the *Price of Anarchy* (PoA) measures the efficiency loss induced by decentralized, self-interested behavior relative to a centralized optimum [31, 32]. In its original formulation, the PoA is defined as the ratio of the optimal objective to that of the worst-case equilibrium,

$$PoA_{\text{std}} = \frac{h_k(\varphi_k(\mathcal{U}))}{h_k(u)}, \quad (16)$$

where $h_k(\varphi_k(\mathcal{U}))$ is the centralized optimum under metric k and u is a decentralized outcome (e.g., a worst-case Nash equilibrium); for a nonnegative metric this ratio is at least 1, with a value close to 1 indicating that equilibrium behavior preserves the centralized objective and larger values indicating greater inefficiency. For consistency with the Price of Fairness and Price of Efficiency above, we report instead an equivalent normalized loss fraction, so that larger values uniformly indicate greater loss. For a chosen performance metric k (e.g., PF or MM),

$$PoA = 1 - \frac{h_k(u)}{h_k(\varphi_k(\mathcal{U}))}. \quad (17)$$

Since φ_k maximizes h_k , we have $h_k(u) \leq h_k(\varphi_k(\mathcal{U}))$, so the ratio is at most 1 and a lower $h_k(u)$ indicates greater loss due to non-cooperative behavior. When h_k is nonnegative, $PoA \in [0, 1]$, with 0 denoting full preservation of the centralized objective and 1 its complete erosion; for metrics that may take negative values, such as proportional fairness, the bound on h_k should be verified for the application at hand. Equilibrium analyses of self-interested agents arise broadly across decision and service systems, including strategic route choice in queueing networks [33].

The Price of Choice

Rodríguez-González et al. [17] extend the Price of Fairness (PoF) and the Price of Efficiency (PoE) with the *Price of Choice* (PoC), a comparative metric that quantifies trade-offs across allocation schemes relative to a chosen baseline—whether that baseline reflects α -fairness, efficiency, or any other normative criterion. Unlike PoF or PoE, which rely on fixed ideological references (e.g., utilitarian or egalitarian), the PoC generalizes the notion of cross-scheme evaluation by allowing any allocation scheme to serve as the reference.

Let $\mathcal{K}$ denote the set of all allocation schemes under consideration. Each scheme $k \in \mathcal{K}$ produces an optimal allocation $\varphi_k(\mathcal{U})$ according to its own fairness principle. Define $\hat{\mathcal{K}} \subset \mathcal{K}$ as the subset of schemes characterized by scalar fairness metrics $h_k(\cdot)$ (e.g., Social Welfare, Proportional Fairness) and $\tilde{\mathcal{K}} \subset \mathcal{K}$ as those defined geometrically or procedurally (e.g., Kalai–Smorodinsky, Max-Min Fairness). The PoC compares an *alternative* scheme k' against a *reference* scheme k , measuring how much adopting k' would diverge from the best possible outcome of k .

If $k \in \hat{\mathcal{K}}$, the PoC is defined as:

$$PoC_{k,k'} = \frac{h_k(\varphi_{k'}(\mathcal{U})) - h_k(\varphi_k(\mathcal{U}))}{h_k(\neg\varphi_k(\mathcal{U})) - h_k(\varphi_k(\mathcal{U}))}, \quad (18)$$

where $\neg\varphi_k(\mathcal{U})$ denotes the worst-case reference allocation from the perspective of scheme k . Since a global worst case may be unbounded or ill-defined—for instance when h_k is unbounded below, or when distances over a continuous feasible set have no finite maximum—we define $\neg\varphi_k(\mathcal{U})$ as the least favorable allocation within the available pool of candidate schemes under comparison, or one deliberately constructed to be at least as unfavorable as any of them. With this reference, no compared alternative $\varphi_{k'}(\mathcal{U})$ is more unfavorable than $\neg\varphi_k(\mathcal{U})$, so $PoC_{k,k'} \in [0, 1]$ across the schemes evaluated.

For schemes without explicit scalar metrics ($k \in \tilde{\mathcal{K}}$), the PoC instead relies on normalized distance:

$$PoC_{k,k'} = \frac{\|\varphi_{k'}(\mathcal{U}) - \varphi_k(\mathcal{U})\|_{\mathcal{L}}}{\|\neg\varphi_k(\mathcal{U}) - \varphi_k(\mathcal{U})\|_{\mathcal{L}}},$$

where $\|\cdot\|_{\mathcal{L}}$ is an $\mathcal{L}$ -norm (e.g., Manhattan or Euclidean) chosen based on the problem context. As in the scalar case, $\neg\varphi_k(\mathcal{U})$ is taken as the least favorable allocation within the pool of compared schemes (here, the most distant from $\varphi_k(\mathcal{U})$), keeping $PoC_{k,k'} \in [0, 1]$ across the schemes evaluated..

By construction, the PoC normalizes the criteria across heterogeneous scales. Intuitively, it measures how far the alternative scheme’s outcome falls between the best and worst performance attainable under the reference scheme k : the deviation of $\varphi_{k'}(\mathcal{U})$ from k ’s ideal, scaled by the full range between k ’s best and worst attainable performances. Hence, $PoC = 0$ corresponds to perfect alignment with k ’s allocation logic, while $PoC = 1$ indicates maximal divergence—comparable to k ’s own worst allocation. The normalization is mathematically equivalent to a standard min–max transformation when the “best” and “worst” correspond to the extrema of a dataset, and it can flexibly accommodate both benefit-type and cost-type objectives through proper definition of these reference points.

In this sense, the PoC quantifies the “cost” of adopting one fairness paradigm over another, providing a coherent, interpretable, and ideologically neutral framework for cross-scheme comparison. It allows decision-makers to assess how sensitive fairness outcomes are to the chosen fairness notion itself—highlighting not only efficiency–equity trade-offs but also the meta-level trade-offs between competing conceptions of fairness.

2.7 Valuation Criteria in the Problem of Fair Division

Fair division criteria formalize the outcome properties of an allocation under participants’ subjective valuations and entitlement levels. Each participant $i \in \mathcal{N}$ possesses an entitlement level, which can be represented by a weight $w_i \in [0, 1]$ such that $\sum_i w_i = 1$ (equal entitlements correspond to $w_i = \frac{1}{n}$). Each participant also has a valuation function $v_i : \mathcal{X} \rightarrow [0, 1]$ that maps any feasible allocation x_i to its perceived value relative to the total resource C .

Proportionality

An allocation is proportional if

$$v_i(x_i) \geq v_i(C) \cdot w_i \quad \forall i \in \mathcal{N} \quad (19)$$

Equal entitlements yield proportional division [15]; heterogeneous w_i yield weighted proportional division [34].

Super Proportionality

A stronger requirement is

$$v_i(x_i) > v_i(C) \cdot w_i \quad \forall i \in \mathcal{N}. \quad (20)$$

Unlike proportionality, a super-proportional division need not exist: if all participants share identical valuations, the best attainable outcome gives each exactly their entitled share, so a super-proportional division exists only when at least two participants value the resource differently [15]. This condition is especially relevant in the homogeneous setting considered here, where identical per-unit valuations preclude super-proportionality.

Envy-Free Division

No participant prefers another's bundle under their own valuation [35–37]:

$$\frac{v_i(x_i)}{w_i} \geq \frac{v_i(x_k)}{w_k} \quad \forall i \neq k. \quad (21)$$

Strong Envy-Free Division

A stronger requirement is that each participant strictly prefers their own bundle [38]:

$$\frac{v_i(x_i)}{w_i} > \frac{v_i(x_k)}{w_k} \quad \forall i \neq k. \quad (22)$$

Strong envy-freeness implies both envy-freeness and super-proportionality, and exists if and only if no two participants share identical valuations.

Equitable Division

With equal entitlements, equitability requires equal perceived value across participants [39]:

$$v_i(x_i) = v_k(x_k) \quad \forall i \neq k. \quad (23)$$

Whereas envy-freeness compares each participant's valuation across bundles, equitability equalizes participants' valuations of their *own* shares, so that all participants are equally satisfied in their own terms. With heterogeneous entitlements, equitability may be infeasible alongside

entitlement guarantees.

2.8 Sensitivity Properties of Fair Allocations: Axioms of Monotonicity

Sensitivity properties articulate how fair allocations respond to exogenous changes—resources, population, and technology. Segal-Halevi and Sziklai [40] and Thomson [41] discuss the following:

1. **Resource monotonicity:** With more total resources, the resulting allocation (under the same fairness rule) should not make participants worse off; with fewer, not better off.
2. **Population monotonicity:** If the population shrinks, remaining participants should be weakly better off; if it grows, original participants should not be strictly better off solely due to entry.
3. **Technology monotonicity:** If the allocation possibilities expand—through an improved production technology or a richer set of feasible divisions—no participant should be made worse off under the same rule.

These properties need not always cohere with other desiderata. For a heterogeneous connected cake, Segal-Halevi and Sziklai [42] show that each monotonicity property is incompatible with proportionality and Pareto optimality: no Pareto-optimal proportional rule can be resource- or population-monotonic. In the homogeneous, disconnected setting, the behavior of utilitarian (social welfare) allocations under changing resources depends on the curvature of utilities. When utilities are concave (diminishing returns), the utilitarian optimum tends to spread additional resources across participants, so increasing the total pool does not make any participant worse off; when utilities are convex (increasing returns), the optimum concentrates resources on the most efficient converters, and enlarging the pool may reallocate away from some participant, violating resource monotonicity.

