Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

Fourier series for Henstock-Kurzweil integrable functions

Manuel Bello-Hernandez

mbello@unirioja.es

Universidad de La Rioja

Pablo Saenz-Lopez-Guerra

Universidad de La Rioja

Research Article

Keywords: Fourier series, Henstock-Kurzweil integral, Pointwise convergence, Uniform convergence, Dini condition, Localization principle, The van der Corput lemma

Posted Date: August 6th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-10402501/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

Fourier series for Henstock-Kurzweil integrable functions

M. Bello-Hernández P. Sáenz-López-Guerra

July 18, 2026

Abstract

We study the convergence of Fourier series for functions that are Henstock–Kurzweil integrable but not Lebesgue integrable. The lack of Lebesgue integrability is caused by a singular behaviour at the origin, whereas the oscillation near the origin provides the cancellation needed for Henstock–Kurzweil integrability. Together with a Dini-type condition or a bounded-variation condition around the point under consideration, this ensures convergence of the Fourier series. We also prove a localization result for this class of functions.

Keywords Fourier series, Henstock-Kurzweil integral, Pointwise convergence, Uniform convergence, Dini condition, Localization principle, The van der Corput lemma

1 Introduction

The convergence problem for Fourier series has played a central role in the development of real and harmonic analysis. For a periodic integrable function f, the symmetric partial sums are

$$S_n(f,x) = \sum_{|k|\le n} \widehat{f}(k)\mathrm{e}^{2\pi ikx}, \qquad \widehat{f}(k) = \int_0^1 f(t)\mathrm{e}^{-2\pi ikt}\,dt,$$

whenever the coefficient integral is meaningful. The basic problem is to determine whether $S_n(f,x)$ converges to $f(x)$, or to the usual midpoint value at a jump discontinuity.

Classical Lebesgue-space theory provides both positive and negative results. Kolmogorov constructed a Lebesgue integrable function whose Fourier series diverges almost everywhere [13]. Thus ordinary pointwise convergence cannot be expected for arbitrary L^1 functions. On the other hand, Carleson's theorem provides almost-everywhere convergence for L^2 functions, and Hunt extended this to L^p, $1 < p < \infty$ [3, 11]. Hence $p = 1$ is a genuine endpoint obstruction for ordinary Fourier summation.

The present work is motivated by the question of what remains true beyond the Lebesgue integral. The Henstock–Kurzweil integral, also known as the gauge integral, is a nonabsolute integral extending the Riemann and Lebesgue integrals and closely related to the primitive form of the fundamental theorem of calculus. Its modern development goes back to Kurzweil's 1957 paper and to Henstock's work, including his 1961 monograph [14, 9]. It is especially well adapted to highly oscillatory improper integrals and to functions that are derivatives but need not be Lebesgue integrable.

The functions considered in this paper are designed to lie beyond the Lebesgue L^1 framework while still being accessible to Henstock–Kurzweil integration and Fourier analysis. A typical model is

$$f(x) = x^{-a}h(x)\cos(x^{-b} - \theta), \qquad 0 < x < 1,$$

where $b > 0$, $\theta \in \mathbb{R}$, and h is of bounded-variation near the origin. The singular factor x^{-a} may destroy absolute integrability, in particular for $a \ge 1$, whereas the rapid oscillation of $\cos(x^{-b} - \theta)$ may still compensate for this singularity.

Thus the genuinely non-Lebesgue range relevant to our results is

$$1 \leq a < 1 + \frac{b}{2}.$$

Here the lower bound indicates the possible failure of Lebesgue integrability, while the upper bound is the threshold naturally produced by the second derivative van der Corput estimates on dyadic intervals near the origin.

Fourier analysis in the Henstock–Kurzweil setting differs substantially from the L^1 theory. A difficulty in studying Fourier series of Henstock–Kurzweil integrable functions is that, in general, the Fourier coefficients of such functions need only be $o(n)$ as n goes to infinity (see, for example, [15] and [25]). Talvila ([25]) studied Fourier series for the continuous primitive integral and considered the Banach space of periodic distributions that are distributional derivatives of continuous functions. This space contains periodic Lebesgue and Henstock–Kurzweil integrable functions, and many statements are formulated with the Alexiewicz norm in place of the L^1 norm [25]. In the same direction, Khadjiev proved that the norm of the widest continuous integral is equivalent to the Alexiewicz norm and that the corresponding integral is wider than the Lebesgue integral [12].

A closely related classical problem is the so-called coefficient problem for convergent trigonometric series. If

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges everywhere to a function f, then Cantor's uniqueness theorem shows that the coefficients are uniquely determined by the sum. The remaining question is whether they can be recovered from f by Fourier-type formulae, possibly with respect to an integral more general than the Lebesgue integral.

This problem was treated in depth by Denjoy [5]. Later developments replaced Denjoy's transfinite construction by several Perron-type and symmetric integrals, including the Marcinkiewicz–Zygmund integral, Burkill's SCP-integral, James' P^2-integral, and the Preiss–Thomson AS-integral. This line of research provides a natural background for Boas' monograph on integrability theorems for trigonometric transforms, although Boas' emphasis is on integrability properties of trigonometric transforms rather than on pointwise convergence of Fourier partial sums (see [2]).

The same coefficient-recovery viewpoint has also been extended beyond the trigonometric system. In the dyadic setting, generalized Denjoy-, Perron-, and Kurzweil–Henstock-type integrals have been used to recover coefficients of Haar and Walsh series from their sums. The works of Skvortsov and collaborators, for instance, study multidimensional dyadic Kurzweil–Henstock and Perron-type integrals for Haar and Walsh expansions, reducing the coefficient problem to the recovery of a primitive from an appropriate derivative associated with a dyadic differentiation basis. These developments show that nonabsolute integration is not merely a technical enlargement of Lebesgue integration, but a structural tool in the theory of orthogonal expansions: it provides the right language for recovering coefficients when the sum of a convergent orthogonal series lies outside the classical Lebesgue framework (see [21], [22]).

The aim of this paper is more modest than a general Carleson-type theorem for Henstock–Kurzweil integrable functions. We isolate a concrete family of oscillatory singularities for which ordinary Fourier partial sums can be controlled locally. The singularity lies at the origin, whereas convergence is studied at points $x_0 \in (0, 1)$. The guiding principle is that the oscillation near the origin supplies enough cancellation to make the singular contribution vanish, while the local behaviour near x_0 is governed by standard Fourier hypotheses such as bounded variation, Dini regularity, or localization.

Throughout the paper, if f is defined on $(0, 1)$, its 1-periodic extension is denoted by the same symbol. The notation $S_n(f, x)$ refers to the symmetric Fourier partial sum of this

periodic extension. The coefficient integrals are understood in the Henstock–Kurzweil sense whenever they are not Lebesgue integrals.

We prove the following results.

Theorem 1 (BV hypotheses around the convergence point)**.** *Let $a > 0$, $b > 0$, $\theta \in \mathbb{R}$, and let*

$$f(x) = g(x)\cos(x^{-b} - \theta), \qquad 0 < x < 1.$$

We assume that:

(i) $g \in L^1([\epsilon, 1])$ *for all* $\epsilon > 0$*;*

(ii) *there exists* $L \neq 0$ *such that* $\lim_{x\to 0^+} x^a g(x) = L$*;*

(iii) *the function* $h(x) = x^a g(x)$ *is of bounded variation in* $(0, \eta)$ *for some* $\eta > 0$*;*

(iv) g *coincides with a function of bounded variation in a neighbourhood of* $x_0 \in (0, 1)$*;*

(v) $a < 1 + b/2$.

Then for the 1*-periodic extension of the function f, we have*

$$\lim_{n\to\infty} S_n(f, x_0) = \frac{f(x_0+) + f(x_0-)}{2}.$$

Moreover, if g is continuous in the above neighbourhood, the convergence is uniform on compact subsets of this neighbourhood and the limit is f.

This theorem is proved in Section 3. In Section 4 we state an analogous convergence result replacing the local bounded-variation assumption at the convergence point by a Dini-type condition and prove a localization principle in the corresponding Henstock-Kurzweil integrable functions. We give examples which show that the above results are sharp.

2 Auxiliary results

In this section we state several auxiliary results. The first one is a well-known version of van der Corput estimate for oscillatory integrals. The classical van der Corput lemma (see [23, Chapter VIII]) states that if ϕ is a real-valued function of class $\mathcal{C}^k([c, d])$ and that $|\phi^{(k)}(x)| \geq C$ for every $x \in [c, d]$. Then

$$\left| \int_c^d e^{i\lambda\phi(x)}\, dx \right| \leq c_k \lambda^{-1/k}, \tag{1}$$

where c_k depends only on k and C, but not on λ or on ϕ, whenever one of the following two situations holds:

i) $k \geq 2$.

ii) $k = 1$ and ϕ' is monotone on $[c, d]$.

We can extend the van der Corput result by considering continuously differentiable "amplitudes" ψ.

Lemma 2. *Let $I = (c, d)$, where c and d may be finite or infinite. Let $\Phi \in \mathcal{C}^2(I)$ be a real-valued function such that Φ' does not vanish on I. Let $\psi \in \mathcal{C}^1(I)$. Suppose that*

$$q(x) := \frac{\psi(x)}{\Phi'(x)}$$

is absolutely continuous on every compact subinterval of I, that $q' \in L^1(I)$, and that the one-sided limits $q(c+)$ and $q(d-)$ exist. Then

$$\left| \int_c^d \psi(x) e^{i\Phi(x)}\, dx \right| \le |q(d-)| + |q(c+)| + \int_c^d |q'(x)|\, dx. \tag{2}$$

Proof. Let $[\alpha, \beta] \subset I$. Integrating by parts,

$$\int_\alpha^\beta \psi(x) e^{i\Phi(x)}\, dx = \left[\frac{\psi(x)}{i\Phi'(x)} e^{i\Phi(x)} \right]_\alpha^\beta - \int_\alpha^\beta e^{i\Phi(x)} \frac{d}{dx} \left(\frac{\psi(x)}{i\Phi'(x)} \right) dx. \tag{3}$$

Taking absolute values,

$$\left| \int_\alpha^\beta \psi(x) e^{i\Phi(x)}\, dx \right| \le |q(\beta)| + |q(\alpha)| + \int_\alpha^\beta |q'(x)|\, dx.$$

We finish the proof by passing to the limit[1] $\alpha \to c+$, $\beta \to d-$, obtaining (2). □

To prove Theorem 1 we have to apply van der Corput lemma in a nonstationary zone where we need the following result.

