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A Residual-Certified Product-Integration Method for Nonlinear Caputo–Volterra Equations

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Abstract

We study an implicit product-integration scheme for nonlinear Caputo initial-value problems written as weakly singular Volterra equations. The Abel kernel is integrated exactly against a piecewise-linear interpolant of the nonlinear term, producing a lower-triangular nonlinear system with a nonzero diagonal. A computable diagonal condition gives discrete existence and uniqueness without the stronger global Banach restriction. Under this condition, a positive triangular majorant gives convergence of global and nodewise Picard iterations and a componentwise a posteriori bound for any inexact discrete vector. The bound is obtained on the same mesh by forward substitution and separates iteration error from discretization error. A blockwise Volterra estimate yields a mesh-independent resolvent bound and second-order convergence on sufficiently fine meshes. If the solution has a power-type singularity near the initial point, the error on a graded mesh is of order $M^{-\min\{r\sigma, 2\}}$; order two follows when $r\sigma \geq 2$. Numerical tests compare the method with fractional Adams predictor–evaluate–correct–evaluate and left-endpoint product integration. The tests cover smooth, singular, and non-manufactured problems and report estimator effectivity, convergence rates, iteration counts, and execution times.

Keywords: Caputo fractional equation; weakly singular Volterra equation; product integration; nonlinear solver; residual certification; graded mesh.

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1 Introduction

Caputo initial-value problems can be written as nonlinear Volterra equations with Abel-type kernels. Standard references are Podlubny [1], Kilbas–Srivastava–Trujillo [2], Lubich [3], and Brunner [4]. Product integration, collocation, fractional Adams methods, smoothing transformations, and graded meshes have all been used for such equations; see, for example, Orsi [5], Brunner–Pedas–Vainikko [6], Jumarhon–McKee [7], Diethelm–Ford–Freed [8], Najafi [9], Liang–Brunner [10], Katani–McKee [11], and Ma–Stynes [12]. Second-order implicit product-trapezoidal rules are classical [7], and extrapolation estimators are known for nonlinear weakly singular Volterra equations [13, 14]. The quadrature rule and the general idea of a posteriori estimation are therefore not claimed as new here.

The paper considers a narrower question. Can the current-node correction of a higher-order product-integration rule be kept while retaining a causal implementation and a computable bound for an inexact nonlinear solve? A left-endpoint rule is strictly causal and can be advanced by substitution; treating that recurrence as a global Picard process adds no nonlinear computation. The piecewise-linear rule used below is different because the value at the current node appears nonlinearly.

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The resulting system is lower triangular. Its diagonal coefficient is known explicitly, which leads to a mesh-local solvability test. The paper proves the following results.

- (i) Exact weights are obtained for piecewise-linear product integration on nonuniform meshes.
- (ii) The discrete problem is solvable when $k_{i-1}^\alpha L_i / \Gamma(\alpha + 2) < 1$ at every node, even when the usual global Banach constant is larger than one.
- (iii) Both global and nodewise Picard iterations converge under the same local condition.
- (iv) A positive triangular resolvent gives a componentwise error bound for any inexact discrete vector.
- (v) The error bound separates the effect of terminating the nonlinear iteration from the quadrature error.
- (vi) A blockwise discrete-Volterra estimate gives a mesh-independent resolvent bound and closes the second-order convergence proof.
- (vii) If $|g''(t)| \leq Ct^{\sigma-2}$, the graded-mesh error is $O(M^{-\min\{r\sigma, 2\}})$.
- (viii) Numerical tests report estimator effectivity, convergence rates, iteration counts, and wall-clock times, including a problem without a prescribed exact solution.

The analysis keeps the componentwise Volterra structure instead of replacing it by one maximum-norm Lipschitz constant. The main result is the combination of the diagonal solvability test, the weighted contraction argument, the triangular residual bound, and the graded-mesh error estimate.

2 Continuous problem and Volterra majorant

Let $0 < \alpha < 1$, $J = [0, T]$, and consider

$${}^C D_{0+}^\alpha u(t) = F(t, u(t)), \quad u(0) = u_0. \quad (2.1)$$

Assume that $F \in C(J \times \mathbb{R})$ and

$$|F(t, x) - F(t, y)| \leq L(t)|x - y|, \quad L \in C(J, [0, \infty)). \quad (2.2)$$

Then (2.1) is equivalent to

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} F(\tau, u(\tau)) d\tau. \quad (2.3)$$

Define the positive Volterra operator

$$(Av)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} L(\tau) v(\tau) d\tau.$$

Since $0 \leq A^n v \leq L_*^n I^{n\alpha} v$ for $v \geq 0$,

$$\|A^n\|_\infty \leq \frac{L_*^n T^{n\alpha}}{\Gamma(n\alpha + 1)},$$

and hence $r(A) = 0$.

Theorem 2.1. Under (2.2), problem (2.3) has a unique solution $u \in C(J)$. The continuous Picard iterates converge for every initial function. Also, for every $v \in C(J)$,

$$|v - u| \leq (I - A)^{-1}|v - \mathcal{T}v| \quad (2.4)$$

pointwise on J , where $\mathcal{T}$ denotes the right-hand side of (2.3).