3 A Unified Optimization Formulation

The schemes, metrics, and criteria reviewed above share a common computational skeleton, built directly from the notation introduced Section 2. Recall that $\mathcal{U}$ denotes the induced feasible utility set, $\bar{u}$ the vector of utopia (component-wise maximal) utilities, and $\underline{u}$ the vector of worst-case (component-wise minimal) utilities attainable for each participant $i \in \mathcal{N}$ (Eq. (1)). Every scheme returns an allocation $\varphi(\mathcal{U}) \in \mathcal{U}$ by optimizing some fairness metric $h_k(\cdot)$ over this set. Making the constraints that define $\mathcal{U}$ explicit, and writing d_i for participant i 's upper attainable utility (i.e., $d_i \equiv \bar{u}_i$ under linear, demand-capped utilities, as in the proof-of-concept that follows) and $\underline{u}_i$ for the corresponding lower bound, every allocation problem considered in this tutorial can be written, without loss of generality, in the canonical form

$$\max(\text{or } \min) \ h_k(u) \quad \text{s.t.} \quad u_i \leq d_i \ \forall i \in \mathcal{N}, \quad \sum_{i \in \mathcal{N}} q_i u_i \leq C, \quad u_i \geq \underline{u}_i \ \forall i \in \mathcal{N} \quad (24)$$

where h_k denotes the allocation scheme’s fairness or efficiency metric. Here, C is the total available resource (the “cake” of Section 2), and q_i is participant i ’s unit consumption rate—the resource cost of delivering one unit of utility to i —which equals 1 for all participants unless otherwise specified; the proof-of-concept in Section 4 later shows how making q_i heterogeneous across participants can break scheme equivalences established under uniform costs. The valuation-based fair-division criteria reviewed above—proportionality, super-proportionality, envy-freeness, strong envy-freeness, and equitability—are optional and can each be incorporated within this same formulation as an additional constraint on u , appended to whichever objective h_k the modeler adopts, whenever the modeler chooses to enforce that criterion; the proof-of-concept in Section 4 illustrates this with $h_k = h_{SW}$, though any objective could be paired with a VFC in the same way. We refer to any such criterion generically as a valuation-based fairness criteria (VFC).

This formulation is what allows fairness, ordinarily treated as a normative or axiomatic question, to be handled with the standard tools of mathematical programming: schemes become objective functions, fairness criteria become constraints, and comparisons across schemes become comparisons across optimal solutions to structurally identical problems. The proof-of-concept in Section 4 instantiates this formulation directly for a stylized cloud computing credit-allocation scenario, where d_i is realized concretely as each participant’s reported demand forecast.

4 Computational Illustration

To demonstrate the operational use of the fairness notions and concepts presented thus far—without committing to domain-specific policies, network topologies, or institutional hierarchies—we adopt a neutral resource: cloud compute credits. The purpose of the example is illustrative rather than empirical: it is designed to expose boundary behavior, equivalences, and divergences among fairness concepts under controlled assumptions. The illustration is fictitious and does not attempt to encompass every element discussed previously (such as the full set of axioms or equilibrium properties), and instead seeks to capture the essential aspects of fair allocation.

Concretely, a platform administrator holds a fixed budget of compute credits and must divide it among competing internal projects, each reporting a forecast of the credits it can productively consume over the allocation period. Cloud credits are a practical yet abstract resource: each credit represents a standardized unit of computational capacity (e.g., CPU-hours or GPU-minutes), and they satisfy the assumptions of Section 2—they are *homogeneous* (one unit is functionally identical to any other), *divisible* (partial credits, e.g., 0.5, can be allocated), and *disconnected* (several non-contiguous allocations are equivalent in value to a single allocation of the same total size). The credit budget plays the role of the cake, the projects are the participants, and each project’s demand forecast bounds its attainable utility, while the setting remains deliberately stylized so that the boundary behavior of each fairness scheme can be isolated and compared.

Setup. We retain the notation established in Section 2 and instantiate the canonical formulation of Section 3 concretely. The total credit pool—analogous to the total cake—is defined as $C = 18$

(in thousands of credits). Let $\mathcal{N} := \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}^1$, with $|\mathcal{N}| = 6$, denote the set of participants to be allocated shares of C . We also include an artificial participant ω to absorb any credits that remain unassigned (e.g., prudential reserve or feasibility leftovers). Let $\mathcal{X} \subseteq \mathbb{R}_{\geq 0}^n$ be the set of feasible allocations, and let $x_i \in \mathcal{X}$ denote the credits assigned to participant $i \in \mathcal{N}$. Each participant provides a demand forecast d_i (Table 1); utilities are linear and capped at demand,

$$u_i(x_i) = \min\{x_i, d_i\},$$

so that $\underline{u}_i = 0$ and $\bar{u}_i = d_i$. Entitlement weights are $w_i \in [0, 1]$ with $\sum_i w_i = 1$. Valuations are the demand-normalized utilities, $v_i(x_i) = u_i(x_i)/d_i = \min\{\frac{x_i}{d_i}, 1\}$, so that $v_i \in [0, 1]$ with $v_i = 1$ when a participant's demand is fully met.

Table 1: Participants' demand forecasts (in thousands).

Participant	d_i
α	2
β	3
γ	4
δ	2.5
ϵ	5
ζ	7

Scenarios. We analyze six scenarios that exercise utility-based schemes, VFCs, inequality measures, and sensitivity principles, while keeping assumptions minimal. In every VFC formulation across all six scenarios, *Social welfare* serves as the overarching objective while the corresponding fair-division criterion is imposed as an additional constraint; per Section 3, any objective h_k could serve this role in general, and we adopt h_{SW} throughout this proof-of-concept for consistency.

1. *Equal entitlements (baseline)*: allocations are computed under each utility-based scheme and each VFC using an optimization model (details provided below). To assess distributional outcomes, we report both Jain's and Gini indices, computed over the absolute utility allocations of the n real participants in $\mathcal{N}$; the artificial sink ω is excluded from all inequality computations, as it represents unassigned residual rather than a participant whose welfare is being evaluated. We also evaluate the Price of Choice (PoC) across schemes and VFCs to quantify cross-scheme deviations under each scheme's own evaluative logic.
2. *Heterogeneous unit costs*: equal entitlements, but participant ϵ 's workloads require premium instances; each unit of utility for ϵ consumes two credits, $q_\epsilon = 2$ (others $q_i = 1$). This breaks scheme equivalences that depend on uniform consumption rates.
3. *Heterogeneous entitlements (baseline costs)*: participants have different entitlement weights w_i (e.g., by scope/criticality). We allocate under each VFC and study inequality metrics.

¹Throughout the proof-of-concept, $\alpha, \beta, \gamma, \delta, \epsilon$, and ζ are participant labels and are unrelated to the α -fairness parameter of Section 2.4; the overloading of α is purely nominal.

Table 2: Acronym encoding for each utility allocation scheme.

Utility allocation scheme	Acronym
Social Welfare	SW
Kalai–Smorodinsky	KS
Max-Min	MM
Proportional Fairness	PF
Theil Index (Entropy)	E
Max-Min Fairness	MMF

Table 3: Numerical encoding for each VFC.

VFC	Encoding
Proportionality	I
Super Proportionality	II
Envy Freeness	III
Strong Envy Freeness	IV
Equitability	V

4. *Resource monotonicity (baseline entitlements)*: vary the total credits C and trace scheme responses.
5. *Population monotonicity (baseline entitlements)*: change $|\mathcal{N}|$ by removing a participant or adding a new participant η with demand d_η .
6. *Price of Anarchy*: one participant strategically inflates its stated demand; we quantify efficiency loss for Social Welfare and Proportional Fairness relative to truthful baselines.

Schemes, criteria, and encodings. Acronyms for schemes and numerical encodings for VFCs are summarized in Tables 2–3. We use the standard n -way extension of Kalai–Smorodinsky by equating proportional gains:

$$\varphi_{KS}(\mathcal{U}) \Rightarrow \frac{u_1 - \underline{u}_1}{\bar{u}_1 - \underline{u}_1} = \frac{u_2 - \underline{u}_2}{\bar{u}_2 - \underline{u}_2} = \dots = \frac{u_n - \underline{u}_n}{\bar{u}_n - \underline{u}_n}.$$

Canonical optimization form. Every scenario that follows is an instance of the canonical formulation introduced in Section 3 (Eq. (24)), with $q_i = 1$ unless otherwise specified (as in Scenario 2).

Modeling strict inequalities. Strict VFCs are implemented using small tolerance parameters to maintain numerical stability while preserving generality. We set $\tau = 0.05$ for *Super proportionality* and *Strong envy freeness*:

$$v_i(x_i) \geq v_i(C) w_i + \tau \quad \forall i \in \mathcal{N}, \quad \frac{v_i(x_i)}{w_i} \geq \frac{v_i(x_k)}{w_k} + \tau \quad \forall i \neq k.$$

Since C exceeds every demand forecast, $v_i(C) = 1$ under the normalization, so the proportional-share term reduces to w_i .

These tolerance-based relaxations allow strict fairness constraints to be represented within a mixed-integer or continuous optimization framework without numerical infeasibility. For broader optimization-based treatments of fairness–efficiency trade-offs, see Karsu and Morton [43], Chen and Hooker [44], and Ogryczak et al. [45].