Lemma 3 (Uniform control of the kernel factor)**.** *Let $K \Subset (0,1)$ be a compact set and choose*

$$0 < \delta < \frac{1}{4} \operatorname{dist}(K, \{0,1\}).$$

For $x \in K$ and $0 \le t \le \delta$, set

$$\chi_x(t) := \frac{1}{\sin \pi(x-t)}.$$

Then

$$\sup_{x \in K} \|\chi_x\|_{\mathcal{C}^1([0,\delta])} < \infty, \tag{4}$$

where, as usual, $\|\chi_x\|_{\mathcal{C}^1([0,\delta])} = \|\chi_x\|_{L^\infty([0,\delta])} + \|\chi_x'\|_{L^\infty([0,\delta])}$. Moreover, if $h \in BV(0,\eta)$ and $0 < \delta \le \eta$, then, with $A = \delta^{-b}$ and

$$H_x(v) := h(v^{-1/b}) \chi_x(v^{-1/b}), \qquad v \ge A,$$

one has

$$\sup_{x \in K} \Big(\|H_x\|_{L^\infty([A,\infty))} + \operatorname{Var}(H_x; [A,\infty)) \Big) < \infty, \tag{5}$$

and also

$$\lim_{R \to \infty} \sup_{x \in K} \operatorname{Var}(H_x; [R,\infty)) = 0. \tag{6}$$

Proof. Let

$$d_K := \operatorname{dist}(K, \{0,1\}) > 0.$$

If $x \in K$ and $0 \le t \le \delta < d_K/4$, then

$$x - t \in [d_K - \delta, 1 - d_K] \subset [3d_K/4, 1 - d_K].$$

Thus $x - t$ stays a positive distance away from the zeros of $\sin(\pi \cdot)$. Consequently

$$m_K := \inf_{x \in K,\, 0 \le t \le \delta} |\sin \pi(x-t)| > 0.$$

[1] Formula (3), together with the existence of the one-sided limits at c and d, and the fact that $q' \in L^1(I)$, ensures the existence of the limit of the left-hand side of the preceding equation.

It follows that

$$|\chi_x(t)| \le m_K^{-1}, \qquad x \in K, \qquad 0 \le t \le \delta.$$

Moreover

$$\chi_x'(t) = \pi \frac{\cos \pi(x-t)}{\sin^2 \pi(x-t)},$$

and therefore

$$|\chi_x'(t)| \le \pi m_K^{-2}, \qquad x \in K, \qquad 0 \le t \le \delta.$$

This proves the uniform $\mathcal{C}^1$ estimate (4).

Write

$$H_x(v) = U(v)V_x(v), \qquad U(v) = h(v^{-1/b}), \qquad V_x(v) = \chi_x(v^{-1/b}).$$

The map $v \mapsto v^{-1/b}$ is monotone decreasing and maps $[A, \infty)$ onto $(0, \delta]$. Therefore composition with this map preserves bounded variation, up to reversal of orientation. In particular,

$$\operatorname{Var}(U; [A, \infty)) = \operatorname{Var}(h; (0, \delta]).$$

Similarly, since the functions χ_x have uniformly bounded $\mathcal{C}^1$ norms,

$$\operatorname{Var}(V_x; [A, \infty)) \le \sup_{x \in K} \|\chi_x'\|_{L^\infty([0,\delta])}\, \delta \le C_K \delta.$$

Using the standard product estimate

$$\operatorname{Var}(UV_x; I) \le \|U\|_{L^\infty(I)} \operatorname{Var}(V_x; I) + \|V_x\|_{L^\infty(I)} \operatorname{Var}(U; I),$$

we obtain the uniform bound (5).

It remains to prove the tail assertion. Since $h \in BV((0, \eta))$, this function has a finite limit at 0+,

$$\operatorname{Var}(U; [R, \infty)) = \operatorname{Var}(h; (0, R^{-1/b}]) \longrightarrow 0, \quad \text{as } R \to \infty.$$

On the other hand, (4) gives

$$\sup_{x \in K} \operatorname{Var}(V_x; [R, \infty)) \le C_K R^{-1/b} \longrightarrow 0, \quad \text{as } R \to \infty.$$

The same product estimate applied on $[R, \infty)$ gives (6). This completes the proof. □

The following result is a version of van der Corput lemma with amplitude of bounded variation.

Lemma 4 (Abel–Stieltjes)**.** *Let $I = [r, s]$ be an interval, let $H \in BV(I)$, and let $B \in L^1(I)$. Define*

$$F(x) = \int_r^x B(u)\, du;$$

then $F(r) = 0$ and

$$\int_r^s H(v)B(v)\, dv = H(s)F(s) - \int_{(r,s]} F(v)\, dH(v).$$

Moreover,

$$\left| \int_r^s H(v)B(v)\, dv \right| \le \sup_{r \le x \le s} |F(x)| \left(|H(s)| + \operatorname{Var}(H; [r, s]) \right).$$

Proof. The function F is absolutely continuous, and hence of bounded variation on I. Since $H \in BV(I)$, the Riemann–Stieltjes integration by parts formula gives

$$\int_r^s H\,dF = H(s)F(s) - H(r)F(r) - \int_{(r,s]} F\,dH.$$

But $F(r) = 0$ and $dF(v) = B(v)\,dv$, whence

$$\int_r^s H(v)B(v)\,dv = H(s)F(s) - \int_{(r,s]} F(v)\,dH(v).$$

Taking absolute values,

$$\begin{aligned}\left|\int_r^s H(v)B(v)\,dv\right| &\le |H(s)|\,|F(s)| + \sup_{r\le x\le s} |F(x)|\,|dH|(I)\\ &\le \sup_{r\le x\le s} |F(x)|\,(|H(s)| + \mathrm{Var}(H;I))\,.\end{aligned}$$

This proves the result. □

Remark 5 (Uniform Abel–Stieltjes step). In applications of the Abel–Stieltjes lemma, the estimates are uniform with respect to $x \in K$ whenever the amplitudes H_x satisfy

$$\sup_{x\in K} (\|H_x\|_\infty + \mathrm{Var}(H_x)) < \infty.$$

Indeed, the lemma gives

$$\left|\int_r^s H_x(v)B_\lambda(v)\,dv\right| \le \sup_{r<s}\left|\int_r^s B_\lambda(v)\,dv\right| (\|H_x\|_\infty + \mathrm{Var}(H_x))\,,$$

and the last factor is uniformly bounded by Lemma 3. Hence any pointwise estimate of the primitive of B_λ which tends to zero uniformly on the relevant intervals gives a uniform estimate after multiplication by H_x.

The main auxiliary estimate is the following second order van der Corput estimate with BV amplitude.

Lemma 6 (Second order van der Corput estimate with BV amplitude)**.** *Let $I = [c,d]$ be a compact interval, and let $\phi \in \mathcal{C}^2(I)$ be a real-valued phase such that*

$$|\phi''(x)| \ge m > 0, \qquad x \in I.$$

Then there exists an absolute constant $C > 0$ such that, for every $\mu \ge 1$ and every amplitude $B \in BV(I)$,

$$\left|\int_c^d B(x)e^{i\mu\phi(x)}\,dx\right| \le Cm^{-1/2}\mu^{-1/2}\left(\|B\|_{L^\infty(I)} + \mathrm{Var}(B;I)\right).$$

Proof. The proof consists in separating the oscillatory estimate from the regularity of the amplitude.

Define, for $x \in [c,d]$,

$$F(x) = \int_c^x e^{i\mu\phi(u)}\,du.$$

Applying the van der Corput estimate (1) for $k = 2$ to all subintervals $[c,x] \subset I$, we get

$$\|F\|_{L^\infty(I)} \le Cm^{-1/2}\mu^{-1/2}. \tag{7}$$

Since $B \in BV(I)$, it induces a finite signed (or complex) Stieltjes measure dB, whose total variation measure will be denoted by $d|B|$. Moreover, F is absolutely continuous and $dF(x) = e^{i\mu\phi(x)}\,dx$. Hence

$$\int_c^d B(x)e^{i\mu\phi(x)}\,dx = \int_c^d B\,dF.$$

By the integration by parts formula for functions of bounded variation,

$$\int_c^d B\,dF = B(d)F(d) - \int_{(c,d]} F\,dB.$$

Using (7), we conclude that

$$\left|\int_c^d B(x)e^{i\mu\phi(x)}\,dx\right| \le Cm^{-1/2}\mu^{-1/2}\left(\|B\|_{L^\infty(I)} + \mathrm{Var}(B;I)\right).$$

This proves the lemma. □

Remark 7 (Uniform van der Corput constants). If $K \Subset (0,1)$ is a compact set, the nonstationary and second-order van der Corput estimates used in the proof are uniform for $x \in K$. The reason is that the oscillatory phases do not depend on x; the parameter x only appears in the amplitude through the factor $\chi_x(t) = 1/\sin\pi(x-t)$. By Lemma 3, after the change of variables $v = t^{-b}$, the corresponding amplitudes have uniformly bounded BV norms and uniformly small tail variation. Therefore the constants in the nonstationary and stationary-region estimates are independent of $x \in K$.

To give the complete argument for an example showing that the localization principle for Fourier series may fail outside L^1, we shall need the asymptotic computation stated in Lemma 10. Its proof requires an explicit evaluation of the contribution arising from a non-degenerate stationary point. We therefore first record the following one-dimensional stationary-phase formula. We include its proof for completeness; the argument reduces the phase locally to a quadratic one by means of the Morse lemma [16, Lemma 2.2] and then uses the classical Fresnel integral.

