Supplementary Material

Bose-Einstein condensation and collective Bogoliubov excitations of microwave photons in superconducting quantum electrodynamics circuits

Navez et al.

Device fabrication

The fabrication of T- and R- resonators was performed by patterning by Reactive Ion Etching (RIE) a 200 nm thick Nb film deposited on a silicon substrate. The flux qubits were realized in Al by two-angle shadow evaporation technique, forming three Josephson junctions loops having lateral dimensions equal to 6.0x4.5 μm^2. Each loop incorporates two identical junctions (0.2x0.87 μm^2), while the third one is scaled of a factor 0.8. Each qubit is shunted by a coplanar capacitor. The critical current density of the fabricated network qubit is equal to 80 A/cm^2. Additional details on the fabrication process and on the flux qubit parameters are reported in [1, 2]. The device layout is shown in Fig. S1.

Experimental setup

To perform the scattering parameters measurements, the SQN network was mounted on a sample holder provided with six SMA connector for both RF and DC input and output signals. A superconducting NbTi filament was wound around the holder to provide the static magnetic field. The microwave signals were provided by a Rohde&Schwartz SMA100B (applied to Port 3 in the scheme shown in Fig. S1), while the transmission spectra of both T- and R- resonators were acquired by an Agilent E5071C VNA. The signal generator and VNA outputs were directed to two RF lines into a Leiden

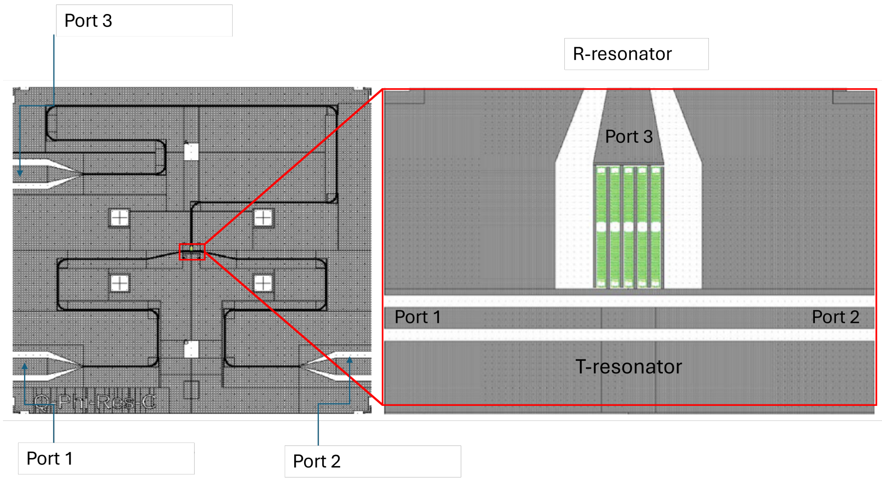

Figure S1: Layout of the device, with a magnification of the SQN coupled to the R- resonator and T-transmission line.

cryostat. In particular, thermal noise coming from the hottest plates and from the environment was attenuated by two 20 dB attenuators located at the 4 K stage and at the 10 mK stage. Including also the losses introduced by the coaxial cables, the total attenuation of the system is about 60 dB in the operation frequency range.

On the output side, cryogenic circulators (operating from 4 to 12 GHz) directed the signal from either the T- or R-resonator to a cryogenic switch, which routed it to the amplification stage. The signal was first amplified by a low-noise HEMT amplifier at 4 K, followed by a room-temperature FET amplifier, yielding a total gain of 60dB. Finally, a splitter distributes the output power to the VNA for the S_{21} measurements and to the Signal Hound SM200B spectrum analyzer for power-spectrum measurements. Additional details on the experimental setup scheme are reported in [1].

Loss parameter estimation of R- resonator

In this section, we describe the methods used to estimate the loaded quality factor Q_l and coupling quality factor Q_c from the fitting procedure of the S_{33} experimental data related to R- resonator.

The SQN device has been represented by the hanger type equivalent circuit shown in Fig. S2.