Computation. All scenarios were implemented in Python and solved using the Gurobi Optimizer (version 9.1.2) [46]. Runs were performed on a machine with an Intel Core i7 processor and 16GB RAM. *Theil Index (Entropy)* was handled via `scipy.optimize` SLSQP for smooth constrained non-linear programs [47].

4.1 Scenario One: Equal Entitlements — Schemes, VFCs, PoC, and Inequality Measures

With $w_i = \frac{1}{n}$, Figure 1 shows allocations by scheme; Figure 2 shows allocations under each VFC. Tables 4–5 report Jain’s and Gini values; Table 6 indicates which VFCs each scheme satisfies.

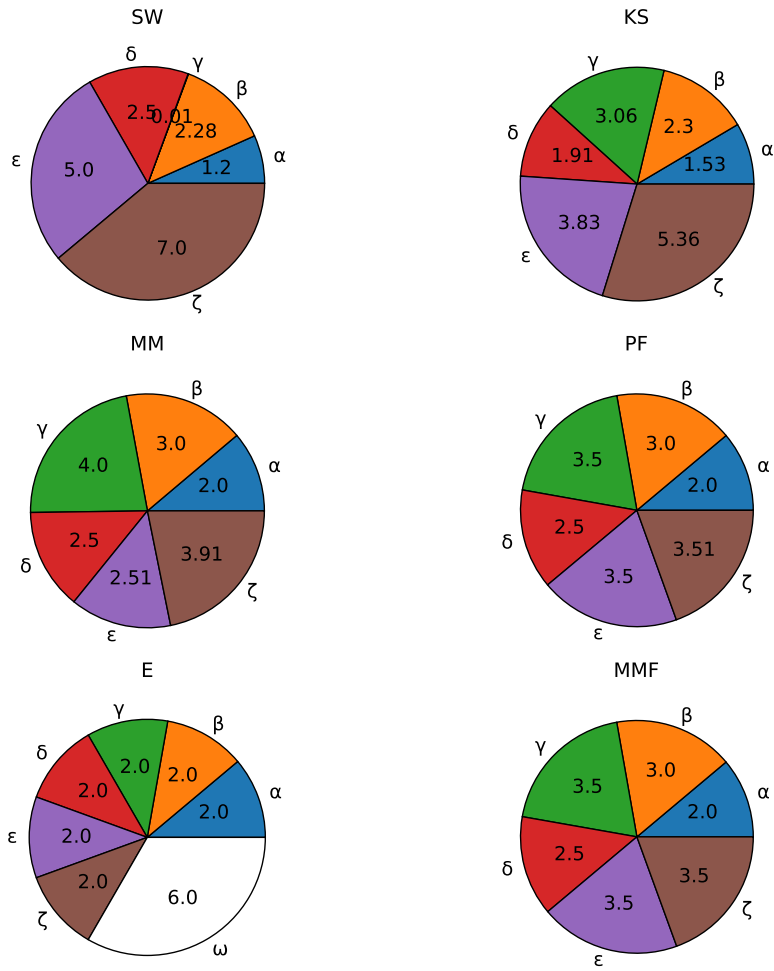


Figure 1: Allocations by utility-based schemes over the credit pool C .

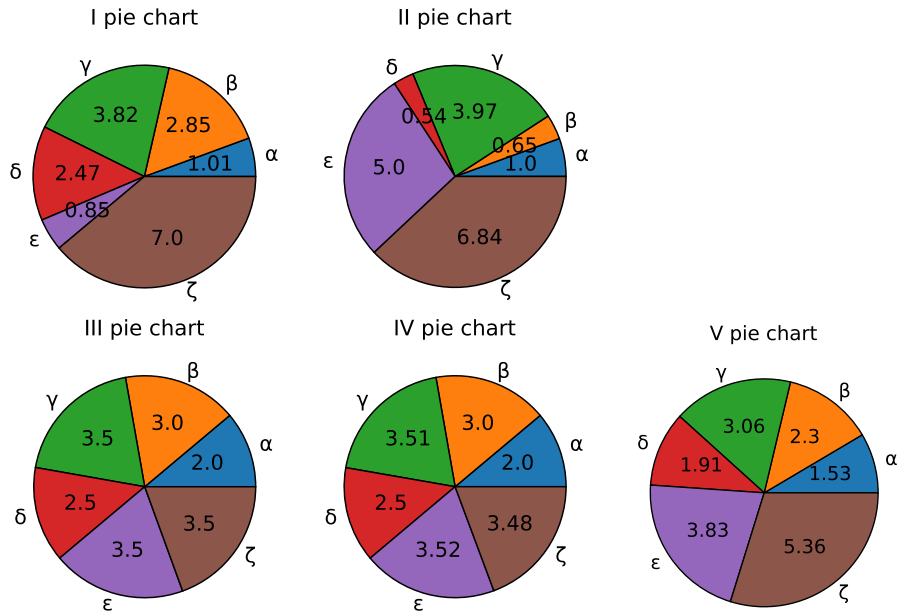


Figure 2: Allocations under VFCs with equal entitlements.

Table 4: Jain’s index and Gini coefficient by utility-based scheme.

$\varphi_f(\mathcal{U})$	Jain’s index	Gini coefficient
Social Welfare	0.62	0.43
Kalai–Smorodinsky	0.84	0.24
Max-min	0.94	0.14
Proportional Fairness	0.96	0.10
Entropy	1.00	0.00
Max-min fairness	0.96	0.10

Table 5: Jain’s index and Gini coefficient by VFC.

VFC	Jain’s index	Gini coefficient
Proportionality	0.68	0.37
Super Proportionality	0.61	0.44
Envy Freeness	0.96	0.10
Strong Envy Freeness	0.96	0.10
Equitability	0.84	0.24

Reading the patterns. The schemes express distinct doctrines. *Proportional fairness* and *Max-min fairness* often look similar because both “lift the floor” while balancing gains; they are not equivalent in general, as Scenario Two will show. *Theil index (entropy)* equalizes utilities exactly, achieving *Jains* = 1 and *Gini* = 0, but can under-utilize the cake C when the equalized utility level is capped by the smallest demand; a convenient closed form for this instance is

$$u_i = \min \left\{ \frac{C}{|\mathcal{N}|}, \min_{k \in \mathcal{N}} d_k \right\} \quad \forall i \in \mathcal{N},$$

with the remainder assigned to ω .

Kalai–Smorodinsky equalizes percentage coverage of demand (proportional gains toward $\bar{u}$), aligning naturally with the *Equitability* VFC. *Social welfare*, solved via the simplex method [48], selects a basis that improves the linear objective most sharply; this can look “irregular” across participants yet remains consistent with maximizing $\sum_i u_i$.

Price of Choice. We compute the PoC using the $\mathcal{L} = 1$ (Manhattan) distance between optimal utility allocation vectors, applying the geometric form of Section 2.6 uniformly across all schemes so that scalar-metric and procedurally-defined schemes are compared on a common basis. Figure 3 shows PoC using each scheme as the reference against all others and the VFC-induced allocations. Unsurprisingly, relative to *Social welfare* (which already maximizes $\sum_i u_i$), many alternative choices have little marginal impact on h_{SW} —except where a scheme deliberately trades utilization for equality (e.g., *Theil index (entropy)*).

Table 6: VFC satisfaction by scheme (equal entitlements).

$\varphi_f(\mathcal{U})$ vs. VFC	I	II	III	IV	V
Social Welfare	×	×	×	×	×
Kalai–Smorodinsky	✓	✓	×	×	✓
Max-min	✓	✓	×	×	×
Proportional Fairness	✓	✓	✓	×	×
Theil Index (Entropy)	✓	✓	✓	×	×
Max-Min Fairness	✓	✓	✓	×	×

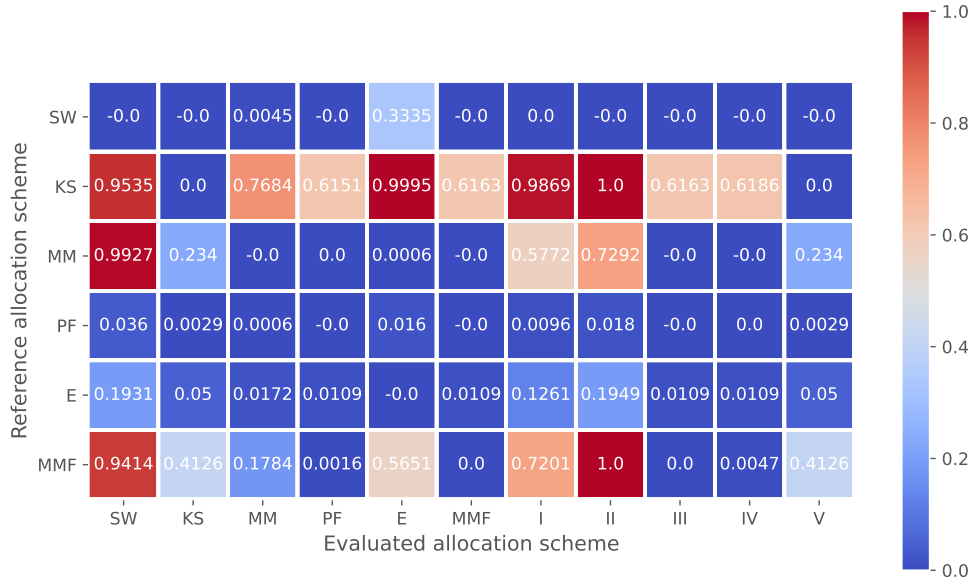


Figure 3: Price of Choice heatmap (Scenario One).