Proposition 8 (One-dimensional stationary phase)**.** *Let $c < d$, let $\phi \in \mathcal{C}^\infty([c,d])$ be a real function, and let $A \in \mathcal{C}_c^\infty((c,d))$. Assume that ϕ has a unique critical point x_* in the interior of* $\operatorname{supp} A$ *and that*

$$\phi'(x_*) = 0, \qquad \phi''(x_*) \neq 0.$$

Then, as $\lambda \to +\infty$,

$$\int_c^d A(x)e^{i\lambda\phi(x)}\,dx = A(x_*)e^{i\lambda\phi(x_*)}e^{i\pi\,\mathrm{sgn}(\phi''(x_*))/4}\left(\frac{2\pi}{\lambda|\phi''(x_*)|}\right)^{1/2} + o(\lambda^{-1/2}). \tag{8}$$

More generally, if $K \subset (c,d)$ is compact and $|\phi'| \ge m > 0$ on K, then every $B \in \mathcal{C}_c^\infty(K)$ satisfies

$$\int_c^d B(x)e^{i\lambda\phi(x)}\,dx = O_N(\lambda^{-N}) \qquad (N \ge 1).$$

Proof. Put $\varepsilon = \operatorname{sgn}\phi''(x_*)$ and $\tau = |\phi''(x_*)|$. By Taylor's formula with integral remainder,

$$2\varepsilon\big(\phi(x) - \phi(x_*)\big) = (x - x_*)^2 q(x),$$

where $q \in \mathcal{C}^\infty$ near x_* and $q(x_*) = \tau > 0$. Hence

$$y = y(x) := (x - x_*)\sqrt{q(x)}$$

is smooth near x_*, satisfies $y(x_*) = 0$ and $y'(x_*) = \sqrt{\tau}$, and is therefore a local diffeomorphism. Moreover,

$$\phi(x) = \phi(x_*) + \frac{\varepsilon}{2} y(x)^2.$$

Choose $\eta \in \mathcal{C}_c^\infty((c,d))$ which equals 1 near x_* and whose support lies in the coordinate neighbourhood. The complementary amplitude $(1-\eta)A$ is supported where $|\phi'|$ is bounded below; repeated integration by parts with

$$L = \frac{1}{i\lambda\phi'(x)} \frac{d}{dx}, \qquad L e^{i\lambda\phi} = e^{i\lambda\phi},$$

shows that its contribution is $O_N(\lambda^{-N})$ for every N.

For the localized part, write $x = x(z) = y^{-1}(z)$ and

$$\widetilde{A}(z) = \eta(x(z)) A(x(z)) x'(z),$$

extended by zero outside the coordinate image. Then $\widetilde{A} \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\widetilde{A}(0) = \frac{A(x_*)}{\sqrt{\tau}},$$

and

$$\int_c^d \eta(x) A(x) e^{i\lambda\phi(x)}\, dx = e^{i\lambda\phi(x_*)} \int_{\mathbb{R}} \widetilde{A}(z) e^{i\varepsilon\lambda z^2/2}\, dz.$$

It remains to use the elementary quadratic-phase formula

$$\int_{\mathbb{R}} B(z) e^{i\varepsilon\lambda z^2/2}\, dz = B(0) e^{i\pi\varepsilon/4} \left(\frac{2\pi}{\lambda}\right)^{1/2} + o(\lambda^{-1/2}), \qquad B \in \mathcal{C}_c^\infty(\mathbb{R}). \tag{9}$$

For completeness, take $\rho \in \mathcal{C}_c^\infty(\mathbb{R})$ with $\rho = 1$ near zero and write $B = B(0)\rho + zB_1$, where $B_1 \in \mathcal{C}_c^\infty(\mathbb{R})$. One integration by parts gives

$$\int_{\mathbb{R}} z B_1(z) e^{i\varepsilon\lambda z^2/2}\, dz = O(\lambda^{-1}).$$

After $w = \sqrt{\lambda} z$, the other term is evaluated by truncating at $|w| \le M$, using dominated convergence on the bounded interval and integration by parts on the two tails, and then applying

$$\int_{\mathbb{R}} e^{i\varepsilon w^2/2}\, dw = e^{i\pi\varepsilon/4}\sqrt{2\pi}.$$

This proves (9), and substitution of $B(0) = A(x_*)/\sqrt{\tau}$ proves (8). □

The stationary-phase formula stated above assumes a smooth, compactly supported amplitude. In the application to the Fourier coefficients below, however, the relevant amplitude is only of bounded variation and the stationary point approaches the singular endpoint as the frequency tends to infinity. We therefore record the following rescaled version, which isolates the leading contribution under precisely the regularity assumptions needed later.

Lemma 9 (Stationary phase with a BV amplitude on a moving scale)**.** *Let $a, b > 0$, set*

$$\alpha = 1 + \frac{b}{2} - a > 0,$$

and let $h \in BV((0,\eta))$ satisfy

$$h(x) \longrightarrow L \qquad (x \to 0+).$$

If $\chi \in C_c^\infty([0,\eta))$ and $\chi = 1$ in a neighbourhood of the origin, then, as $\lambda \to +\infty$,

$$\int_0^\eta \chi(x)h(x)x^{-a}\mathrm{e}^{-i(x^{-b}+\lambda x)}\,dx$$
$$= L\left(\frac{2\pi}{b(b+1)}\right)^{1/2} b^{\alpha/(b+1)}\lambda^{-\alpha/(b+1)}\exp\left(-i(b+1)b^{-b/(b+1)}\lambda^{b/(b+1)} - \frac{i\pi}{4}\right) + o(\lambda^{-\alpha/(b+1)}). \tag{10}$$

Moreover,

$$\int_0^\eta \chi(x)h(x)x^{-a}\mathrm{e}^{i(x^{-b}-\lambda x)}\,dx = o(\lambda^{-\alpha/(b+1)}). \tag{11}$$

Proof. Set

$$x_\lambda = \left(\frac{b}{\lambda}\right)^{1/(b+1)}, \qquad \mu = x_\lambda^{-b} = b^{-b/(b+1)}\lambda^{b/(b+1)},$$

and make the change of variables $x = x_\lambda y$. Since $\lambda x_\lambda = b\mu$, we have

$$x^{-b} + \lambda x = \mu\phi(y), \qquad \phi(y) = y^{-b} + by.$$

The function ϕ has a unique critical point on $(0,\infty)$, namely $y = 1$, and

$$\phi(1) = b+1, \qquad \phi''(1) = b(b+1) > 0.$$

Thus the integral on the left-hand side of (10) becomes

$$x_\lambda^{1-a}\int_0^{\eta/x_\lambda} \chi(x_\lambda y)h(x_\lambda y)y^{-a}\mathrm{e}^{-i\mu\phi(y)}\,dy. \tag{12}$$

Choose $0 < c < 1 < C < \infty$ and a function $\vartheta \in \mathcal{C}_c^\infty((c,C))$ which is equal to 1 in a neighbourhood of 1. We first consider the part of (12) corresponding to the amplitude $\vartheta(y)$. On $[c,C]$ we write

$$h(x_\lambda y) = L + r_\lambda(y), \qquad r_\lambda(y) = h(x_\lambda y) - L.$$

Because $h(x) \to L$ as $x \to 0+$,

$$\|r_\lambda\|_{L^\infty([c,C])} \longrightarrow 0.$$

Furthermore, composition with the increasing map $y \mapsto x_\lambda y$ preserves variation, and hence

$$\mathrm{Var}(r_\lambda;[c,C]) = \mathrm{Var}(h;[cx_\lambda, Cx_\lambda]) \le \mathrm{Var}(h;(0,Cx_\lambda]) \longrightarrow 0.$$

Here the last limit follows from the finiteness of the variation measure of h and the fact that the nested intervals $(0,Cx_\lambda]$ decrease to the empty set. Since $y^{-a}\vartheta(y)$ is smooth on $[c,C]$, the elementary product estimate for functions of bounded variation gives

$$\|r_\lambda y^{-a}\vartheta\|_{L^\infty([c,C])} + \mathrm{Var}(r_\lambda y^{-a}\vartheta;[c,C]) = o(1).$$

The second-order van der Corput estimate with a BV amplitude (Lemma 6) therefore yields

$$\int_c^C r_\lambda(y)y^{-a}\vartheta(y)\mathrm{e}^{-i\mu\phi(y)}\,dy = o(\mu^{-1/2}). \tag{13}$$

For the constant part, the classical one-dimensional stationary-phase formula gives

$$L\int_c^C y^{-a}\vartheta(y)\mathrm{e}^{-i\mu\phi(y)}\,dy = L\left(\frac{2\pi}{\mu\phi''(1)}\right)^{1/2}\exp\left(-i\mu\phi(1) - \frac{i\pi}{4}\right) + o(\mu^{-1/2})$$
$$= L\left(\frac{2\pi}{b(b+1)}\right)^{1/2}\mu^{-1/2}\exp\left(-i(b+1)\mu - \frac{i\pi}{4}\right) + o(\mu^{-1/2}). \tag{14}$$

It remains to control the complement of the stationary region. We claim that

$$\int_0^{\eta/x_\lambda} (1-\vartheta(y))\chi(x_\lambda y)h(x_\lambda y)y^{-a}\mathrm{e}^{-i\mu\phi(y)}\,dy = o(\mu^{-1/2}). \tag{15}$$

Indeed, ϕ' does not vanish on the support of $1-\vartheta$. Split that support into $(0,c]$, $[c,C]$, and $[C,\eta/x_\lambda)$. On the middle interval, $|\phi'|$ is bounded from below, and Abel–Stieltjes integration by parts gives $O(\mu^{-1})$, uniformly in λ, because the amplitudes have uniformly bounded BV norms. Near zero one has $|\phi'(y)| \asymp y^{-b-1}$; the same argument, now applied to $y^{-a}/\phi'(y)$, gives a bound

$$O(\mu^{-1}c^{b+1-a}).$$

At infinity, $\phi'(y) \to b$ and the cut-off $\chi(x_\lambda y)$ restricts the integral to an interval of length $O(x_\lambda^{-1})$. Abel–Stieltjes integration by parts, followed by a dyadic decomposition, gives

$$O(\mu^{-1}C^{1-a})$$

after summation when $a > 1$, with the usual logarithmic modification when $a = 1$; when $0 < a < 1$, the endpoint created by χ is instead estimated before rescaling, where it is separated from the origin and is $O(\lambda^{-1})$. In every case, after multiplication by x_λ^{1-a}, these bounds are $o(x_\lambda^{1-a}\mu^{-1/2})$: first let $\lambda \to \infty$, and then let $c \downarrow 0$ and $C \uparrow \infty$. The condition $\alpha = 1+b/2-a > 0$ is precisely what is needed in the endpoint estimates. This proves (15).

Combining (13), (14), and (15), and observing that

$$x_\lambda^{1-a}\mu^{-1/2} = b^{\alpha/(b+1)}\lambda^{-\alpha/(b+1)},$$

we obtain (10).