Proof. The difference inequality $|\mathcal{T}v - \mathcal{T}w| \leq A|v - w|$ and $r(A) = 0$ imply convergence and uniqueness by the positive Neumann series. Also, $|v - u| \leq |v - \mathcal{T}v| + A|v - u|$; application of the positive inverse $(I - A)^{-1}$ gives (2.4). $\square$

3 Implicit linear product integration on a nonuniform mesh

Let

$$0 = t_0 < t_1 < \cdots < t_M = T, \quad k_j = t_{j+1} - t_j,$$

and write $F_j(U_j) = F(t_j, U_j)$. On $[t_j, t_{j+1}]$ replace $F(\tau, u(\tau))$ by its linear interpolant

$$\frac{t_{j+1} - \tau}{k_j} F_j(U_j) + \frac{\tau - t_j}{k_j} F_{j+1}(U_{j+1}).$$

For $j < i$, set $A_{ij} = t_i - t_j$ and $B_{ij} = t_i - t_{j+1}$. Exact integration gives

$$p_{ij} = \frac{1}{\Gamma(\alpha)k_j} \left[\frac{A_{ij}^{\alpha+1} - B_{ij}^{\alpha+1}}{\alpha + 1} - \frac{B_{ij}(A_{ij}^\alpha - B_{ij}^\alpha)}{\alpha} \right], \quad (3.1)$$

$$q_{ij} = \frac{1}{\Gamma(\alpha)k_j} \left[\frac{A_{ij}(A_{ij}^\alpha - B_{ij}^\alpha)}{\alpha} - \frac{A_{ij}^{\alpha+1} - B_{ij}^{\alpha+1}}{\alpha + 1} \right]. \quad (3.2)$$

Both weights are nonnegative and $p_{ij} + q_{ij} = [A_{ij}^\alpha - B_{ij}^\alpha]/\Gamma(\alpha + 1)$.

The proposed discrete problem is

$$U_0 = u_0, \quad U_i = u_0 + \sum_{j=0}^{i-1} [p_{ij} F_j(U_j) + q_{ij} F_{j+1}(U_{j+1})], \quad 1 \leq i \leq M. \quad (3.3)$$

The last term in the i th equation is implicit, because

$$q_{i,i-1} = \frac{k_{i-1}^\alpha}{\Gamma(\alpha + 2)} =: \kappa_i. \quad (3.4)$$

Remark 3.1. The diagonal term (3.4) distinguishes (3.3) from strictly causal left-endpoint product integration. The system is not an explicit recurrence, yet its lower-triangular structure permits independent certification of accumulated history and current-node nonlinear error.

Let $L_i = L(t_i)$ and define the nonnegative lower-triangular matrix $C_h = (c_{i\ell})_{i,\ell=0}^M$ by

$$(C_h z)_0 = 0, \quad (C_h z)_i = \sum_{j=0}^{i-1} p_{ij} L_j z_j + \sum_{j=0}^{i-1} q_{ij} L_{j+1} z_{j+1}. \quad (3.5)$$

Its diagonal entries are $c_{ii} = \kappa_i L_i$. Hence

$$r(C_h) = \max_{1 \leq i \leq M} \kappa_i L_i =: \mu_h. \quad (3.6)$$

Theorem 3.2 (Mesh-local solvability). Assume (2.2) and $\mu_h < 1$. Then (3.3) has a unique solution U_h . For every initial vector $W^{(0)}$ with $W_0^{(0)} = u_0$, the global iteration $W^{(n+1)} = \mathcal{T}_h W^{(n)}$ converges to U_h . No condition involving $L_* T^\alpha / \Gamma(\alpha + 1)$ is required.

Proof. For componentwise absolute values, $|\mathcal{T}_h V - \mathcal{T}_h W| \leq C_h |V - W|$. Since C_h is lower triangular, its eigenvalues are its diagonal entries; thus $r(C_h) = \mu_h < 1$ and $w := (I - C_h)^{-1} \mathbf{1} = \sum_{n \geq 0} C_h^n \mathbf{1}$ is well defined and satisfies $w \geq \mathbf{1}$. Introduce the weighted maximum norm $\|z\|_w = \max_i |z_i|/w_i$. Since $C_h w = w - \mathbf{1}$, one has

$$C_h w \leq q_h w, \quad q_h := \max_i \left(1 - \frac{1}{w_i}\right) < 1.$$

Consequently $\|\mathcal{T}_h V - \mathcal{T}_h W\|_w \leq q_h \|V - W\|_w$. Banach's theorem gives a unique fixed point and convergence of the global Picard iteration from every starting vector with the prescribed initial component. This argument also identifies an explicit contraction norm and does not invoke the scalar condition $L_* T^\alpha / \Gamma(\alpha + 1) < 1$. $\square$

Corollary 3.3 (Nodewise solver). *Under the assumptions of Theorem 3.2, after $U_0, \dots, U_{i-1}$ have been computed, U_i is the unique fixed point of*

$$z = H_i + \kappa_i F(t_i, z), \quad (3.7)$$

where

$$H_i = u_0 + \sum_{j=0}^{i-1} p_{ij} F_j(U_j) + \sum_{j=0}^{i-2} q_{ij} F_{j+1}(U_{j+1}).$$

The local Picard iteration converges with contraction factor at most $\kappa_i L_i < 1$.

4 Residual and two-level certification

The next theorem gives the algebraic error bound.

Theorem 4.1 (Componentwise algebraic certificate). *Under $\mu_h < 1$, every vector $W \in \mathbb{R}^{M+1}$ with $W_0 = u_0$ satisfies*

$$|W - U_h| \leq (I - C_h)^{-1} |W - \mathcal{T}_h W|. \quad (4.1)$$

The right-hand side is computable by forward substitution and does not require formation of a dense inverse.