Takeaway. No single metric dominates. The *Theil index (entropy)* achieves perfect absolute equality but may sacrifice aggregate utility; *Social Welfare* maximizes utilization but can violate fairness criteria (VFCs). *Proportional Fairness* and *Max-min Fairness* yield robust *Envy Freeness* in this instance, whereas *Kalai–Smorodinsky* aligns most closely with *Equitability* (relative equality). For applied settings, this underscores the role of inequality indices as diagnostic tools rather than as primary optimization objectives. Notably, if Jain’s or Gini indices were computed using relative valuations instead of absolute utilities, they would attain their ideal values under the *Kalai–Smorodinsky* solution—consistent with its equitability principle—while remaining non-optimal under the other schemes.

4.2 Scenario Two: Heterogeneous Unit Costs — Breaking Scheme Equivalences

We now set $q_\epsilon = 2$ (premium instance consumption), with other $q_i = 1$. The capacity constraint becomes

$$u_\alpha + u_\beta + u_\gamma + u_\delta + 2u_\epsilon + u_\zeta \leq C.$$

Figures 4–6 and Tables 7–9 summarize results.

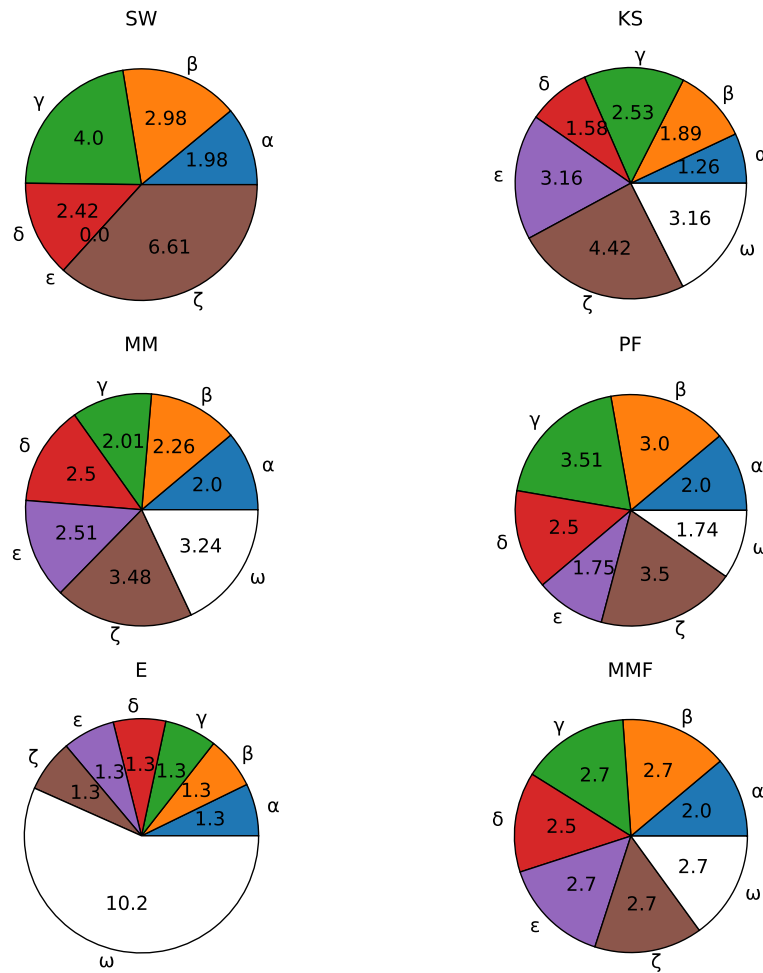


Figure 4: Allocations by scheme with $q_\epsilon = 2$.

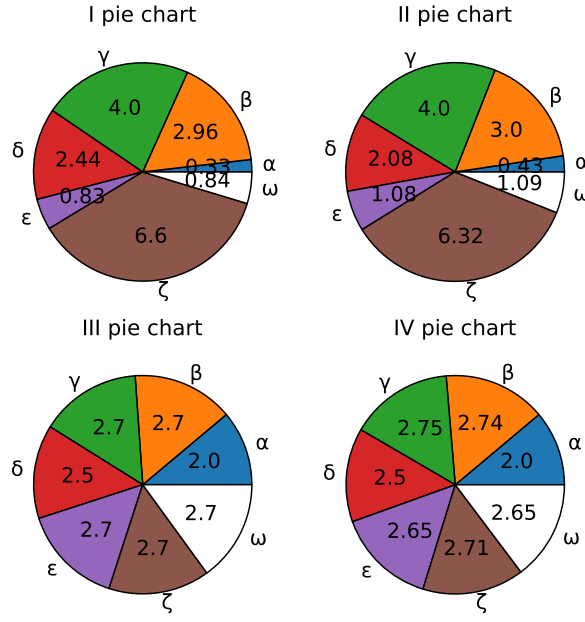


Figure 5: Allocations under VFCs with $q_\epsilon = 2$.

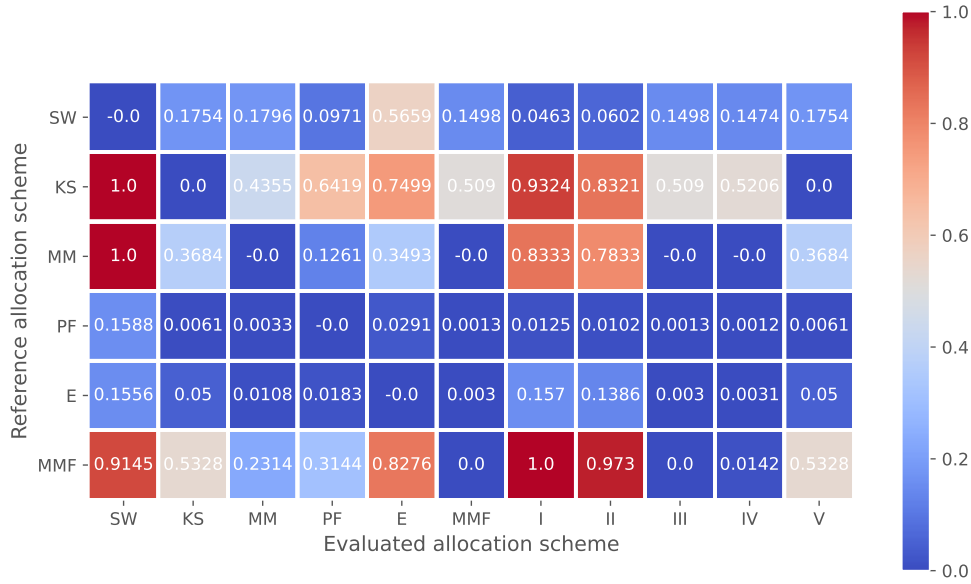


Figure 6: Price of Choice heatmap (Scenario 2).

What changes and why. Under *Social welfare*, assigning credits to participant ϵ is dominated (two credits per one unit of utility), so $u_\epsilon = 0$. *Proportional fairness* and *Max-min fairness* that looked similar in Scenario 1 now diverge: *Proportional fairness* scales ϵ 's allocation by $1/q_\epsilon$ and loses *Envy freeness*; *Max-min fairness* lifts the minimum utility lexicographically and then shares residuals, preserving *Envy freeness*. *Kalai-Smorodinsky* continues to equalize percentage coverage of demands, but the common percentage drops because serving ϵ is credit-expensive; Leftover resources accrue to ω when equal coverage cannot to fully utilize C . *Theil index (entropy)*, once again, perfectly equalizes utilities, now at a lower common level, with substantial residual assigned to ω .

Table 7: Jain’s index and Gini coefficient by scheme ($q_\epsilon = 2$).

$\varphi_f(\mathcal{U})$	Jain’s index	Gini coefficient
Social Welfare	0.69	0.37
Kalai–Smorodinsky	0.84	0.24
Max-min	0.96	0.10
Proportional Fairness	0.94	0.14
Theil Index (Entropy)	1.00	0.00
Max-Min Fairness	0.99	0.04

Table 8: Jain’s index and Gini coefficient by VFC ($q_\epsilon = 2$).

VFC	Jain’s index	Gini coefficient
Proportionality	0.65	0.40
Super Proportionality	0.68	0.39
Envy Freeness	0.99	0.04
Strong Envy Freeness	0.99	0.05
Equitability	0.84	0.24

PoC and inequality. Relative PoC values shift upward when *Social welfare* is the alternative: most schemes allocate something to participant ϵ , but *Social welfare* does not. *Jain/Gini* tend to improve or remain stable because the effective pool available to satisfy utilities shrinks (resources “spent” on ϵ ’s unit cost), dampening dispersion.

4.3 Scenario Three: VFCs with Heterogeneous Entitlements

Keeping unit costs uniform, we vary entitlements w_i (Table 10) and allocate under each VFC. Figure 7 shows allocations; Table 11 reports inequality indices. As expected, shares x_i scale with w_i . *Equitability* can become infeasible with heterogeneous entitlements, consistent with the caveat in Section 2.7.

Table 9: VFC satisfaction by scheme ($q_\epsilon = 2$).