For (11), after the same rescaling the phase is

$$\psi(y) = y^{-b} - by, \qquad \psi'(y) = -b(y^{-b-1}+1),$$

so that ψ' never vanishes on $(0,\infty)$. The preceding Abel–Stieltjes estimates therefore apply on the whole interval, with no stationary region, and yield

$$\int_0^{\eta} \chi(x)h(x)x^{-a}\mathrm{e}^{i(x^{-b}-\lambda x)}\,dx = o\big(x_\lambda^{1-a}\mu^{-1/2}\big) = o\big(\lambda^{-\alpha/(b+1)}\big).$$

This completes the proof. □

Lemma 10 (A stationary phase computation)**.** *Let $\varepsilon > 0$. As $\mu \to +\infty$, one has*

$$\int_0^{\varepsilon} t^{1/2}\cos\left(\frac{1}{t^2}+\mu t\right)\,dt = C_0\mu^{-5/6}\cos\left(3\cdot 2^{-2/3}\mu^{2/3}+\frac{\pi}{4}\right)+o(\mu^{-5/6}),$$

where

$$C_0 = 2^{5/6}\sqrt{\frac{\pi}{3}}.$$

In particular, $C_0 \neq 0$.

Proof. We prove the corresponding asymptotic formula for the complex integral

$$I(\mu) := \int_0^{\varepsilon} t^{1/2}e^{i(1/t^2+\mu t)}\,dt,$$

and then take real parts.

Make the change of variables $t = \mu^{-1/3}u$. Then

$$I(\mu) = \mu^{-1/2}J(\lambda),$$

where

$$J(\lambda) = \int_0^{R_\lambda} u^{1/2} e^{i\lambda\Phi(u)}\, du,$$

$\lambda = \mu^{2/3}$, $R_\lambda = \mu^{1/3}\varepsilon = \lambda^{1/2}\varepsilon$, and

$$\Phi(u) = u + u^{-2}.$$

We first analyse the critical point of the phase. One has

$$\Phi'(u) = 1 - \frac{2}{u^3}, \qquad \Phi''(u) = \frac{6}{u^4}.$$

Hence Φ has a unique positive stationary point, $u_0 = 2^{1/3}$. which is non-degenerate. Since $R_\lambda \to +\infty$ as $\lambda \to \infty$, the point u_0 belongs to $(0, R_\lambda)$ for all sufficiently large λ.

Let $\chi \in C_c^\infty((0,\infty))$ be such that $\chi = 1$ in a small neighbourhood of u_0, and such that Φ' vanishes nowhere on $\operatorname{supp}\chi \setminus \{u_0\}$. We write

$$J(\lambda) = J_{\mathrm{crit}}(\lambda) + J_{\mathrm{non}}(\lambda),$$

where

$$J_{\mathrm{crit}}(\lambda) = \int_0^{R_\lambda} \chi(u) u^{1/2} e^{i\lambda\Phi(u)}\, du$$

and

$$J_{\mathrm{non}}(\lambda) = \int_0^{R_\lambda} (1-\chi(u)) u^{1/2} e^{i\lambda\Phi(u)}\, du.$$

Since χ is compactly supported in $(0,\infty)$, the upper endpoint R_λ lies beyond $\operatorname{supp}\chi$ for all sufficiently large λ. We may therefore apply Proposition 8 to the fixed amplitude $A(u) = \chi(u)u^{1/2}$ and the phase $\Phi(u) = u + u^{-2}$. The unique critical point is $u_0 = 2^{1/3}$, and

$$\Phi(u_0) = 3\cdot 2^{-2/3}, \qquad \Phi''(u_0) = 6\cdot 2^{-4/3}, \qquad A(u_0) = 2^{1/6}.$$

Consequently,

$$J_{\mathrm{crit}}(\lambda) = C_0 \lambda^{-1/2} e^{i(3\cdot 2^{-2/3}\lambda + \pi/4)} + o(\lambda^{-1/2}), \qquad C_0 = 2^{5/6}\sqrt{\frac{\pi}{3}}.$$

It remains to estimate $J_{\mathrm{non}}(\lambda)$; this must be done separately because its interval has the singular endpoint 0 and the moving endpoint R_λ.

Let us prove that

$$J_{\mathrm{non}}(\lambda) = o(\lambda^{-1/2}).$$

This is a nonstationary estimate. We give the details because the upper endpoint R_λ depends on λ.

Near $u = 0$, one has

$$\Phi'(u) = 1 - \frac{2}{u^3} \sim -\frac{2}{u^3}.$$

Thus

$$\frac{u^{1/2}}{\Phi'(u)} = O(u^{7/2}) \qquad (u \to 0^+).$$

Using

$$e^{i\lambda\Phi(u)} = \frac{1}{i\lambda\Phi'(u)} \frac{d}{du} e^{i\lambda\Phi(u)},$$

integration by parts produces no boundary contribution at $u = 0$, and the contribution from a sufficiently small interval near 0 is $O(\lambda^{-1})$.

On any fixed compact subinterval of $(0,\infty)$ avoiding u_0, the quantity $|\Phi'(u)|$ is bounded below by a positive constant. Hence one integration by parts gives again an $O(\lambda^{-1})$ contribution.

Finally, choose $M > 0$ large enough so that $|\Phi'(u)| \geq c > 0$ for all $u \geq M$. On the interval

$$M \leq u \leq R_\lambda$$

we integrate by parts. The boundary terms are bounded by

$$C\lambda^{-1} R_\lambda^{1/2}.$$

Since

$$R_\lambda = \varepsilon \lambda^{1/2},$$

this is

$$O(\lambda^{-1}\lambda^{1/4}) = O(\lambda^{-3/4}) = o(\lambda^{-1/2}).$$

The integral term obtained after integration by parts is estimated in the same way, because for $u \geq M$ the function Φ' is bounded away from zero and

$$\left(\frac{u^{1/2}}{\Phi'(u)}\right)' = O(u^{-1/2}) \qquad (u \to \infty).$$

Therefore

$$\lambda^{-1} \int_M^{R_\lambda} \left| \left(\frac{u^{1/2}}{\Phi'(u)}\right)' \right| du = O(\lambda^{-1} R_\lambda^{1/2}) = O(\lambda^{-3/4}).$$

Hence

$$J_{\text{non}}(\lambda) = o(\lambda^{-1/2}).$$

Combining the critical and non-critical estimates, we obtain

$$J(\lambda) = C_0 \lambda^{-1/2} e^{i(3\cdot 2^{-2/3}\lambda + \pi/4)} + o(\lambda^{-1/2}).$$

Since

$$I(\mu) = \mu^{-1/2} J(\lambda), \qquad \lambda = \mu^{2/3},$$

we have

$$\mu^{-1/2}\lambda^{-1/2} = \mu^{-1/2}\mu^{-1/3} = \mu^{-5/6}.$$

Therefore

$$I(\mu) = C_0 \mu^{-5/6} e^{i(3\cdot 2^{-2/3}\mu^{2/3} + \pi/4)} + o(\mu^{-5/6}).$$

Taking real parts gives

$$\int_0^\varepsilon t^{1/2} \cos\left(\frac{1}{t^2} + \mu t\right) dt = C_0 \mu^{-5/6} \cos\left(3 \cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}\right) + o(\mu^{-5/6}),$$

as claimed. □

3 Proof of Theorem 1

Throughout this section we denote

$$H(v) = h(v^{-1/b})\chi(v^{-1/b}), \tag{16}$$

and assume that the notations and hypotheses of Theorem 1 hold.

Let us consider

$$\Phi_\lambda^+(v) = v + \lambda v^{-1/b}.$$

Then

$$(\Phi_\lambda^+)'(v) = 1 - \frac{\lambda}{b} v^{-1/b-1}$$

and the unique critical point of Φ_λ^+ is given by

$$v_\lambda = \left(\frac{\lambda}{b}\right)^{b/(b+1)}. \tag{17}$$

We need three additional auxiliary lemmas.

Lemma 11. *Under the notations and assumptions of Theorem 1, we have*

$$\lim_{\lambda\to\infty} \int_A^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv = 0.$$

Proof. We split the integral into three regions:

$$[A,\infty) = [A, v_\lambda/2] \cup [v_\lambda/2, 2v_\lambda] \cup [2v_\lambda, \infty).$$

and divide the proof into three steps, corresponding to the three intervals of integration.

Step 1: Bounded nonstationary region

If $A \le v \le \frac{v_\lambda}{2}$, then for λ large enough we get

$$\begin{aligned} |(\Phi_\lambda^+)'(v)| &= \frac{\lambda}{b} v^{-1/b-1} - 1 \\ &\ge \frac{\lambda}{b} v^{-1/b-1}. \end{aligned}$$

Let $m(v) = v^{\alpha-1}$. Then

$$\left|\frac{m(v)}{(\Phi_\lambda^+)'(v)}\right| \lesssim \lambda^{-1} v^{\alpha-1} v^{1/b+1} = \lambda^{-1} v^{\alpha+1/b}.$$

Moreover,

$$\left|\left(\frac{m(v)}{(\Phi_\lambda^+)'(v)}\right)'\right| \lesssim \lambda^{-1} v^{\alpha+1/b-1}.$$

Consequently, if

$$A \le r < s \le v_\lambda/2,$$

the nonstationary phase lemma (Lemma 2) gives

$$\begin{aligned} \left|\int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv\right| &\lesssim \lambda^{-1} s^{\alpha+1/b} + \lambda^{-1} r^{\alpha+1/b} + \lambda^{-1} \int_r^s v^{\alpha+1/b-1}\, dv \\ &\lesssim \lambda^{-1} s^{\alpha+1/b} + \lambda^{-1} r^{\alpha+1/b}. \end{aligned}$$

Since $s \le v_\lambda/2$, it follows that

$$\left|\int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv\right| \lesssim \lambda^{-1} v_\lambda^{\alpha+1/b} + O(\lambda^{-1}).$$

From (17) we have

$$\lambda^{-1} v_\lambda^{\alpha+1/b} \asymp \lambda^{-1} \lambda^{\frac{b}{b+1}(\alpha+1/b)} = \lambda^{\frac{b\alpha+1}{b+1}-1} = \lambda^{\frac{b(\alpha-1)}{b+1}}.$$

Since in our case $\alpha = \frac{a-1}{b} < 1/2 < 1$,

$$\lim_{\lambda\to\infty} \sup_{A\le r<s\le v_\lambda/2} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| = 0.$$