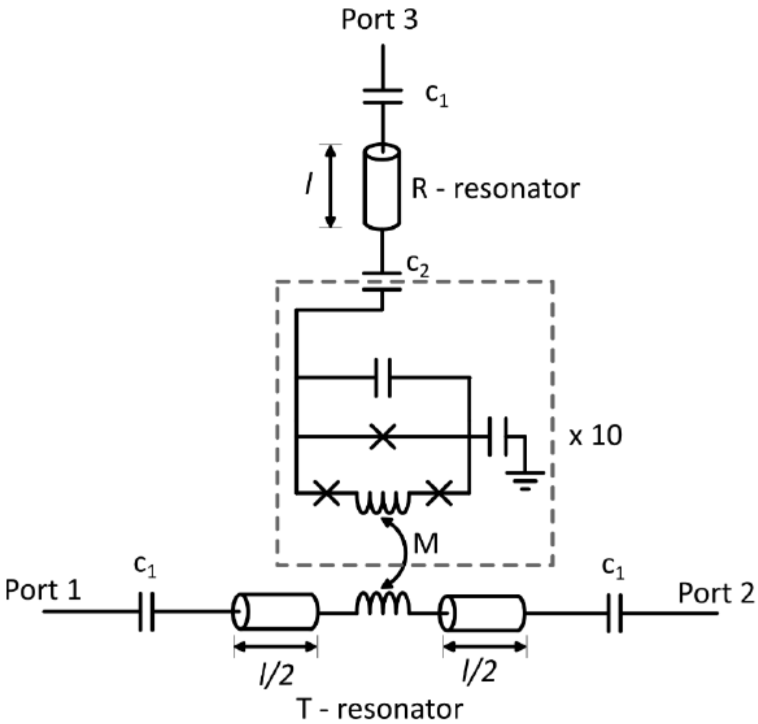

Figure S2: Equivalent circuit of the SQN setup.

It consists of two finite-length transmission lines acting as microwave resonators. The readout resonator (T-resonator), spanning a total length l, is defined between ports 1 and 2, while the signal resonator (R-resonator), also of length l, is connected to port 3 and can be considered as a necklace-type resonator (see [3]). Both resonators are capacitively terminated at their ends and support standing-wave modes at frequencies corresponding to integer multiples of *(l/2)*. The T type geometry arrangement of the two resonators ensures weak mutual coupling in the absence of the flux qubit network [1]. The reflection coefficient S_{33} of the R resonator is described by [3]:

$$S_{33} \approx A e^{-i(\omega\tau+\varphi)} \cdot \left(1 - \frac{\frac{Q_l}{|Q_c|} e^{i\phi}}{1 + 2iQ_l \left(\frac{f}{f_r} - 1 \right)} \right) \tag{S1}$$

where f_r is the resonance frequency, A is the damping coefficient caused by the finite-length feedlines, $e^{-\omega\tau}$ is the frequency dependent phase factor, τ is the time constant, $e^{i\varphi}$ is the constant phase delay because of the imperfect calibration of the cable delay. $Ae^{-i(\omega\tau+\varphi)}$-global factor originated from the finite length of the feedlines - we define as a - fitting coefficient. Note in this expression the opposite imaginary sign convention of i used in microwave engineering that differs from the one used in quantum mechanics (see [4] where the imaginary unit j is opposite in our notation $i = -j$).

The process involves several steps, including data preprocessing, interpolation, model definition, fitting, and evaluation of results. In the first step, we preprocessed the data by loading the experimental measurements of S_{33} as a function of frequency from a raw file (see inset of Fig. S3). The data were filtered to include only measurements within a specified frequency window, focusing on the interval of interest around the resonance dip. In this case, the frequency window used was $[7.7422, 7.7445]$ GHz (Fig.S3).

Next, we interpolated the reduced data using a cubic spline. This interpolation created a smooth curve that better represents the experimental data, providing a continuous representation essential for an accurate fitting. We used *Powell's Dog Leg* fitting method ([5]) to fit the model to the interpolated data (Fig. S3). This method optimizes the model parameters to minimize the discrepancy between the experimental data and the theoretical model. The fitting was performed using the `scipy` ([6]) library, which identified the optimal parameters that best fit the data.