$\varphi_f(\mathcal{U})$ vs. VFC	I	II	III	IV	V
Social Welfare	×	×	×	×	×
Kalai–Smorodinsky	✓	✓	×	×	✓
Max-min	✓	✓	×	×	×
Proportional Fairness	✓	✓	×	×	×
Entropy	✓	×	✓	×	×
Max-min fairness	✓	✓	✓	×	×

Table 10: Participants' entitlements to the total credit pool.

Participant i	w_i
α	10%
β	8%
γ	7%
δ	15%
ϵ	10%
ζ	50%

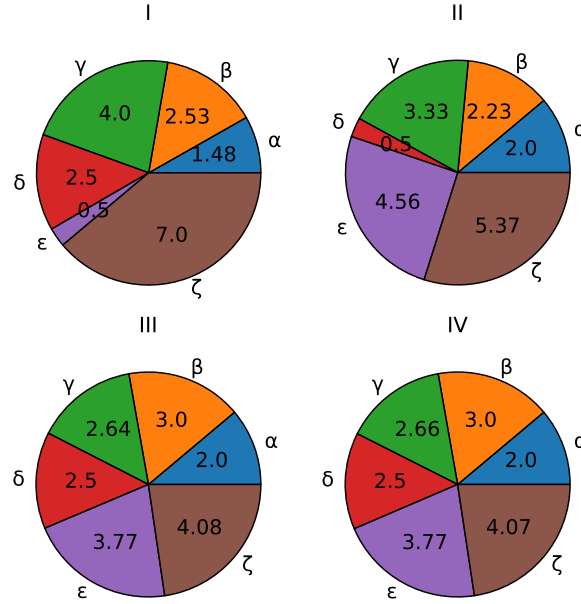


Figure 7: Allocations under VFCs with heterogeneous entitlements.

Table 11: Jain’s index and Gini coefficient under heterogeneous entitlements.

Fairness criterion	Jain’s index	Gini coefficient
Proportionality	0.67	0.37
Super Proportionality	0.77	0.31
Envy Freeness	0.94	0.14
Strong Envy Freeness	0.95	0.13
Equitability	0.34	0.57

4.4 Scenario Four: Resource Monotonicity

We vary C and trace allocations (legend in Figure 8; results in Figure 9).

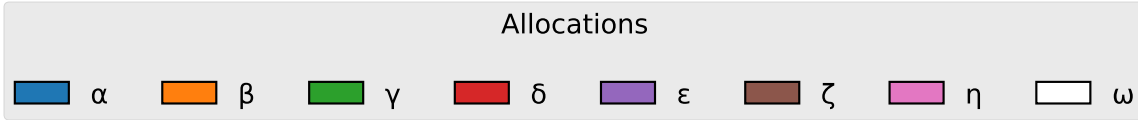


Figure 8: Legend for sensitivity plots in Figures 9 and 10.

Most schemes display resource-monotone behavior: as C increases, no participant becomes worse off. Specifically, *Kalai–Smorodinsky*, *Proportional fairness*, *Max-min fairness*, *Envy freeness*, *Strong envy freeness*, and *Equitability* are resource-monotone throughout the full range of C ; participant-level allocations supporting this are reported in Appendix A (Table 14 and 15).

Theil index (entropy) is also resource-monotone, but exhibits a distinctive plateau: all participant allocations are equalized at $u_i = \min\{C/|\mathcal{N}|, \min_k d_k\} = 2$ from $C = 12$ onward, at which point the per-participant equalization level reaches $\min_i d_i = 2$ and can no longer increase. Additional credits beyond $C = 12$ accumulate entirely in the residual ω rather than being distributed—a structural consequence of the entropy objective, which maximizes equality and has no incentive to utilize capacity beyond the equalization point.

Max-min is resource-monotone in participant allocations but similarly leaves residual capacity in ω at intermediate C values (Table 14, e.g. $\omega = 0.08$ at $C = 18$). This occurs because the Max-min objective optimizes only the minimum utility across participants and provides no gradient signal to utilize remaining capacity once the worst-off participant has been served at their demand cap; any residual beyond that threshold is irrelevant to the objective and remains unassigned.

By contrast, three schemes violate resource monotonicity: *Social welfare* and the *Proportionality* and *Super proportionality* criteria. As visible in both Figure 9 and Tables 14–15, their participant-level bands fluctuate rather than growing smoothly—particularly at low-to-moderate C —with individual allocations dropping as C increases (e.g., α under *Social welfare* falls from

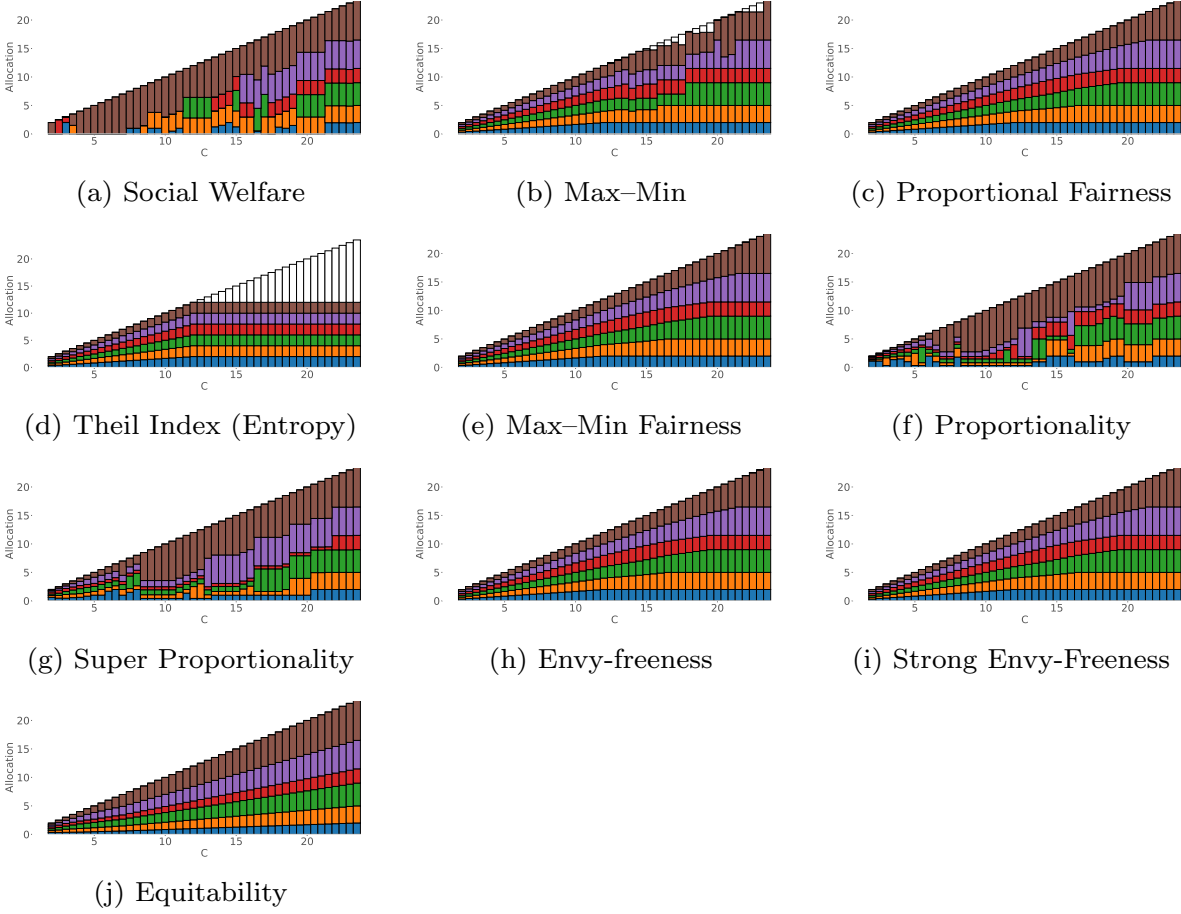


Figure 9: Sensitivity to the total cake C across schemes.

2.00 at $C = 12$ to 0 at $C = 14$, and α under *Proportionality* falls from 2.00 at $C = 6$ to 0.33 at $C = 10$). For *Social welfare* as a standalone scheme, this reflects the multiplicity of optima in a linear objective: the simplex method selects an extreme point of the feasible region, which can shift discontinuously as C expands, reallocating resources lumpily across participants with no mechanism to protect incumbents. For the *Proportionality* and *Super proportionality* criteria, the same mechanism operates: both are solved by maximizing *Social welfare* subject to their respective constraints (Section 4.1), so the underlying utilitarian objective inherits the same non-monotone behavior. Because proportionality and super-proportionality impose only lower bounds on participants' shares, the objective retains substantial freedom to reshuffle remaining capacity, and violations arise for the same structural reason as in pure *Social welfare*. All three schemes converge to the demand-saturated allocation at $C = 23.5$, where every participant receives their full demand.

4.5 Scenario Five: Population Monotonicity

To analyze *population monotonicity*, we first remove participant α to study population shrinkage and then restore the original instance and introduce a new participant η with demand $d_\eta = 10$ (thousand units of resource) to examine population growth. The detailed participant-level allocations are reported in Appendix B (Table 16) and illustrated in Figure 10. Recall that population monotonicity requires all incumbent participants to move in the same direction when the population changes: following the departure of a participant, no remaining participant

should be worse off, whereas after the arrival of a new participant, no incumbent should become better off.