Applying now the Abel–Stieltjes lemma (Lemma 4) with

$$B(v) = v^{\alpha-1} e^{i\Phi_\lambda^+(v)},$$

we obtain

$$\begin{aligned} &\left| \int_A^{v_\lambda/2} H(v) v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| \\ &\qquad \le \sup_{A\le r<s\le v_\lambda/2} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| \left(\|H\|_\infty + \mathrm{Var}(H; [A,\infty)) \right). \end{aligned}$$

Since the factor in parentheses is bounded independently of λ, we conclude

$$\lim_{\lambda\to\infty} \int_A^{v_\lambda/2} H(v) v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv = 0.$$

Step 2: Non-stationary phase on an unbounded interval

If

$$v \ge 2v_\lambda,$$

then there exists a constant $c_b > 0$ such that

$$(\Phi_\lambda^+)'(v) \ge c_b. \tag{18}$$

Moreover,

$$(\Phi_\lambda^+)''(v) = \frac{\lambda}{b} \left(1 + \frac{1}{b}\right) v^{-1/b-2}.$$

Again, let

$$m(v) = v^{\alpha-1}.$$

By (18), we have

$$\left| \frac{m(v)}{(\Phi_\lambda^+)'(v)} \right| \lesssim v^{\alpha-1}.$$

Also,

$$\begin{aligned} \left| \left(\frac{m(v)}{(\Phi_\lambda^+)'(v)} \right)' \right| &\le C v^{\alpha-2} + C\lambda v^{\alpha-1} v^{-1/b-2} \\ &= C v^{\alpha-2} + C\lambda v^{\alpha-1/b-3}. \end{aligned}$$

Thus, by Lemma 2 for $2v_\lambda \le r < s \le \infty$,

$$\begin{aligned} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| &\lesssim r^{\alpha-1} + \int_r^\infty v^{\alpha-2}\, dv + \lambda \int_r^\infty v^{\alpha-1/b-3}\, dv \\ &\lesssim r^{\alpha-1} + \lambda r^{\alpha-1/b-2}. \end{aligned}$$

Since $r \ge 2v_\lambda$ and (17), we have

$$\lambda r^{-1/b-1} \lesssim 1.$$

Hence

$$\lambda r^{\alpha-1/b-2} = r^{\alpha-1} \lambda r^{-1/b-1} \lesssim r^{\alpha-1}.$$

Thus

$$\left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| \lesssim r^{\alpha-1}.$$

In particular,

$$\sup_{2v_\lambda \le r < s \le \infty} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv \right| \lesssim v_\lambda^{\alpha-1}.$$

Since $\alpha = \frac{a-1}{b} < \frac{1}{2} < 1$, by the Abel–Stieltjes lemma (Lemma 4), we conclude

$$\lim_{\lambda\to\infty} \int_{2v_\lambda}^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv = 0.$$

Step 3: Stationary phase

Now we consider $v \in [v_\lambda/2, 2v_\lambda]$. Denote

$$I_{\lambda,\mathrm{crit}} := \int_{v_\lambda/2}^{2v_\lambda} H(v) v^{\alpha-1} e^{i\Phi_\lambda^+(v)}\, dv.$$

We perform the change of variables

$$v = v_\lambda w, \qquad \frac{1}{2} \le w \le 2.$$

Then

$$\Phi_\lambda^+(v_\lambda w) = v_\lambda \left(w + b w^{-1/b} \right).$$

Define

$$\Psi(w) = w + b w^{-1/b}.$$

Then

$$\Phi_\lambda^+(v_\lambda w) = v_\lambda \Psi(w).$$

Thus

$$I_{\lambda,\mathrm{crit}} = v_\lambda^\alpha \int_{1/2}^2 B_\lambda(w) e^{i v_\lambda \Psi(w)}\, dw. \tag{19}$$

where

$$B_\lambda(w) = H(v_\lambda w) w^{\alpha-1}, \qquad \frac{1}{2} \le w \le 2.$$

We next examine the rescaled phase. For $1/2 \le w \le 2$,

$$\Psi''(w) \ge \left(1 + \frac{1}{b}\right) 2^{-1/b-2} =: c_b > 0.$$

We now estimate the BV-norm of B_λ uniformly in λ. Since $w \mapsto w^{\alpha-1}$ is $\mathcal{C}^1([1/2,2])$, and $w \mapsto v_\lambda w$ is increasing,

$$\|H(v_\lambda w)\|_{L^\infty(1/2,2)} = \|H\|_{L^\infty(v_\lambda/2, 2v_\lambda)},$$

and using the elementary product estimate for functions of bounded variation,

$$\|B_\lambda\|_\infty + \mathrm{Var}(B_\lambda; [1/2,2]) \le C_{\alpha,H} \tag{20}$$

for all sufficiently large λ.

We may now apply Lemma 6 on the interval $[1/2, 2]$, with

$$\phi = \Psi, \qquad \mu = v_\lambda.$$

This gives

$$\left|\int_{1/2}^{2} B_\lambda(w) e^{iv_\lambda \Psi(w)}\, dw\right| \le C v_\lambda^{-1/2} \left(\|B_\lambda\|_\infty + \mathrm{Var}(B_\lambda; [1/2, 2])\right).$$

Using (20), we obtain

$$\left|\int_{1/2}^{2} B_\lambda(w) e^{iv_\lambda \Psi(w)}\, dw\right| \le C_{b,\alpha,H} v_\lambda^{-1/2}. \tag{21}$$

Therefore, from (19) and (21) it follows

$$\lim_{\lambda\to\infty} I_{\lambda,\mathrm{crit}} = 0.$$

Combining the three steps for the positive phase, we get the statement of the lemma. □

Lemma 12 (Uniform singular estimate)**.** *Let $a > 0$, $b > 0$, and*

$$a < 1 + \frac{b}{2}.$$

Let $h \in BV((0,\eta))$ and assume that $h(t)$ has a finite limit as $t \to 0^+$. Let $K \Subset (0,1)$ and choose

$$0 < \delta < \min\left\{\eta, \frac{1}{4}\operatorname{dist}(K, \{0,1\})\right\}.$$

For $x \in K$ put

$$\chi_x(t) = \frac{1}{\sin \pi(x-t)}.$$

Then, for each fixed $\theta \in \mathbb{R}$,

$$\lim_{\lambda\to\infty} \sup_{x\in K} \left|\int_0^\delta h(t) t^{-a} \cos(t^{-b} - \theta) \chi_x(t) \mathrm{e}^{i\lambda t}\, dt\right| = 0.$$

The same conclusion holds with $\mathrm{e}^{-i\lambda t}$ in place of $\mathrm{e}^{i\lambda t}$.

Proof. We indicate the modifications needed to make the preceding pointwise estimates uniform with respect to $x \in K$. Put

$$\alpha := \frac{a-1}{b}.$$

The condition $a < 1 + b/2$ is equivalent to

$$\alpha < \frac{1}{2}.$$

With $v = t^{-b}$, the integral becomes a linear combination of integrals of the form

$$\int_A^\infty H_x(v) v^{\alpha-1} \mathrm{e}^{i\Phi_\lambda^\pm(v)}\, dv, \qquad A = \delta^{-b},$$

where

$$H_x(v) = h(v^{-1/b})\chi_x(v^{-1/b}), \qquad \Phi_\lambda^\pm(v) = \pm v + \lambda v^{-1/b}.$$

The phase shift θ contributes only unimodular constants.

By Lemma 3 the functions H_x have uniformly bounded BV norms on $[A,\infty)$ and uniformly small tail variation. On the non-stationary pieces, Remarks 5 and 7 give exactly the same estimates as in the pointwise proof, uniformly in $x \in K$. On the stationary piece for

$$\Phi_\lambda^+(v) = v + \lambda v^{-1/b},$$

we write

$$v = v_\lambda w, \qquad v_\lambda = \left(\frac{\lambda}{b}\right)^{b/(b+1)}, \qquad \frac{1}{2} \le w \le 2.$$

Then

$$\Phi_\lambda^+(v_\lambda w) = v_\lambda \Psi(w), \qquad \Psi(w) = w + bw^{-1/b},$$

and Ψ'' is bounded below by a positive constant on $[1/2, 2]$. The second-order van der Corput estimate with BV amplitude gives, uniformly in $x \in K$,

$$\left| \int_{v_\lambda/2}^{2v_\lambda} H_x(v) v^{\alpha-1} \mathrm{e}^{i\Phi_\lambda^+(v)}\, dv \right| \le C v_\lambda^{\alpha-1/2}.$$

Since $\alpha < 1/2$, this tends to zero. The phase Φ_λ^- has no stationary point and is treated only by the non-stationary estimates. This proves the uniform convergence to zero. □

Let

$$\Phi_\lambda^-(v) = -v + \lambda v^{-1/b}.$$

Then

$$(\Phi_\lambda^-)'(v) = -1 - \frac{\lambda}{b} v^{-1/b-1} < 0.$$

Thus, there is no stationary point. So the integrals involving $\Phi_\lambda^-(v)$ are handled as a non-stationary phase case. Let

$$J_\lambda^- = \int_A^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^-(v)}\, dv,$$

Lemma 13. *We have*

$$\lim_{\lambda\to\infty} \int_A^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^-(v)}\, dv = 0.$$

Proof. We split

$$[A, \infty) = [A, R] \cup [R, \infty),$$

where $R > A$ will be chosen large.