Proof. By adding and subtracting $\mathcal{T}_h W$,

$$|W - U_h| \leq |W - \mathcal{T}_h W| + |\mathcal{T}_h W - \mathcal{T}_h U_h| \leq |W - \mathcal{T}_h W| + C_h |W - U_h|.$$

The inverse $(I - C_h)^{-1}$ is positive because $r(C_h) < 1$. This proves (4.1). Since $I - C_h$ is lower triangular with positive diagonal, the certificate is obtained by a forward solve. $\square$

Corollary 4.2 (Nodewise tolerance propagation). *Suppose that a sequential nonlinear solver accepts W_i with local residual*

$$\varepsilon_i = |W_i - H_i(W_0, \dots, W_{i-1}) - \kappa_i F(t_i, W_i)|, \quad 1 \leq i \leq M,$$

and set $\varepsilon = (0, \varepsilon_1, \dots, \varepsilon_M)^\top$. Then

$$|W - U_h| \leq (I - C_h)^{-1} \varepsilon. \quad (4.2)$$

Hence, if the forward solve $(I - C_h)B = \varepsilon$ gives $\|B\|_\infty \leq \tau$, the accepted nodewise iterates meet the prescribed global algebraic tolerance $\|W - U_h\|_\infty \leq \tau$.

Proof. For a sequentially assembled vector, the i th component of $W - \mathcal{T}_h W$ is exactly the signed local residual in the displayed formula. Hence $|W - \mathcal{T}_h W| = \varepsilon$, and Theorem 4.1 gives (4.2). $\square$

Remark 4.3 (Single-grid character). The bound in (4.2) uses only the accepted vector, its local residuals, and one triangular forward solve. It does not require an exact solution, an asymptotic error expansion, or a second computation on a refined mesh. This distinguishes the algebraic certificate from extrapolation-based discretization-error estimators [13, 14].

Let $P_h u = (u(t_0), \dots, u(t_M))$ and denote the consistency vector by

$$\xi_h = |P_h u - \mathcal{T}_h(P_h u)|.$$

Theorem 4.4 (Two-level error separation). *For any computed vector W ,*

$$|P_h u - W| \leq (I - C_h)^{-1}(\xi_h + |W - \mathcal{T}_h W|). \quad (4.3)$$

The first contribution represents the discretization error, whereas the second quantifies the error introduced by terminating the nonlinear iteration.

Proof. Use $|P_h u - W| \leq |P_h u - U_h| + |U_h - W|$. The first term satisfies $|P_h u - U_h| \leq \xi_h + C_h |P_h u - U_h|$, while the second is bounded by Theorem 4.1. $\square$

Discrete Gronwall inequalities for weakly singular kernels are classical [16, 17, 18]. The next lemma gives the form needed for the present nonuniform triangular matrix.

Lemma 4.5 (Uniform blockwise resolvent bound). *Set $S = L_* T^\alpha / \Gamma(\alpha + 1)$. Choose $\Delta > 0$ so that*

$$\theta := \frac{L_*(2\Delta)^\alpha}{\Gamma(\alpha + 1)} < 1,$$

and assume $k_{\max} \leq \Delta$. Then there is a constant $K = K(\alpha, L_, T, \Delta)$, independent of the mesh and of M , such that*

$$\|(I - C_h)^{-1} \mathbf{1}\|_\infty \leq K.$$

One admissible constant is obtained from $K_0 = 1$ and

$$K_m = \max \left\{ K_{m-1}, \frac{1 + SK_{m-1}}{1 - \theta} \right\}, \quad 1 \leq m \leq B_* := \left\lceil \frac{T}{\Delta} \right\rceil + 1,$$

by taking $K = K_{B_}$.*

Proof. Let $z = (I - C_h)^{-1} \mathbf{1}$, so that $z = \mathbf{1} + C_h z$ and $z \geq 0$. Partition the nodes into the time slabs $((m-1)\Delta, m\Delta]$, with t_0 assigned to the first slab, and let Z_m be the largest component of z through slab m . For a node t_i in slab m , split the row of C_h according to whether the nodal component z_ℓ belongs to the current slab or to an earlier slab. The product-integration basis attached to a current-slab node is supported on at most its two adjacent mesh intervals. Hence its leftmost support point is no earlier than $t_\ell - k_{\max}$, and

$$t_i - (t_\ell - k_{\max}) \leq \Delta + k_{\max} \leq 2\Delta.$$

Because the piecewise-linear basis is nonnegative and forms a partition of unity on each interval, the sum of all current-slab coefficients in that row is bounded by

$$\frac{L_*}{\Gamma(\alpha)} \int_{\max\{0, t_i - 2\Delta\}}^{t_i} (t_i - \tau)^{\alpha-1} d\tau \leq \theta.$$

The coefficients multiplying earlier-slab components have total row sum at most S . Therefore

$$z_i \leq 1 + SZ_{m-1} + \theta Z_m,$$

and consequently

$$Z_m \leq \max \left\{ Z_{m-1}, \frac{1 + SZ_{m-1}}{1 - \theta} \right\}.$$

There are at most B_* slabs. Induction gives $Z_m \leq K_m$, proving the stated mesh-independent bound. $\square$

Theorem 4.6 (Reliability and norm efficiency). *Let $\eta_{\text{alg}}(W) = \|(I - C_h)^{-1}|W - \mathcal{T}_h W|\|_\infty$. Under the hypotheses of [Theorem 4.5](#),*

$$\|W - U_h\|_\infty \leq \eta_{\text{alg}}(W) \leq K(1 + S)\|W - U_h\|_\infty.$$

Hence the estimator is reliable and efficient with a mesh-independent constant.