Most allocation rules satisfy this property in our instance. Specifically, *Kalai–Smorodinsky*, *Proportional fairness*, *Max-min*, *Max-min fairness*, *Theil index (entropy)*, *Envy freeness*, *Strong envy freeness*, and *Equitability* are population-monotone: incumbent participants weakly improve when α exits and weakly deteriorate (or remain unchanged) when η enters. These schemes differ only in how they accommodate the newcomer. Some preserve incumbent allocations as much as possible, whereas the envy-based and coverage-based schemes distribute the reduction more evenly across participants, producing the nearly uniform compression observed in panels (i)–(j) of Figure 10. A notable case is the *Max-min* scheme: when α exits, the resulting allocation leaves 1.76 thousand credits unassigned in ω (Table 16). This occurs because the Max-min objective optimizes only the minimum utility across participants and provides no gradient signal to utilize remaining capacity once the worst-off participant has been served at their demand; any residual beyond that point is irrelevant to the objective and is left in ω . Similarly, the *Theil index (entropy)* scheme systematically under-utilizes the pool, assigning each participant only $\min\{C/|\mathcal{N}|, \min_k d_k\}$ and leaving the remainder in ω across all population configurations.

Only three schemes violate population monotonicity, and notably they are among the same criteria that fail resource monotonicity in Scenario 4.4: *Social welfare*, *Proportionality*, and *Super proportionality*. When α leaves, *Social welfare* reduces δ ’s allocation from 2.50 to 0, despite the decrease in population, and *Proportionality* lowers γ ’s allocation from 3.82 to 1.00. Conversely, when η enters, *Super proportionality* increases α ’s allocation from 1.00 to 2.00 even though the population has grown. These violations arise from the model structure described in Section 4.1. In each case, the corresponding valuation-based criterion is imposed as a constraint while *Social welfare* is retained as the objective. Because proportionality and super-proportionality provide only lower bounds on participants’ allocations, the utilitarian objective retains sufficient flexibility to redistribute the remaining capacity and thereby inherits the non-monotonic behavior of the *Social welfare* solution itself. By contrast, envy-freeness and equitability constrain the relative structure of the allocation strongly enough to prevent such adverse reallocations. Hence, population monotonicity in this setting is governed by the extent to which each criterion restricts the common utilitarian objective, indicating that the observed violations are structural rather than artifacts of the particular instance considered.

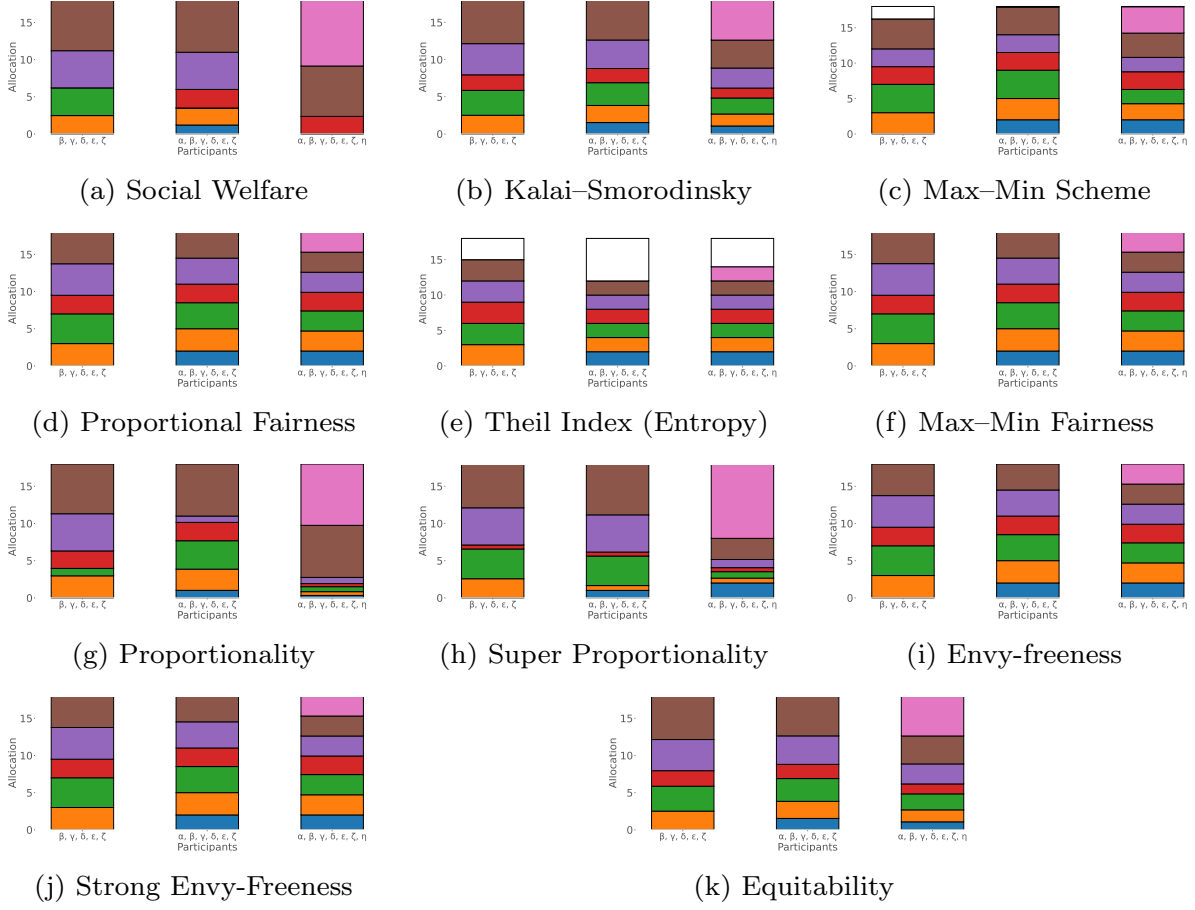


Figure 10: Sensitivity to population size $|\mathcal{N}|$ across schemes.

4.6 Scenario Six: Strategic Overstatement and PoA-Inspired Efficiency Loss

Suppose participant α inflates d_α from 2 to 7 while $C = 18$. Table 12 reports allocations under truthful and deceitful inputs for *Social welfare* and *Proportional fairness*; Table 13 reports the resulting PoA.

Table 12: Participant-level allocations under truthful and misreported demand (d_α inflated from 2 to 7). *Optimizer output* shows what the solver assigns under the reported demand; *productive consumption* shows what each participant can actually use (capped at true demand d_i). The difference for α under misreporting is waste—credits allocated but not productively consumable. Productive utilization is the ratio of total productive consumption to $C = 18$.

Scheme	Input	α	β	γ	δ	ϵ	ζ	α waste	Productive util. %
<i>Social welfare</i>	Truthful	1.20	2.28	0.01	2.50	5.00	7.00	0.00	100.00
	Misreported ($d_\alpha = 7$)	2.00	1.00	3.99	1.01	5.00	0.00	5.00	72.22
<i>Proportional fairness</i>	Truthful	2.00	3.00	3.50	2.50	3.50	3.51	0.00	100.00
	Misreported ($d_\alpha = 7$)	2.00	3.00	3.14	2.50	3.12	3.13	1.12	93.83

Both schemes are solved under the misreported demand $d_\alpha = 7$; the optimizer treats this as α 's productive capacity and allocates the full pool $C = 18$ accordingly. The efficiency loss arises because α 's true productive demand is only $d_\alpha^{\text{true}} = 2$: any allocation beyond 2 cannot be

Table 13: PoA-inspired efficiency loss from unilateral demand overstatement, measured relative to truthful centralized optima per (17).

PoA_{SW} (%)	PoA_{PF} (%)
27.78	5.22

productively consumed and is effectively wasted. The allocation received by other participants therefore reflects a world in which part of the cake was squandered by the misreporting agent.

Under *Social welfare*, the optimizer awards α all 7 units claimed. Of these, only 2 are productively consumed; the remaining 5 are wasted, reducing productive utilization to 72.22% and inducing a 27.78% efficiency loss. The distortion to others is severe: ζ 's allocation collapses from 7.00 to 0 and β 's from 2.28 to 1.00—changes more adverse than any legitimate reduction in C would produce, since resource monotonicity (Scenario 4.4) protects incumbents under genuine pool changes but offers no protection against strategic misreports. α 's realized productive gain is modest: only +0.80 units (1.20 $\rightarrow$ 2.00), achieved at substantial cost to others.

Under *Proportional fairness*, the optimizer allocates 3.12 units to α ; only 2 are productively consumed, wasting 1.12 units and reducing productive utilization to 93.83%. Crucially, α 's productive allocation is identical under truthful and misreported inputs (2.00 in both cases): the misreport yields *no productive gain for the manipulator*. Other participants absorb small losses, and the log-product objective degrades by 5.22%. The contrast with *Social welfare* is sharp: *Proportional fairness* removes the incentive to misreport entirely, while *Social welfare* rewards it with a +0.80 unit gain at substantial cost to others.

Strictly speaking, this scenario does not compute a full Price of Anarchy over equilibrium behavior. Rather, it reports a PoA-inspired loss induced by unilateral demand overstatement relative to the truthful centralized benchmark.