On $A \le v \le R$, we have

$$|(\Phi_\lambda^-)'(v)| \ge \frac{\lambda}{b} R^{-1/b-1}.$$

Therefore

$$\left| \frac{v^{\alpha-1}}{(\Phi_\lambda^-)'(v)} \right| \lesssim_{A,R} \lambda^{-1}.$$

Moreover,

$$\left| \left(\frac{v^{\alpha-1}}{(\Phi_\lambda^-)'(v)} \right)' \right| \lesssim_{A,R} \lambda^{-1}.$$

Hence

$$\sup_{A\le r<s\le R} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^-(v)}\, dv \right| \lesssim_{A,R} \lambda^{-1}.$$

By the Abel–Stieltjes lemma (Lemma 4),

$$\left| \int_A^R H(v) v^{\alpha-1} e^{i\Phi_\lambda^-(v)}\, dv \right| \lesssim_{A,R} \lambda^{-1} \left(\|H\|_\infty + \mathrm{Var}(H; [A, R]) \right).$$

Consequently, for each fixed R,

$$\lim_{\lambda\to\infty} \int_A^R H(v) v^{\alpha-1} e^{i\Phi_\lambda^-(v)}\, dv = 0.$$

On the tail, $v \geq R$, we use the fact that

$$|(\Phi_\lambda^-)'(v)| \geq 1.$$

As before,

$$\left| \frac{v^{\alpha-1}}{(\Phi_\lambda^-)'(v)} \right| \lesssim v^{\alpha-1}.$$

Moreover,

$$|(\Phi_\lambda^-)''(v)| = C_b \lambda v^{-1/b-2}.$$

Again,

$$\left| \left(\frac{v^{\alpha-1}}{(\Phi_\lambda^-)'(v)} \right)' \right| \lesssim v^{\alpha-2} + \lambda v^{\alpha-1/b-3} \frac{1}{|(\Phi_\lambda^-)'(v)|^2}.$$

Since

$$|(\Phi_\lambda^-)'(v)| = 1 + \frac{\lambda}{b} v^{-1/b-1},$$

the second term is uniformly controlled. Indeed,

$$\lambda v^{\alpha-1/b-3} \frac{1}{|(\Phi_\lambda^-)'(v)|^2} = v^{\alpha-2} \frac{\lambda v^{-1/b-1}}{\left(1 + \frac{\lambda}{b} v^{-1/b-1}\right)^2} \lesssim v^{\alpha-2}.$$

Therefore,

$$\left| \left(\frac{v^{\alpha-1}}{(\Phi_\lambda^-)'(v)} \right)' \right| \lesssim v^{\alpha-2},$$

uniformly in λ. It follows that

$$\sup_{R \leq r < s < \infty} \left| \int_r^s v^{\alpha-1} e^{i\Phi_\lambda^-(v)} \, dv \right| \lesssim R^{\alpha-1}.$$

Here we use $\alpha < 1$, which holds in our setting since $\alpha < 1/2$.

By the Abel–Stieltjes lemma (Lemma 4),

$$\left| \int_R^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^-(v)} \, dv \right| \leq C R^{\alpha-1} \left(\|H\|_\infty + \mathrm{Var}(H; [R, \infty)) \right).$$

Since $\alpha < 1$, from Lemma 3 the tail can be made uniformly small in λ by choosing R sufficiently large. □

Lemma 14 (Oscillatory lemma with BV amplitude)**.** *Let $b > 0$, $a > 0$, $\theta \in \mathbb{R}$, and suppose that $a < 1 + b/2$. Let $h \in BV(0, \eta)$, for some $\eta > 0$, and suppose that there exists*

$$\lim_{t \to 0^+} h(t) = L.$$

Let $\chi \in \mathcal{C}^1([0, \eta])$. Then

$$\lim_{\lambda \to \infty} \int_0^\delta h(t) t^{-a} \cos(t^{-b} - \theta) \chi(t) e^{i\lambda t} \, dt = 0,$$

for every $0 < \delta \leq \eta$.

Proof. Let

$$I_\lambda = \int_0^\delta h(t) t^{-a} \cos(t^{-b} - \theta) \chi(t) e^{i\lambda t}\, dt, \tag{22}$$

and

$$\alpha := \frac{a-1}{b} < \frac{1}{2}.$$

After the change of variables $v = t^{-b}$, we obtain

$$I_\lambda = \frac{1}{b} \int_{\delta^{-b}}^\infty H(v) v^{\alpha-1} \cos(v-\theta)\, e^{i\lambda v^{-1/b}}\, dv, \tag{23}$$

where

$$H(v) := h(v^{-1/b}) \chi(v^{-1/b}).$$

According to hypotheses, $\lim_{v\to\infty} H(v) = L\chi(0)$, and by Lemma 3, $H \in BV([A,\infty))$, with $A = \delta^{-b}$.

Writing the cosine function as a linear combination of exponentials, the problem can be reduced to studying integrals of the form

$$J_\lambda^\pm = \int_A^\infty H(v) v^{\alpha-1} e^{i\Phi_\lambda^\pm(v)}\, dv, \tag{24}$$

where $\Phi_\lambda^\pm(v) = \pm v + \lambda v^{-1/b}$.

By Lemma 13 we have

$$\lim_{\lambda\to\infty} J_\lambda^- = 0, \tag{25}$$

and by Lemma 11, we obtain

$$\lim_{\lambda\to\infty} J_\lambda^+ = 0. \tag{26}$$

The statement of the lemma follows immediately from (22)–(26). □

For ease of reading, we state the following lemma separately. The proof may first be carried out for continuous F, using equicontinuity and integration by parts, and then extended to arbitrary $F \in L^1$ by a standard approximation argument.

Lemma 15. *Let K be compact and let $A_x \in \mathcal{C}^1(\Omega)$ satisfy*

$$\sup_{x\in K} \|A_x\|_{\mathcal{C}^1(\Omega)} < \infty$$

with Ω a compact interval away from the singularity. If $F \in L^1(\Omega)$, then

$$\lim_{\lambda\to\infty} \sup_{x\in K} \left| \int_\Omega F(u) A_x(u) e^{i\lambda u}\, du \right| = 0.$$

Proof of Theorem 1. We write

$$D_n(u) = \frac{\sin((2n+1)\pi u)}{\sin(\pi u)}$$

for the Dirichlet kernel in the normalization adapted to the interval $[0,1]$. Thus

$$S_n(f, x_0) = \int_0^1 f(t) D_n(x_0 - t)\, dt,$$

where the integral is understood in the Henstock–Kurzweil sense (or as an improper Riemann integral) if it is not absolutely convergent near the origin.

Choose

$$0 < \delta < \frac{1}{2}\min\{x_0, 1 - x_0, \eta\}.$$

We decompose

$$S_n(f, x_0) = I_{n,0} + I_{n,\mathrm{reg}} + I_{n,x_0}, \tag{27}$$

where

$$I_{n,0} = \int_0^{\delta} f(t) D_n(x_0 - t)\, dt, \quad I_{n,x_0} = \int_{|t-x_0|<\delta} f(t) D_n(x_0 - t)\, dt, \tag{28}$$

and $I_{n,\mathrm{reg}}$ denotes the remaining part of the integral. The three pieces are disjoint because $\delta < x_0/2$.

By assumption, g coincides with a function of bounded variation in a neighbourhood of x_0. Since $t \mapsto \cos(t^{-b} - \theta)$ is $\mathcal{C}^1$ on compact subintervals of $(0, 1)$, the product

$$f(t) = g(t)\cos(t^{-b} - \theta)$$

also coincides with a function of bounded variation in a possibly smaller neighbourhood of x_0. Set

$$F_\delta(t) = f(t)\mathbf{1}_{\{|t-x_0|<\delta\}},$$

extended periodically. Then

$$I_{n,x_0} = S_n(F_\delta, x_0).$$

Since F_δ coincides with f in a neighbourhood of x_0, and since f is of bounded variation in that neighbourhood, F_δ is also of bounded variation near x_0. Hence Jordan's criterion applies and yields (see [6, Chapter 1, Section 2, Theorem 1.2])

$$\lim_{n\to\infty} I_{n,x_0} = \frac{F_\delta(x_0+) + F_\delta(x_0-)}{2} = \frac{f(x_0+) + f(x_0-)}{2}. \tag{29}$$

On the regular part we are separated both from the singular point 0 and from the pole of the translated kernel at $t = x_0$. Consequently $f \in L^1$ there, since $g \in L^1([\epsilon, 1])$ for all $\epsilon > 0$ and $|\cos(t^{-b} - \theta)| \le 1$. Let

$$\Omega = (0, 1) \setminus \big((0, \delta) \cup (x_0 - \delta, x_0 + \delta)\big).$$

Since

$$0 < \delta < \frac{1}{2}\min\{x_0, 1 - x_0, \eta\},$$

the interval $(x_0 - \delta, x_0 + \delta)$ is compactly contained in $(0, 1)$, and Ω is separated from both 0 and x_0. Moreover, $\sin(\pi(x_0 - t))$ is bounded away from zero on Ω. Hence

$$t \mapsto \frac{1}{\sin(\pi(x_0 - t))}$$

is bounded on Ω and smooth on each connected component of Ω. Hence

$$I_{n,\mathrm{reg}} = \int_\Omega A(t)\sin((2n+1)\pi(x_0 - t))\, dt, \qquad A \in L^1(\Omega).$$

The Riemann–Lebesgue lemma (see Lemma 15) gives

$$\lim_{n\to\infty} I_{n,\mathrm{reg}} = 0. \tag{30}$$

It remains to prove that

$$\lim_{n\to\infty} I_{n,0} = 0. \tag{31}$$

Since $\delta < x_0/2$, the denominator of $D_n(x_0 - t)$ does not vanish on $(0, \delta)$. Put $\chi(t) = 1/\sin(\pi(x_0 - t))$. Then $\chi \in \mathcal{C}^1([0, \delta])$, and

$$I_{n,0} = \int_0^\delta h(t) t^{-a} \cos(t^{-b} - \theta) \chi(t) \sin((2n+1)\pi(x_0 - t))\, dt.$$

Writing the cosine function as a linear combination of exponentials, to get (31) it is enough to prove that

$$\lim_{\lambda \to \infty} \int_0^\delta h(t) t^{-a} \cos(t^{-b} - \theta) \chi(t) e^{\pm i\lambda t}\, dt = 0,$$

which holds true by Lemma 14.

The relations (27)–(31) prove the first part of Theorem 1.