Proof. Reliability is [Theorem 4.1](#). Also, $|W - \mathcal{T}_h W| \leq |W - U_h| + C_h|W - U_h|$, so $\|W - \mathcal{T}_h W\|_\infty \leq (1 + \|C_h\|_\infty)\|W - U_h\|_\infty \leq (1 + S)\|W - U_h\|_\infty$. Now apply [Theorem 4.5](#) and positivity. $\square$

Theorem 4.7 (Second-order consistency). *Let $g(t) = F(t, u(t)) \in C^2[0, T]$ and $k_{\max} = \max_j k_j$. Then*

$$0 \leq (\xi_h)_i \leq \frac{\|g''\|_\infty}{8\Gamma(\alpha + 1)} t_i^\alpha k_{\max}^2, \quad 0 \leq i \leq M. \quad (4.4)$$

Hence, if $k_{\max} \leq \Delta$ with Δ as in [Theorem 4.5](#),

$$\|P_h u - U_h\|_\infty \leq \frac{K \|g''\|_\infty T^\alpha}{8\Gamma(\alpha + 1)} k_{\max}^2. \quad (4.5)$$

Hence, the method converges with order two on every sufficiently fine admissible mesh.

Proof. On $[t_j, t_{j+1}]$, the linear-interpolation remainder obeys $|g(\tau) - I_1 g(\tau)| \leq \|g''\|_\infty k_j^2/8$. Inserting this bound into the fractional integral and summing the subintervals gives (4.4). Apply [Theorem 4.4](#) with $W = U_h$ and then [Theorem 4.5](#). $\square$

Theorem 4.8 (Graded-mesh rate for initial singularities). *Let $t_j = T(j/M)^r$ with $r \geq 1$. Assume that g is continuous on $[0, T]$, twice continuously differentiable on $(0, T]$, and for some $0 < \sigma \leq 2$,*

$$|g(t) - g(0)| \leq C_g t^\sigma, \quad |g''(t)| \leq C_g t^{\sigma-2} \quad (0 < t \leq T).$$

Then, for all sufficiently large M ,

$$\|P_h u - U_h\|_\infty \leq C M^{-\min\{r\sigma, 2\}},$$

where C is independent of M . Specifically, second order is recovered whenever $r\sigma \geq 2$.

Proof. On the first interval, continuity with exponent σ gives $\|g - I_1 g\|_{L^\infty(0, t_1)} \leq C t_1^\sigma = O(M^{-r\sigma})$. For $j \geq 1$, the interpolation remainder and the graded-mesh estimates $k_j \leq C r M^{-r}(j+1)^{r-1}$ and $t_j \asymp (j/M)^r$ yield

$$\|g - I_1 g\|_{L^\infty(t_j, t_{j+1})} \leq C k_j^2 t_j^{\sigma-2} \leq C M^{-r\sigma} (j+1)^{r\sigma-2} \leq C M^{-\min\{r\sigma, 2\}}.$$

Integration against the positive Abel kernel therefore gives the same bound for $\|\xi_h\|_\infty$. The uniform resolvent estimate of [Theorem 4.5](#) and [Theorem 4.4](#) with $W = U_h$ complete the proof. $\square$

Remark 4.9 (Why the local condition is weaker). The standard continuous maximum-norm criterion is expressed through $S = L_* T^\alpha / \Gamma(\alpha + 1)$. In comparison, on a uniform mesh,

$$\mu_h = \frac{h^\alpha}{\Gamma(\alpha + 2)} \max_i L_i.$$

Thus S may exceed one while $\mu_h < 1$ after an ordinary mesh refinement. The reason is that S collapses the entire Volterra history into one number, whereas μ_h measures only the nonlinear diagonal generated by the current subinterval.

5 Comparison with related methods

We state the relation to earlier work explicitly. Jumarhon–McKee [7] already derived explicit and implicit product Euler schemes and a second-order product-trapezoidal scheme for nonlinear weakly singular Volterra systems. Katani–McKee [11] recently combined product integration with a smoothing change of variables and proved convergence for nonsmooth solutions. Najafi [9] used smoothing and quasilinearization to avoid nonlinear algebraic systems. These works establish that neither linear interpolation of the Abel kernel nor treatment of initial singularity is new by itself.

A posteriori estimation also has precedents. Lü–Huang [13] and Han–He–Liu–Lü [14] derived extrapolation estimators from asymptotic error expansions. Their estimators concern discretization error and require the asymptotic regime and additional approximations. In comparison, Theorems 4.1 and 4.2 provides a residual certificate for an arbitrary inexact vector on one fixed mesh. It bounds the algebraic error caused by incomplete nonlinear solution, is componentwise, and is computed by one lower-triangular forward solve. The subsequent two-level estimate adds the consistency contribution without conflating these two error sources.

Table 1 records the claim boundary against the closest lines of work. The table lists the results already available and the points proved here.

Closest line of work	Established result	Distinction of the present analysis
Implicit product integration [7]	First- and second-order explicit/implicit schemes with convergence analysis	The quadrature rule is treated as classical; the new layer is the exact diagonal test, weighted global contraction, and residual propagation for inexact nonlinear solves.
Smoothing or graded treatment [9, 11, 10, 12]	Recovery of accuracy for nonsmooth solutions through smoothing, collocation, or graded meshes	The graded-mesh rate is coupled to the same positive resolvent that certifies the nonlinear algebraic error on the untransformed triangular system.
Extrapolation estimators [13, 14]	A posteriori discretization-error estimates from asymptotic expansions and multiple approximations	The certificate is single-grid, residual-based, componentwise, and valid for any accepted inexact iterate; no error expansion or refined companion solve is required.
Fractional Adams and explicit PI [8, 15]	Established causal baselines with discretization convergence	The current-node nonlinearity is retained and certified; algebraic and discretization errors appear separately in one computable bound.