4.7 Key Insights

First, modeling choices matter. Homogeneous, disconnected resources make it natural to use the cake-cutting lens and to operationalize fairness via scheme-specific objectives with VFCs as constraints. The artificial sink participant ω serves as a pragmatic device for absorbing unallocated residual, arising from three distinct causes: equality targets that plateau before exhausting C (*Theil index (entropy)*), feasibility slack or tie-breaking, and objectives that provide no gradient signal to utilize capacity beyond the worst-off participant's demand (*Max-min*).

Second, no single fairness doctrine dominates. *Social welfare* maximizes aggregate utility; *Kalai–Smorodinsky* maximizes equal percentage coverage of demand, ensuring *Equitability*; *Proportional fairness* and *Max-min fairness* balance equity and efficiency and, under equal unit costs and equal entitlements, secure *Envy freeness*; the *Theil index (entropy)* achieves perfect equality but may leave resources unutilized. Inequality measures such as Jain's index and the Gini coefficient are best interpreted as diagnostics rather than optimization objectives: Jain's index $J \in [\frac{1}{n}, 1]$ equals $J = \frac{k}{n}$ when k of n participants share equally and the remainder receive nothing, and both indices must specify whether they are computed over absolute allocations or relative valuations, as their interpretation and ideal values differ accordingly.

Third, sensitivity principles are decision-relevant and structurally informative. The same three schemes—*Social welfare*, *Proportionality*, and *Super proportionality*—violate both resource monotonicity (Scenario 4.4) and population monotonicity (Scenario 4.5) for the same structural reason: all three are governed by a utilitarian objective subject to only weak fairness constraints, which leaves the optimizer free to reshuffle capacity non-monotonically as parameters change. Structured schemes (*Kalai–Smorodinsky*, *Proportional fairness*, *Max-min fairness*, *Envy freeness*, *Strong envy freeness*, *Equitability*) satisfy both axioms throughout. Separately, heterogeneous unit costs (q_i) can break apparent scheme equivalences and alter VFC satisfaction, as *Proportional fairness* losing *Envy freeness* under $q_e = 2$ illustrates (Scenario 4.2).

Finally, strategic behavior has measurable and scheme-dependent costs. Under *Social welfare*, demand overstatement is rewarded: α gains +0.80 productive units at substantial cost to others, with 5.00 units wasted and a 27.78% efficiency loss. Under *Proportional fairness*, overstatement yields the manipulator no productive gain whatsoever—the misreport distorts the allocation but confers no benefit to α —while still imposing a 5.22% system-wide loss and wasting 1.12 units. This contrast illustrates that robustness to strategic behavior is a structural property of the fairness objective, not a general feature of fairness-aware allocation. The Price of Choice complements these diagnostics by enabling ex ante comparison of schemes under their own evaluative logic, helping practitioners identify allocations that are computationally tractable, fairness-defensible, and manipulation-resistant.

5 Conclusions

We presented a structured tutorial on fairness in resource division and allocation, translating theoretical fairness concepts into operational decision-making settings. We focused on the allocation of homogeneous, divisible, and disconnected resources, employing the cake-cutting analogy as both a conceptual and mathematical foundation. Through this lens, we outlined utility allocation schemes, formal fairness metrics, and inequality indices such as the Gini coefficient and Jain’s index, situating them within a broader efficiency–fairness trade-off perspective, and showed that these schemes and criteria reduce to a single unified optimization formulation, in which fairness doctrines become objective functions and fair-division criteria become constraints on a common feasible set.

We examined core measures of efficiency loss and divergence across fairness logics—the Price of Fairness, the Price of Efficiency, and the Price of Anarchy—and employed the Price of Choice (PoC) [17] as a comparative metric to quantify the alignment or tension between alternative allocation schemes under each scheme’s own evaluative logic. Our analysis also revisited classical fair-division criteria based on participants’ subjective valuations, including proportionality, super-proportionality, envy-freeness, strong envy-freeness, and equitability, and examined sensitivity principles governing how allocations respond to changes in resource availability, population size, and unit costs.

A proof-of-concept grounded in cloud-compute credit allocation illustrated how these notions can be operationalized. Beyond confirming well-known properties of individual schemes, the scenarios surfaced structural findings that cut across the analysis. The same three constructs—*Social welfare*, *Proportionality*, and *Super proportionality*—violated both resource and population monotonicity for a common structural reason: a utilitarian objective subject to only weak fairness

constraints leaves the optimizer free to reshuffle capacity non-monotonically as parameters change, whereas structured schemes with tighter constraints consistently satisfied both axioms. The strategic overstatement scenario further revealed that this structural distinction extends to manipulation resistance: *Social welfare* rewards misreporting with a realized productive gain, while *Proportional fairness* removes the incentive entirely, yielding the manipulator no productive benefit while still incurring a system-wide loss.

For Operations Research practitioners, the central implication is that fairness should not be invoked as a generic modeling aspiration. It must be specified operationally—as an objective, a constraint, a diagnostic metric, or a sensitivity requirement—because each choice encodes a different normative and computational commitment, with measurable consequences for efficiency, inequality, monotonicity, and robustness to strategic behavior.

This work consolidates diverse perspectives on fairness into a single tutorial reference. It emphasizes that fairness is not a single definable target but a multidimensional construct shaped by context, constraints, and competing ethical priorities. Several directions remain open. The most immediate is extending this analysis to settings where participants’ utility functions are uncertain or heterogeneous across units—as the subjective valuation model in Section 2 anticipates—since fairness guarantees under known utilities may break down under uncertainty in ways the current metrics do not capture. Beyond that, extensions to heterogeneous, indivisible, and connected resource environments, and to decentralized settings where global coordination and local autonomy coexist, would elevate fairness from a static modeling assumption to a dynamic guiding principle for equitable, context-aware resource allocation.

6 Declarations

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Competing Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. Samuel Rodríguez-González is employed as an Operations Research Scientist at American Airlines; this work was conducted independently of that employment and does not reflect the views or positions of American Airlines.

Data Availability

This study did not analyze real-world or proprietary data. The proof-of-concept in this paper uses stylized, fictitious data constructed by the authors for illustrative purposes. The Python and Gurobi implementation used to generate the computational results is available from the corresponding author upon reasonable request.

Author Contributions

Samuel Rodríguez-González: Conceptualization, Methodology, Software, Formal Analysis, Writing – Original Draft, Writing – Review & Editing, Visualization. Andrés D. González: Conceptualization, Methodology, Supervision, Writing – Review & Editing, Funding Acquisition. Camilo Gómez: Conceptualization, Methodology, Writing – Review & Editing.

7 Declaration of Generative AI and AI-Assisted Technologies in the Writing Process

The authors used a generative AI-based large language model assistant during manuscript preparation, for tasks including drafting and revising prose, organizing the LaTeX document structure, and cross-checking numerical results against source computational outputs. All model formulations, computational experiments, and their interpretation are the authors’ own. The authors have reviewed and edited all AI-assisted content and take full responsibility for the accuracy and integrity of this publication.

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Appendix A: Resource Monotonicity Results

Table 14: Participant-level allocations across representative values of C (Part 1 of 2): utility-based schemes. Values in thousands of credits; ω is the unallocated residual; each row sums to C . Bold entries mark resource-monotonicity violations (allocation decreases as C increases). The *Theil (Entropy)* scheme plateaus at $C = 12$ when $C/|\mathcal{N}| = \min_i d_i = 2$; residual ω grows thereafter. See Part 2 (Table 15) for valuation-based criteria.

Scheme	C	α	β	γ	δ	ϵ	ζ	ω
Social Welfare	$C = 6$	0.00	0.00	0.00	0.00	0.00	6.00	0.00
	$C = 10$	0.00	2.96	0.01	0.01	0.01	7.00	0.00
	$C = 12$	0.00	2.81	3.61	0.00	0.00	5.58	0.00
	$C = 14$	1.52	3.00	0.01	2.45	0.01	7.00	0.00
	$C = 18$	1.20	2.28	0.01	2.50	5.00	7.00	0.00
	$C = 20$	0.00	2.98	3.92	2.48	4.96	5.65	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Kalai–Smorodinsky	$C = 6$	0.51	0.77	1.02	0.64	1.28	1.79	0.00
	$C = 10$	0.85	1.28	1.70	1.06	2.13	2.98	0.00
	$C = 12$	1.02	1.53	2.04	1.28	2.55	3.57	0.00
	$C = 14$	1.19	1.79	2.38	1.49	2.98	4.17	0.00
	$C = 18$	1.53	2.30	3.06	1.91	3.83	5.36	0.00
	$C = 20$	1.70	2.55	3.40	2.13	4.26	5.96	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Max-min	$C = 6$	1.00	1.00	1.00	1.00	1.00	1.00	0.00
	$C = 10$	1.67	1.67	1.67	1.67	1.67	1.65	0.00
	$C = 12$	2.00	2.00	2.00	2.00	2.00	2.00	0.00
	$C = 14$	2.00	2.00	2.01	2.01	2.51	3.47	0.00
	$C = 18$	2.00	3.00	4.00	2.50	2.51	3.91	0.08
	$C = 20$	2.00	3.00	4.00	2.50	5.00	3.48	0.02
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Proportional Fairness	$C = 6$	1.01	1.00	1.00	1.00	1.00	1.00	0.00
	$C = 10$	1.67	1.66	1.67	1.67	1.66	1.66	0.00
	$C = 12$	2.00	2.00	2.00	2.01	2.00	2.00	0.00
	$C = 14$	2.00	2.39	2.41	2.41	2.39	2.39	0.00
	$C = 18$	2.00	3.00	3.50	2.50	3.50	3.51	0.00
	$C = 20$	2.00	3.00	4.00	2.50	4.25	4.25	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Theil (Entropy)	$C = 6$	1.00	1.00	1.00	1.00	1.00	1.00	0.00
	$C = 10$	1.67	1.67	1.67	1.67	1.67	1.67	0.00
	$C = 12$	2.00	2.00	2.00	2.00	2.00	2.00	0.00
	$C = 14$	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	$C = 18$	2.00	2.00	2.00	2.00	2.00	2.00	6.00
	$C = 20$	2.00	2.00	2.00	2.00	2.00	2.00	8.00
	$C = 23.5$	2.00	2.00	2.00	2.00	2.00	2.00	11.50
Max-min Fairness	$C = 6$	1.00	1.00	1.00	1.00	1.00	1.00	0.00
	$C = 10$	1.67	1.67	1.67	1.67	1.67	1.65	0.00
	$C = 12$	2.00	2.00	2.00	2.00	2.00	2.00	0.00
	$C = 14$	2.00	2.40	2.40	2.40	2.40	2.40	0.00
	$C = 18$	2.00	3.00	3.50	2.50	3.50	3.50	0.00
	$C = 20$	2.00	3.00	4.00	2.50	4.25	4.25	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00