The uniform convergence on compact subsets follows from Lemma 3, Lemma 6, and Lemma 12, and the preceding uniformity remarks (Remark 5 and Remark 7). Indeed, if K is contained in a compact subinterval of the neighbourhood where g is continuous and of bounded variation, then the local part is uniformly convergent by the uniform Dirichlet–Jordan theorem. The regular part is uniformly small by the Riemann–Lebesgue lemma (see Lemma 15), since it is separated from both 0 and the pole of the Dirichlet kernel. Finally, the singular part is uniformly small by Lemma 12, because $1/\sin \pi(x - t)$ has uniformly bounded $\mathcal{C}^1$ norm for $x \in K$ and $0 \le t \le \delta$. □

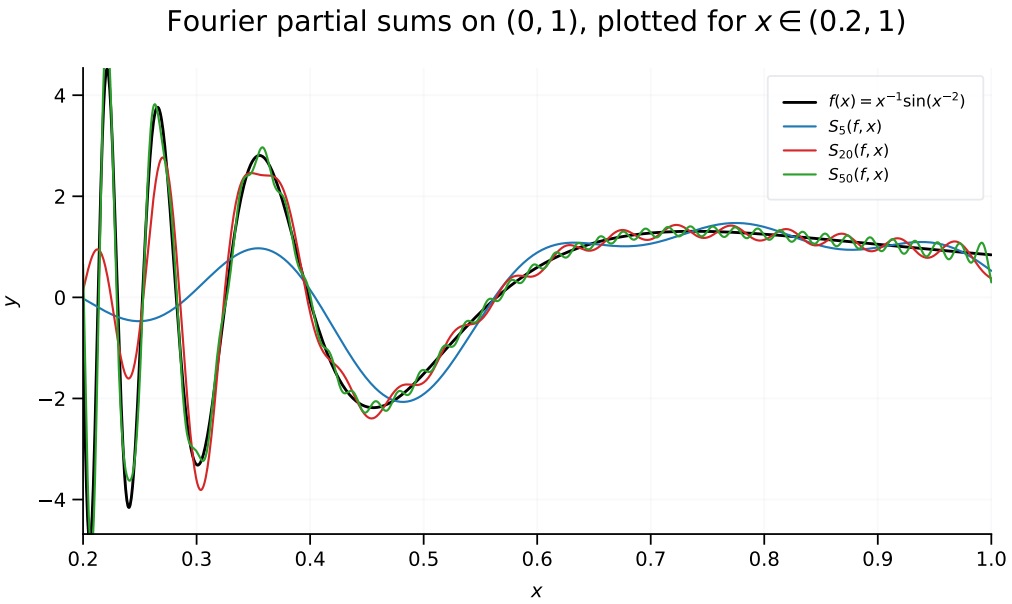


Figure 1: Graphs of $f(x) = \frac{\sin(1/x^2)}{x}$ (black) and $S_N(f, x)$ for $N = 5, 20, 50$.

After the proof of Theorem 1, it is useful to clarify the role of the assumptions on the parameters a and b, and in particular to distinguish between the range in which the model functions are merely Henstock–Kurzweil integrable and the more restrictive range required by the Fourier estimates used in the theorem.

Remark 16. The only possible difficulty occurs at the origin. Let us first consider the model function

$$f(x) = x^{-a} \cos(x^{-b} - \theta), \qquad 0 < x < 1,$$

where $b > 0$ and $\theta \in \mathbb{R}$ are fixed. In order to study the behaviour near 0, we make the change of variables

$$u = x^{-b}, \qquad x = u^{-1/b}, \qquad dx = -\frac{1}{b} u^{-1/b-1}\, du.$$

Then

$$\int_0^1 x^{-a}\cos(x^{-b}-\theta)\,dx = \frac{1}{b}\int_1^\infty u^{a/b}u^{-1/b-1}\cos(u-\theta)\,du$$
$$= \frac{1}{b}\int_1^\infty u^{\frac{a-1}{b}-1}\cos(u-\theta)\,du.$$

Thus the question is reduced to the convergence of the oscillatory improper integral

$$\int_1^\infty u^{\alpha-1}\cos(u-\theta)\,du, \qquad \alpha = \frac{a-1}{b}.$$

By Dirichlet's test, applied to the bounded primitive of $\cos(u-\theta)$ and to the monotone factor $u^{\alpha-1}$, this integral converges whenever

$$\alpha - 1 < 0, \qquad \text{that is,} \qquad a < b+1.$$

Conversely, if $\alpha \geq 1$, the factor $u^{\alpha-1}$ does not tend to zero. Hence the oscillations of $\cos(u-\theta)$ are not sufficient to produce convergence of the improper integral. Therefore the model integral near the origin converges as an improper, and hence as a Henstock–Kurzweil, integral precisely in the range

$$a < b+1.$$

Absolute Lebesgue integrability imposes a stronger condition. Indeed, after the same change of variables,

$$\int_0^\varepsilon x^{-a}\left|\cos(x^{-b}-\theta)\right|\,dx = \frac{1}{b}\int_{\varepsilon^{-b}}^\infty u^{\frac{a-1}{b}-1}\left|\cos(u-\theta)\right|\,du.$$

Since the function $|\cos(u-\theta)|$ has positive mean over each period, the latter integral converges if and only if

$$\frac{a-1}{b} - 1 < -1, \qquad \text{that is,} \qquad a < 1.$$

Consequently,

$$\int_0^\varepsilon x^{-a}\left|\cos(x^{-b}-\theta)\right|\,dx < \infty \quad \Longleftrightarrow \quad a < 1.$$

Thus the model functions exhibit three different regimes:

$a < 1$:	$f \in L^1(0,1)$;
$1 \leq a < b+1$:	$f \notin L^1(0,1)$, but f is Henstock–Kurzweil integrable;
$a \geq b+1$:	the improper integral at the origin does not converge, and hence this model does not give a Henstock–Kurzweil integrable function on $(0,1)$.

In particular, the genuinely non-Lebesgue range in which oscillation compensates for the singularity is

$$1 \leq a < b+1.$$

This is the natural range for the mere Henstock–Kurzweil integrability of the model integral. In the Fourier estimates considered below, a stronger restriction may appear; for instance, the second derivative van der Corput estimates on dyadic intervals near the origin naturally lead to the threshold

$$a < 1 + \frac{b}{2}.$$

Thus the condition $a < 1 + b/2$ should be understood not as the sharp threshold for Henstock–Kurzweil integrability of the model function, but as the threshold required by the Fourier estimates used in the sequel.

Remark 17. Figure 1 displays the graph of $f(x) = \sin(1/x^2)/x$, corresponding to the case $a = 1$, $b = 2$, and $\theta = \pi/2$ in Theorem 1, together with its Fourier partial sums for $n = 5$, 20, 50. As can be seen, increasing the order of the partial sum improves the approximation to the function.

4 Further convergence results

In this section we present further convergence results for the Fourier series of the class of functions considered in Theorem 1. The following result is proved in the same way as Theorem 1; the only change in the argument is that the justification of relation (29) must be carried out using Dini's theorem (see [6, Chapter 1, Section 2, Theorem 1.1]).

Theorem 18 (Dini's condition around the convergence point)**.** *Let $a > 0$, $b > 0$, $\theta \in \mathbb{R}$, and let $f(x) = g(x)\cos(1/x^b - \theta)$, $0 < x < 1$. Let $x_0 \in (0,1)$. We assume that:*

(i) *$g \in L^1([\epsilon, 1])$ for all $\epsilon > 0$;*

(ii) *there exists $L \neq 0$ such that $\lim_{x\to 0^+} x^a g(x) = L$;*

(iii) *$h(x) = x^a g(x)$ is of bounded variation on $(0, \eta)$ for some $\eta > 0$;*

(iv) *there exist $f(x_0+)$, $f(x_0-)$ and*

$$\int_0^\delta \left| \frac{f(x_0+t) + f(x_0-t) - f(x_0+) - f(x_0-)}{t} \right| dt < \infty,$$

for some $0 < \delta < \min\{x_0, 1 - x_0\}$;

(v) *and $a < 1 + b/2$.*

Then for the 1-periodic extension of the function f, we have

$$\lim_{n\to\infty} S_n(f, x_0) = \frac{f(x_0+) + f(x_0-)}{2}.$$

When the functions do not belong to L^1, the localization principle for Fourier series may fail. Even within the class of Henstock–Kurzweil integrable functions, this principle is not valid in general, as the following example shows.

Under the corresponding hypotheses, a localization principle also holds

Theorem 19 (Localization principle)**.** *Let $j = 1, 2$, let $b_j > 0$, $a_j > 0$, $\theta_j \in \mathbb{R}$, and*

$$f_j(x) = g_j(x)\cos(\frac{1}{x^{b_j}} - \theta_j), \qquad 0 < x < 1.$$

We assume that, for $j = 1, 2$,

(i) *$g_j \in L^1([\epsilon, 1])$ for all $\epsilon > 0$;*

(ii) *there exist $L_j \neq 0$ such that*

$$\lim_{x\to 0^+} x^{a_j} g_j(x) = L_j;$$

(iii) *$h_j(x) = x^{a_j} g_j(x)$ is of bounded variation in $(0, \eta_j)$ for some $\eta_j > 0$;*

(iv) *and $a_j < 1 + b_j/2$.*

If there exists an open interval $V \Subset (0,1)$ such that

$$f_1(x) = f_2(x), \qquad x \in V,$$

then, for every compact set $K \Subset V$,

$$\lim_{n\to\infty} \sup_{x\in K} |S_n(f_1, x) - S_n(f_2, x)| = 0.$$

In particular, for all $x \in V$ we have

$$\lim_{n\to\infty} (S_n(f_1, x) - S_n(f_2, x)) = 0, \qquad x \in V.$$

Proof. Let $F = f_1 - f_2$. Since $f_1 = f_2$ on V, we have $F = 0$ on V. Let $K \Subset V$ and choose $\chi \in \mathcal{C}^1[0,1]$, interpreted periodically if necessary, such that

$$\chi = 1 \quad \text{on a neighbourhood of } K, \qquad \operatorname{supp}\chi \subset V.$$

Since $F = 0$ on V, we have $\chi F = 0$, and therefore, for $x \in K$,

$$S_n(F,x) = \int_0^1 F(u)(1-\chi(u))D_n(x-u)\,du.$$

Now the denominator $\sin(\pi(x-u))$ is uniformly separated from zero on the support of $1-\chi$ for $x \in K$. Hence

$$A_x(u) = \frac{1-\chi(u)}{\sin(\pi(x-u))}$$

defines a family bounded in $\mathcal{C}^1[0,1]$:

$$\sup_{x\in K} \|A_x\|_{\mathcal{C}^1([0,1])} < \infty.$$

Using

$$D_n(x-u) = \frac{\sin((2n+1)\pi(x-u))}{\sin(\pi(x-u))},$$

we write $S_n(F,x)$ as a linear combination of terms of the form

$$e^{\pm i(2n+1)\pi x}\int_0^1 F(u)A_x(u)e^{\mp i(2n+1)\pi u}\,du.$$

Since $F = f_1 - f_2$, it is enough to estimate each f_j separately. Near the origin, Lemma 14 applies to each oscillatory singularity with the amplitude A_x; away from the origin, the integrand is an L^1 function and Lemma 15 can be applied. We obtain

$$\lim_{n\to\infty}\sup_{x\in K}|S_n(F,x)| = 0.$$

Equivalently,

$$\lim_{n\to\infty}\sup_{x\in K}|S_n(f_1,x) - S_n(f_2,x)| = 0.$$

This proves the stated localization principle. □

5 Examples

In this section we illustrate the scope and limitations of the preceding results. We begin by examining the threshold $a = 1 + b/2$. After extending the moving-scale stationary-phase asymptotic of Lemma 9 to this endpoint, we construct an example for which the stationary contribution to the Fourier coefficients is of order one and the Fourier series fails to converge at a point away from the singularity. This proves that the strict condition $a < 1 + b/2$ in the convergence theorems is sharp. We next show that, for general Henstock–Kurzweil integrable functions, the classical localization principle may fail even when all Fourier coefficients are well defined. We conclude by deriving the precise asymptotic behaviour of the Fourier coefficients for the oscillatory singularities considered in this paper in the subcritical range $a < 1 + b/2$.