Table 1: Position of the present contribution relative to the closest established approaches.

Hence, we do not claim a new product-trapezoidal formula, a first use of graded meshes, or the first a posteriori estimate for weakly singular Volterra equations. The claim made here is limited to the combination of an exact mesh-local diagonal criterion, a weighted contraction proof, a single-grid componentwise residual certificate for incomplete nonlinear solves, and a mesh-independent convergence/efficiency analysis for that certificate.

6 Implementation

The computation is carried out one node at a time.

Algorithm 1: certified implicit linear product integration.

1. Construct the mesh and weights (3.1)–(3.2); verify $\mu_h < 1$.
2. Set $U_0 = u_0$.
3. For $i = 1, \dots, M$, form H_i from already accepted nodal values and solve (3.7) by local Picard iteration (or Newton’s method).
4. Assemble a possibly inexact vector W , its residual $R = |W - \mathcal{T}_h W|$, and solve $(I - C_h)B = R$ by forward substitution.
5. Accept the nonlinear solve when $\|B\|_\infty$ is below the prescribed algebraic tolerance. When a consistency majorant is available, use (4.3) for a total nodal error bound.

The history evaluation costs $O(M^2)$ operations and $O(M^2)$ storage if all weights are retained; weights may instead be generated row by row using $O(M)$ storage. The certification step has the same asymptotic complexity as a single history evaluation. A reference implementation that reproduces all numerical tables is supplied as Online Resource 1. Fast convolution and sum-of-exponentials acceleration are compatible with the estimator but are outside the present scope.

7 Numerical verification

For the numerical experiments, we set

$$F(t, z) = q(t) - a(t)z + b(t) \tanh z, \quad L(t) = a(t) + b(t),$$

and solve the current-node equation by local Picard iteration to a tolerance of 10^{-14} . All experiments were generated directly from (3.1)–(3.3); Online Resource 1 reproduces every entry in Tables 3 to 5. In all tests the global number $S = L_* T^\alpha / \Gamma(\alpha + 1)$ exceeds one, whereas the mesh-local number μ_h is below one. Tables give the numerical values; figures are used only for convergence, estimator, and cost trends. Full machine-readable outputs are supplied in Online Resource 1.

7.1 Certificate evaluation

The four certificate models use the coefficients listed in Table 2. Their global Banach numbers all exceed the classical threshold $S = 1$. Model P4 has the exact solution $u(t) = 1 + t^3$, with q obtained by substitution and ${}^C D^{0.65} t^3 = \Gamma(4)t^{2.35}/\Gamma(3.35)$.

Model	α	T	u_0	$a(t)$	$b(t)$	S
P1	0.50	1.0	1.0	$1 + t^2$	$(2 + t)/3$	3.385
P2	0.75	1.5	0.2	$2 + t$	$1 + t/2$	7.743
P3	0.90	1.0	−0.5	$3 + t/2$	$2/(1 + t)$	5.199
P4	0.65	1.0	1.0	$1.5 + 0.5t$	$0.8 + 0.3t$	3.444

Table 2: Parameters for the four certificate models. Models P1–P3 use the stated forcing functions, while P4 is constructed from the exact solution $u(t) = 1 + t^3$.

For the certificate experiment, the reference discrete vector was computed with the nodewise solver, whereas $W^{(n)}$ was generated by the global Picard iteration on a uniform mesh with $M = 400$. The bound is $\|(I - C_h)^{-1}|W^{(n)} - \mathcal{T}_h W^{(n)}|\|_\infty$.

Model	n	residual	observed algebraic error	certified bound	effectivity
P1	10	1.740×10^{-3}	1.217×10^{-3}	9.292×10^{-3}	7.64
P1	20	5.998×10^{-8}	4.593×10^{-8}	1.621×10^{-7}	3.53
P2	10	4.583×10^{-1}	3.043×10^{-1}	2.960	9.73
P2	20	5.462×10^{-5}	4.128×10^{-5}	1.343×10^{-4}	3.25
P3	10	5.465×10^{-4}	4.452×10^{-4}	9.092×10^{-4}	2.04
P3	20	1.175×10^{-11}	1.039×10^{-11}	1.534×10^{-11}	1.48
P4	10	7.789×10^{-4}	5.839×10^{-4}	1.633×10^{-3}	2.80
P4	20	2.380×10^{-9}	1.921×10^{-9}	3.849×10^{-9}	2.00

Table 3: Residual, algebraic error, certified bound, and effectivity for the global Picard iterates. The certified bound exceeds the observed algebraic error in every case.