Table 15: Participant-level allocations across representative values of C (Part 2 of 2): valuation-based criteria. Values in thousands of credits; ω is the unallocated residual; each row sums to C . All criteria are solved by maximizing Social Welfare subject to the corresponding fairness constraint. Bold entries mark resource-monotonicity violations. Under equal entitlements, *Kalai–Smorodinsky* and *Equitability* coincide; their rows are identical by construction. See Part 1 (Table 14) for utility-based schemes.

Scheme	C	α	β	γ	δ	ϵ	ζ	ω
Proportionality	$C = 6$	2.00	0.51	0.68	0.42	1.39	1.01	0.00
	$C = 10$	0.33	0.50	0.67	0.49	1.00	7.00	0.00
	$C = 12$	0.33	0.50	0.67	2.50	1.00	7.00	0.00
	$C = 14$	0.99	0.50	3.52	2.03	1.00	5.95	0.00
	$C = 18$	1.01	2.85	3.82	2.47	0.85	7.00	0.00
	$C = 20$	0.99	3.00	3.67	2.45	4.80	5.10	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Super Proportionality	$C = 6$	1.70	0.65	0.87	0.54	1.08	1.16	0.00
	$C = 10$	0.43	0.65	0.87	0.54	1.08	6.42	0.00
	$C = 12$	0.43	2.08	0.87	0.54	1.08	7.00	0.00
	$C = 14$	1.00	0.65	0.87	0.54	5.00	5.94	0.00
	$C = 18$	1.00	0.65	3.97	0.54	5.00	6.84	0.00
	$C = 20$	1.00	2.96	3.97	0.54	5.00	6.53	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Envy Freeness	$C = 6$	1.00	1.00	1.00	1.00	1.00	1.00	0.00
	$C = 10$	1.67	1.67	1.67	1.67	1.67	1.65	0.00
	$C = 12$	2.00	2.00	2.00	2.00	2.00	2.00	0.00
	$C = 14$	2.00	2.40	2.40	2.40	2.40	2.40	0.00
	$C = 18$	2.00	3.00	3.50	2.50	3.50	3.50	0.00
	$C = 20$	2.00	3.00	4.00	2.50	4.25	4.25	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Strong Envy Freeness	$C = 6$	1.01	1.01	0.99	1.01	0.99	0.99	0.00
	$C = 10$	1.67	1.68	1.66	1.68	1.66	1.66	0.00
	$C = 12$	2.00	2.02	1.99	2.01	1.99	1.98	0.00
	$C = 14$	2.00	2.42	2.39	2.41	2.39	2.39	0.00
	$C = 18$	2.00	3.00	3.51	2.50	3.52	3.48	0.00
	$C = 20$	2.00	3.00	4.00	2.50	4.28	4.22	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00
Equitability	$C = 6$	0.53	0.80	1.07	0.67	1.33	1.60	0.00
	$C = 10$	0.85	1.28	1.70	1.06	2.13	2.98	0.00
	$C = 12$	1.02	1.53	2.04	1.28	2.55	3.57	0.00
	$C = 14$	1.19	1.79	2.38	1.49	2.98	4.17	0.00
	$C = 18$	1.53	2.30	3.06	1.92	3.83	5.36	0.00
	$C = 20$	1.70	2.55	3.40	2.13	4.25	5.96	0.00
	$C = 23.5$	2.00	3.00	4.00	2.50	5.00	7.00	0.00

Appendix B: Population Monotonicity Results

Table 16: Participant-level allocations under population change: exit (α removed), baseline, and entry (η added with $d_\eta = 10$). Values in thousands of credits; ω is the unallocated residual, and each row sums to the total pool $C = 18$. A scheme is population-monotone if, on exit, no remaining participant falls below baseline, and on entry, no incumbent rises above baseline. Bold entries mark monotonicity violations. Under equal entitlements, *Kalai–Smorodinsky* and *Equitability* coincide, so their rows are identical by construction.

Scheme	Population	α	β	γ	δ	ϵ	ζ	η	ω
Social Welfare	$\mathcal{N} \setminus \{\alpha\}$	–	2.47	3.73	0.00	5.00	6.80	–	0.00
	$\mathcal{N}$ (base)	1.20	2.28	0.01	2.50	5.00	7.01	–	0.00
	$\mathcal{N} \cup \{\eta\}$	0.00	0.00	0.00	2.36	0.00	6.78	8.86	0.00
Kalai–Smorodinsky	$\mathcal{N} \setminus \{\alpha\}$	–	2.51	3.35	2.09	4.19	5.86	–	0.00
	$\mathcal{N}$ (base)	1.53	2.30	3.06	1.91	3.83	5.37	–	0.00
	$\mathcal{N} \cup \{\eta\}$	1.07	1.61	2.15	1.34	2.69	3.76	5.38	0.00
Max-min	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	4.00	2.50	2.51	4.23	–	1.76
	$\mathcal{N}$ (base)	2.00	3.00	4.00	2.50	2.51	3.91	–	0.08
	$\mathcal{N} \cup \{\eta\}$	2.00	2.27	2.01	2.50	2.03	3.43	3.72	0.04
Proportional Fairness	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	4.00	2.50	4.24	4.26	–	0.00
	$\mathcal{N}$ (base)	2.00	3.00	3.50	2.50	3.50	3.50	–	0.00
	$\mathcal{N} \cup \{\eta\}$	2.00	2.70	2.70	2.50	2.70	2.70	2.70	0.00
Theil (Entropy)	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	3.00	3.00	3.00	3.00	–	3.00
	$\mathcal{N}$ (base)	2.00	2.00	2.00	2.00	2.00	2.00	–	6.00
	$\mathcal{N} \cup \{\eta\}$	2.00	2.00	2.00	2.00	2.00	2.00	2.00	4.00
Max-min Fairness	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	4.00	2.50	4.25	4.25	–	0.00
	$\mathcal{N}$ (base)	2.00	3.00	3.50	2.50	3.50	3.50	–	0.00
	$\mathcal{N} \cup \{\eta\}$	2.00	2.70	2.70	2.50	2.70	2.70	2.70	0.00
Proportionality	$\mathcal{N} \setminus \{\alpha\}$	–	2.96	1.00	2.35	5.00	6.69	–	0.00
	$\mathcal{N}$ (base)	1.01	2.85	3.82	2.47	0.85	7.00	–	0.00
	$\mathcal{N} \cup \{\eta\}$	0.33	0.50	0.67	0.42	0.83	7.00	8.25	0.00
Super Proportionality	$\mathcal{N} \setminus \{\alpha\}$	–	2.57	4.00	0.54	5.00	5.89	–	0.00
	$\mathcal{N}$ (base)	1.00	0.65	3.97	0.54	5.00	6.84	–	0.00
	$\mathcal{N} \cup \{\eta\}$	2.00	0.65	0.87	0.54	1.10	2.84	10.00	0.00
Envy Freeness	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	4.00	2.50	4.25	4.25	–	0.00
	$\mathcal{N}$ (base)	2.00	3.00	3.50	2.50	3.50	3.50	–	0.00
	$\mathcal{N} \cup \{\eta\}$	2.00	2.70	2.70	2.50	2.70	2.70	2.70	0.00
Strong Envy Freeness	$\mathcal{N} \setminus \{\alpha\}$	–	3.00	4.00	2.50	4.27	4.23	–	0.00
	$\mathcal{N}$ (base)	2.00	3.00	3.51	2.50	3.51	3.48	–	0.00
	$\mathcal{N} \cup \{\eta\}$	2.00	2.71	2.72	2.50	2.69	2.69	2.69	0.00
Equitability	$\mathcal{N} \setminus \{\alpha\}$	–	2.51	3.35	2.09	4.19	5.86	–	0.00
	$\mathcal{N}$ (base)	1.53	2.30	3.06	1.91	3.83	5.37	–	0.00
	$\mathcal{N} \cup \{\eta\}$	1.07	1.61	2.15	1.34	2.69	3.76	5.38	0.00