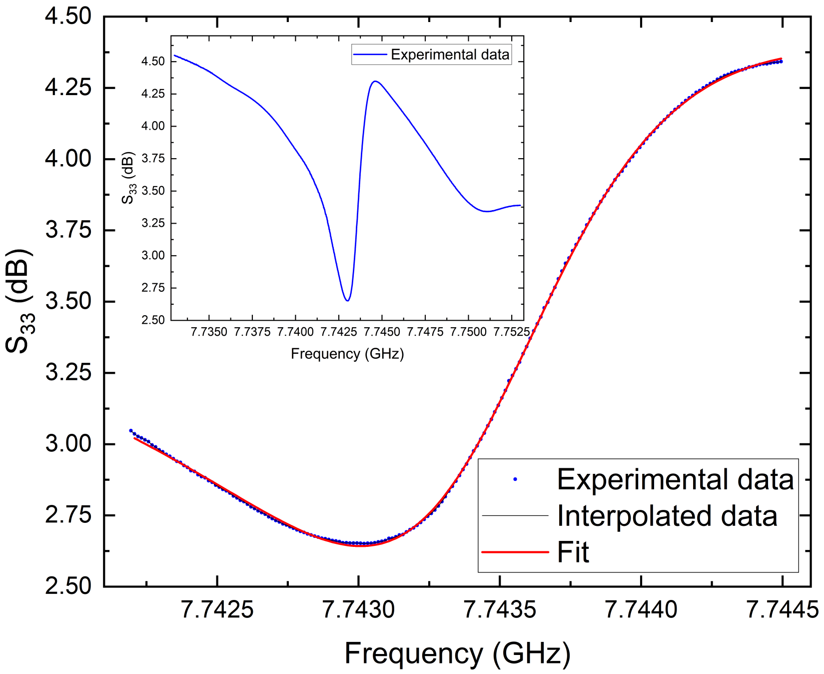

Figure S3: Plot of the experimental data, interpolation, and the obtained fit of the S_{33} parameter within the frequency window $[7.7422, 7.7445]$ GHz. The inset shows the plot of S_{33} as a function of frequency in the whole frequency range

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The fitting results included the optimal model parameters and their uncertainties. In addition, the Root Mean Square Error (RMSE) was calculated to estimate the quality of the fit. The optimal parameters obtained are:

$Q_c = 24530.93 \pm 0.38$
$Q_l = 4682.71 \pm 0.05$
$a = 1.574 \pm 0.00002$
$f_r = 7.743538 \pm 0.000000$
$\phi = -1.054 \pm 0.00002$

The RMSE for the interpolated data was $RMSE_{int} = 0.005348$, while for the experimental data it was $RMSE_{exp} = 0.005453$.

Finally, the covariance matrix obtained from the fitting process is shown in Figure S4.

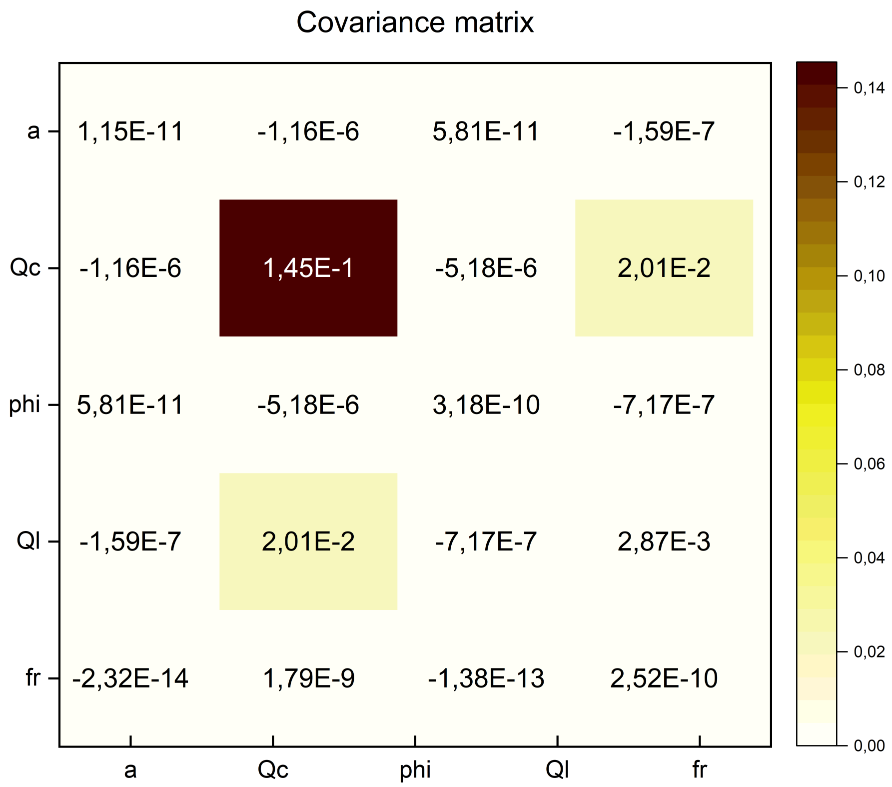

Figure S4: Covariance matrix of the fitted parameters.

References

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