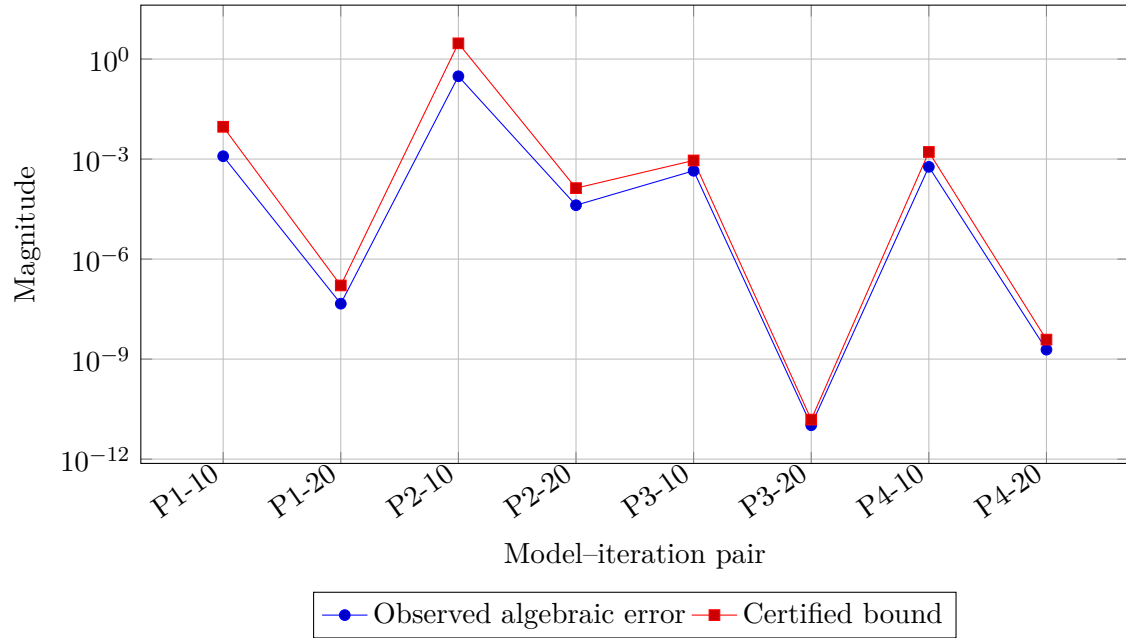


Figure 1: Observed algebraic errors and certified upper bounds. Reliability is maintained for every model, and the bound generally sharpens as the nonlinear iteration proceeds.

Figure 1 shows that the bound exceeds the observed error for every model and iteration level. The effectivity decreases as the iteration enters its asymptotic regime. the bound remains valid even though $\|C_h\|_\infty > 1$ in these tests; only the triangular spectral condition $\mu_h < 1$ is needed.

7.2 Comparative convergence on a smooth benchmark

For P4, Table 4 compares the proposed method with the classical fractional Adams predictor–evaluate–correct–evaluate (PECE) algorithm and the explicit left-endpoint product-integration rule. All methods use the same uniform mesh and nonlinear function. The proposed method is second order, while the Adams rate approaches $1 + \alpha = 1.65$ and the left-endpoint formula is first order. The convergence curves are displayed in Figure 2. At $M = 800$, the maximum error of the proposed method is about 52 times smaller than that of Adams PECE and more than 2800 times smaller than that of the left-endpoint rule.

M	proposed implicit PI error	order	Adams PECE error	order	left-endpoint PI error	order	max. local it.
25	3.365×10^{-4}	–	6.716×10^{-3}	–	3.238×10^{-2}	–	18
50	8.453×10^{-5}	1.99	1.960×10^{-3}	1.78	1.579×10^{-2}	1.04	14
100	2.120×10^{-5}	2.00	5.884×10^{-4}	1.74	7.774×10^{-3}	1.02	12
200	5.311×10^{-6}	2.00	1.801×10^{-4}	1.71	3.849×10^{-3}	1.01	10
400	1.329×10^{-6}	2.00	5.584×10^{-5}	1.69	1.912×10^{-3}	1.01	9
800	3.326×10^{-7}	2.00	1.747×10^{-5}	1.68	9.522×10^{-4}	1.01	8

Table 4: Maximum nodal errors and observed orders for the smooth benchmark P4. The final column gives the largest number of nodewise Picard iterations required on each mesh.

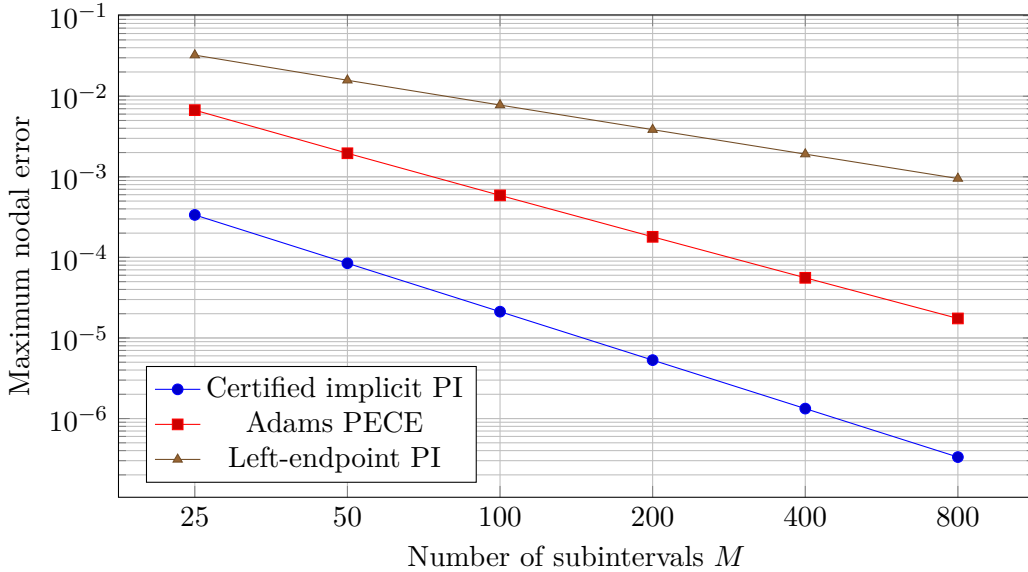


Figure 2: Error convergence for the smooth benchmark P4. The certified implicit product-integration method follows the predicted second-order slope and converges more rapidly than the two baseline schemes.

7.3 Initial singularity and graded mesh

To examine performance below the C^2 regularity threshold, let $\alpha = 0.65$ and prescribe $u(t) = 1 + t^{1.2}$. Then ${}^C D^{0.65} t^{1.2} = \Gamma(2.2)t^{0.55}/\Gamma(1.55)$, so the integrand has an initial derivative singularity. The same a and b as in P4 are used, and q is again determined by substitution. On

the graded mesh $t_i = T(i/M)^r$, $r = 2$, second order is recovered, whereas the uniform-grid order is about 1.2; the contrasting slopes are shown in Figure 3.

M	uniform mesh, $r = 1$		graded mesh, $r = 2$	
	maximum error	order	maximum error	order
25	3.163×10^{-3}	—	1.393×10^{-4}	—
50	1.424×10^{-3}	1.15	3.515×10^{-5}	1.99
100	6.334×10^{-4}	1.17	8.837×10^{-6}	1.99
200	2.796×10^{-4}	1.18	2.217×10^{-6}	1.99
400	1.228×10^{-4}	1.19	5.556×10^{-7}	2.00

Table 5: Maximum nodal errors and observed orders for the initially singular benchmark on uniform and graded meshes.

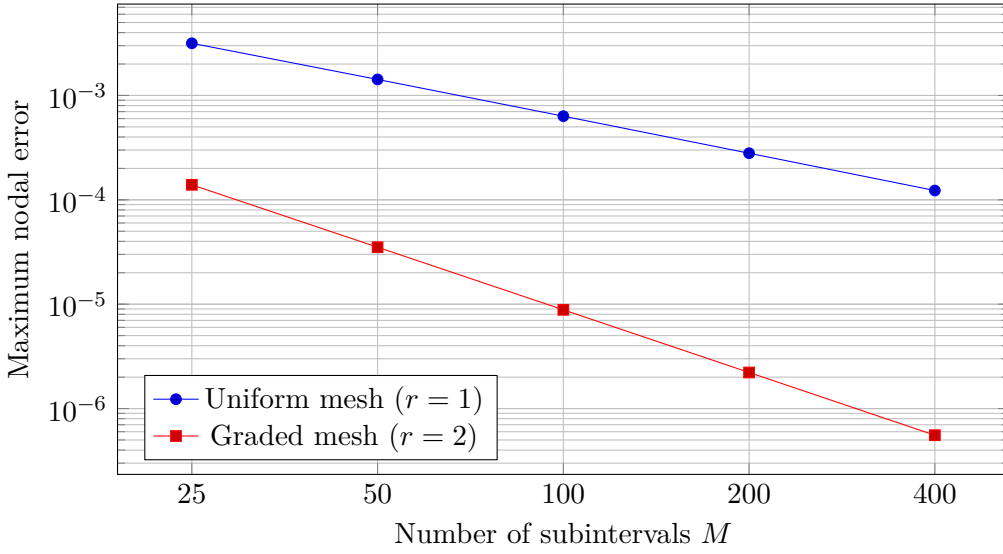


Figure 3: Convergence for the initially singular benchmark. Mesh grading restores the predicted second-order rate, whereas the uniform mesh remains limited by the reduced initial regularity.

7.4 Non-manufactured nonlinear benchmark and computational cost

The final experiment does not rely on a manufactured exact solution. We take

$$\alpha = 0.72, \quad T = 2, \quad u_0 = 0.3,$$

$$q(t) = 1 + \frac{1}{2} \sin(2t), \quad a(t) = 2 + \frac{1}{2}t, \quad b(t) = \frac{1.2}{1+t}.$$

Thus $F(t, z) = q(t) - a(t)z + b(t) \tanh z$. Here $S = 7.581 > 1$, while $\mu_h = 0.160, 0.097, 0.059, 0.036$ for $M = 100, 200, 400, 800$, respectively. Since no exact solution is imposed, the reference vector is the proposed solution on a uniform mesh with $M_{\text{ref}} = 6400$. The maximum difference between the $M = 3200$ solution and the restriction of the $M = 6400$ solution is 9.638×10^{-7} , which serves as a refinement check for the reference value.

Table 6 reports errors at common nodes and median wall-clock times over three runs. Timings include mesh and weight construction for the proposed method and were obtained with the NumPy implementation in Online Resource 1 on the same software path for every method. These timings are reported as reproducible implementation-level measurements, not as hardware-independent performance claims. The certified implicit method is more expensive than PECE

and left-endpoint product integration, as expected from its nonlinear current-node solves and weight construction, but its iteration count decreases under refinement and its error is below the left-endpoint rule. Figure 4 displays the corresponding error and timing data. PECE is faster for this smooth problem. The advantage of the present scheme is the proved nonlinear solvability test and residual bound, not lower execution time in every problem.

M	certified implicit PI			Adams PECE		left-endpoint PI	
	error	time (s)	max. it.	error	time (s)	error	time (s)
100	1.930×10^{-4}	0.0121	13	2.665×10^{-4}	0.0033	2.696×10^{-3}	0.0017
200	7.338×10^{-5}	0.0240	11	7.733×10^{-5}	0.0069	1.338×10^{-3}	0.0041
400	2.735×10^{-5}	0.0463	9	2.482×10^{-5}	0.0158	6.659×10^{-4}	0.0091
800	9.951×10^{-6}	0.1074	8	9.375×10^{-6}	0.0358	3.321×10^{-4}	0.0232

Table 6: Accuracy, execution time, and nonlinear iteration counts for the non-manufactured benchmark. Errors are measured at common nodes against the certified reference solution with $M_{\text{ref}} = 6400$; reported times are medians of three runs.

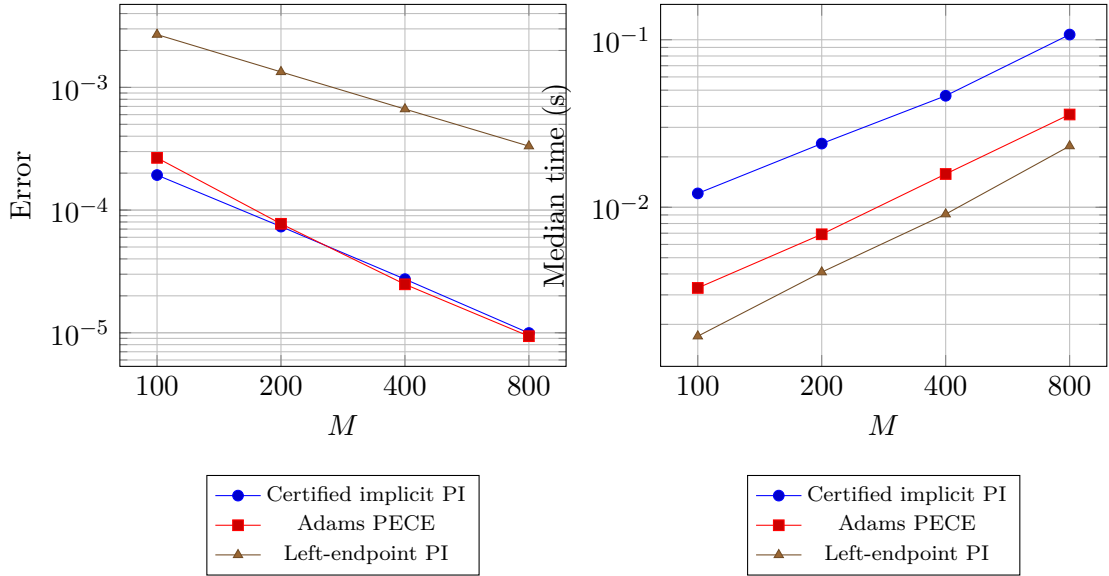


Figure 4: Accuracy and computational cost for the non-manufactured benchmark. The proposed method substantially improves on left-endpoint product integration in accuracy, while the timing panel reports the additional cost of implicit nonlinear solution and certification.

8 Scope and limitations

The piecewise-linear product-integration formula is classical. The results proved here concern the nonlinear triangular system generated by that formula. The diagonal coefficient gives the local condition (3.6); the associated weighted norm proves contraction when the global maximum-norm condition fails; and the positive triangular resolvent bounds the error of an inexact nonlinear vector. Equation (4.3) then separates iteration error from discretization error. The mesh-independent resolvent estimate and the graded-mesh theorem complete the stability and convergence analysis.

The numerical tests examine estimator reliability, comparison with two standard time-stepping methods, recovery of the expected rate on a graded mesh, and performance on a nonlinear problem without an exact solution. The method retains the $O(M^2)$ history cost of direct product integration. Variable-order equations, adaptive choice of the grading exponent, compressed history terms, and parallel implementations are not treated here.

9 Conclusion

An implicit piecewise-linear product-integration scheme has been analysed for nonlinear Caputo–Volterra equations. The lower-triangular system has a diagonal factor that can be checked at each mesh point. If this factor is smaller than one, the discrete problem has a unique solution, and both global and nodewise Picard iterations converge. The same triangular majorant gives a componentwise residual bound for an inexact discrete vector. A two-level estimate separates nonlinear iteration error from discretization error. The blockwise resolvent estimate gives second-order convergence for smooth data, while a graded mesh gives the rate $M^{-\min\{r\sigma, 2\}}$ for solutions with an initial power singularity.

The computations support these results. The residual bound is valid for partially converged nonlinear iterates, the smooth test has order two, and mesh grading restores order two in the singular example. The non-manufactured problem shows the behaviour of the method when no exact solution is available. The timing data also record the extra cost of the implicit solve and the residual calculation. The method therefore supplies a second-order product-integration discretization together with a proved stopping test for the nonlinear algebraic equations.

Statements and Declarations

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Author contributions. Thokchom Chhatrajit Singh was responsible for conceptualization, methodology, formal analysis, software, validation, investigation, writing–original draft, and writing–review and editing. The author read and approved the final manuscript.

Data availability. No external dataset was used. All numerical results were generated from the equations, parameters, and meshes specified in the manuscript and can be reproduced with the accompanying code.

Code availability. The R implementation used to generate the numerical results, tables, and figures is provided as Online Resource 1.

Supplementary information. Online Resource 1 contains the complete R source code, execution instructions, and machine-readable outputs for the residual-certification, convergence, graded-mesh, method-comparison, non-manufactured, and computational-timing experiments.

Use of generative artificial intelligence. During the preparation of this work, the author used ChatGPT to improve the manuscript’s readability and language. After using this tool, the author carefully reviewed all mathematical statements, proofs, computations, and conclusions and accepts full responsibility for the content of the manuscript.

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