5.1 Sharpness of the condition $a < 1 + b/2$

The decay obtained in Lemma 9 is governed by the exponent $1 + b/2 - a/(b+1)$. Thus, as a approaches the threshold $1 + b/2$, the stationary contribution becomes progressively larger, and at the endpoint it is expected to be of order one. To show that the strict inequality in Theorem 1 is sharp, we first extend the asymptotic formula of Lemma 9 to this limiting case. Although the decay factor disappears, the rescaling still transforms the moving stationary point into a fixed non-degenerate one, and the bounded-variation argument for the amplitude remains valid. The estimates on the nonstationary regions also persist at the endpoint. This gives the following result.

Lemma 20 (Endpoint stationary phase with a BV amplitude)**.** *Let $b > 0$ and set*

$$a = 1 + \frac{b}{2}.$$

Let $h \in BV((0,\eta))$ satisfy

$$\lim_{x\to 0+} h(x) = L,$$

and let $\chi \in C_c^\infty([0,\eta))$ be such that $\chi = 1$ in a neighbourhood of the origin. Then, as $\lambda \to +\infty$,

$$\begin{aligned}\int_0^\eta \chi(x)h(x)x^{-1-b/2}e^{-i(x^{-b}+\lambda x)}\,dx \\ = L\left(\frac{2\pi}{b(b+1)}\right)^{1/2}\exp\left(-i(b+1)b^{-b/(b+1)}\lambda^{b/(b+1)} - \frac{i\pi}{4}\right) + o(1).\end{aligned} \tag{32}$$

Moreover,

$$\int_0^\eta \chi(x)h(x)x^{-1-b/2}e^{i(x^{-b}-\lambda x)}\,dx = o(1). \tag{33}$$

Proof. The argument is the same rescaling argument as in Lemma 9, but we include the details needed at the endpoint. Set

$$x_\lambda = \left(\frac{b}{\lambda}\right)^{1/(b+1)}, \qquad \mu = x_\lambda^{-b} = b^{-b/(b+1)}\lambda^{b/(b+1)},$$

and make the change of variables $x = x_\lambda y$. Since $\lambda x_\lambda = b\mu$, we have

$$x^{-b} + \lambda x = \mu\phi(y), \qquad \phi(y) = y^{-b} + by.$$

The function ϕ has a unique critical point on $(0,\infty)$, namely $y = 1$, and

$$\phi(1) = b + 1, \qquad \phi''(1) = b(b+1) > 0.$$

Moreover, since $a = 1 + b/2$,

$$x_\lambda^{1-a}\mu^{-1/2} = x_\lambda^{-b/2}x_\lambda^{b/2} = 1.$$

Choose $0 < c < 1 < C < \infty$ and $\vartheta \in C_c^\infty((c,C))$ such that $\vartheta = 1$ in a neighbourhood of 1. On $[c,C]$, write

$$h(x_\lambda y) = L + r_\lambda(y), \qquad r_\lambda(y) = h(x_\lambda y) - L.$$

Since $h(x) \to L$ as $x \to 0+$,

$$\|r_\lambda\|_{L^\infty([c,C])} \longrightarrow 0.$$

Furthermore, composition with the increasing map $y \mapsto x_\lambda y$ preserves variation, and hence

$$\mathrm{Var}(r_\lambda;[c,C]) = \mathrm{Var}(h;[cx_\lambda, Cx_\lambda]) \le \mathrm{Var}(h;(0,Cx_\lambda]) \longrightarrow 0.$$

Consequently,

$$\|r_\lambda(y)y^{-1-b/2}\vartheta(y)\|_{L^\infty([c,C])} + \mathrm{Var}\Big(r_\lambda(y)y^{-1-b/2}\vartheta(y);[c,C]\Big) = o(1).$$

The second-order van der Corput estimate with a BV amplitude (Lemma 6) therefore gives

$$\int_c^C r_\lambda(y)y^{-1-b/2}\vartheta(y)e^{-i\mu\phi(y)}\,dy = o(\mu^{-1/2}).$$

For the constant part, the one-dimensional stationary-phase formula (the van der Corput formula with $k=1$) yields

$$L\int_c^C y^{-1-b/2}\vartheta(y)e^{-i\mu\phi(y)}\,dy = L\left(\frac{2\pi}{\mu b(b+1)}\right)^{1/2}\exp\left(-i(b+1)\mu - \frac{i\pi}{4}\right) + o(\mu^{-1/2}).$$

It remains to estimate the complement of the stationary region. On this complement, ϕ' does not vanish. The Abel–Stieltjes estimates used in the proof of Lemma 9 apply without change. Near $y=0$, one has

$$|\phi'(y)| \asymp y^{-b-1},$$

and the corresponding bound contains the positive power

$$c^{\,b+1-a} = c^{\,b/2}.$$

At infinity, $\phi'(y)\to b$, and the corresponding bound contains

$$C^{\,1-a} = C^{-b/2}.$$

Thus, after first letting $\lambda\to\infty$, then $c\downarrow 0$, and finally $C\uparrow\infty$, the nonstationary complement is $o(\mu^{-1/2})$. Notice that both powers remain strictly favourable at the endpoint because $b>0$. Multiplying the resulting asymptotic formula by the factor x_λ^{1-a} and using

$$x_\lambda^{1-a}\mu^{-1/2} = 1$$

gives

$$\begin{aligned}\int_0^\eta \chi(x)h(x)x^{-1-b/2}e^{-i(x^{-b}+\lambda x)}\,dx \\ = L\left(\frac{2\pi}{b(b+1)}\right)^{1/2}\exp\left(-i(b+1)b^{-b/(b+1)}\lambda^{b/(b+1)} - \frac{i\pi}{4}\right) + o(1),\end{aligned}$$

which proves (32). For (33), the same rescaling leads to the phase

$$\psi(y) = y^{-b} - by, \qquad \psi'(y) = -b(y^{-b-1}+1),$$

which has no critical point on $(0,\infty)$. The preceding Abel–Stieltjes estimates can therefore be applied on the whole interval. They give

$$\int_0^\eta \chi(x)h(x)x^{-1-b/2}e^{i(x^{-b}-\lambda x)}\,dx = o(1),$$

and the proof is complete. □

We now apply Lemma 20 to show that the strict inequality in Theorem 1 cannot, in general, be replaced by the endpoint condition $a \le 1 + b/2$. Consider

$$f(x) = x^{-3/2}\sin\frac{1}{x}, \qquad 0<x<1.$$

This corresponds to

$$b = 1, \qquad a = \frac{3}{2} = 1 + \frac{b}{2}, \qquad g(x) = x^{-3/2},$$

and

$$h(x) = x^a g(x) \equiv 1.$$

Thus h is of bounded variation near the origin and

$$\lim_{x \to 0+} x^a g(x) = 1.$$

Moreover, g is smooth, and hence of bounded variation, in a neighbourhood of every point of $(0, 1)$. Therefore all the hypotheses of Theorem 1 are satisfied at every interior point except for the strict inequality $a < 1 + b/2$. We shall prove that the Fourier series of the 1-periodic extension of f does not converge at $x_0 = \frac{1}{2}$.

We work with the 1-periodic Fourier convention

$$\widehat{f}(n) = \int_0^1 f(x) \mathrm{e}^{-2\pi i n x}\, dx, \qquad n \in \mathbb{Z},$$

where, for the present non-absolutely integrable function, the integral near 0 is understood as an improper oscillatory integral, equivalently as a Henstock–Kurzweil integral. The symmetric partial sums are

$$S_N(f, x) = \sum_{|n| \le N} \widehat{f}(n) \mathrm{e}^{2\pi i n x}.$$

Let

$$f(x) = x^{-3/2} \sin \frac{1}{x}, \qquad 0 < x < 1.$$

Then this is of the form

$$f(x) = g(x) \sin \frac{1}{x^b}, \qquad b = 1, \qquad g(x) = x^{-3/2}.$$

If $a = 3/2$, then

$$a = 1 + \frac{b}{2}, \qquad h(x) = x^a g(x) = x^{3/2} x^{-3/2} = 1.$$

Therefore h is of bounded variation on every interval $(0, \eta)$ and

$$\lim_{x \to 0^+} x^a g(x) = 1 \neq 0.$$

Moreover g is $\mathcal{C}^1$, hence of bounded variation, in a neighbourhood of every point $x_0 \in (0, 1)$. In particular all local bounded-variation hypotheses at $x_0 = 1/2$ are satisfied. The only difference with the theorem is precisely that the strict inequality $a < 1 + b/2$ is replaced by the endpoint equality $a = 1 + b/2$.

First, we find asymptotics of the positive Fourier coefficients. For $n \ge 1$ write

$$\widehat{f}(n) = \int_0^1 x^{-3/2} \sin \frac{1}{x}\, \mathrm{e}^{-2\pi i n x}\, dx.$$

Using

$$\sin \frac{1}{x} = \frac{\mathrm{e}^{i/x} - \mathrm{e}^{-i/x}}{2i},$$

we split

$$\widehat{f}(n) = \frac{1}{2i}(I_n^+ - I_n^-),$$

where

$$I_n^+ = \int_0^1 x^{-3/2} \mathrm{e}^{i/x} \mathrm{e}^{-2\pi i n x}\, dx, \qquad I_n^- = \int_0^1 x^{-3/2} \mathrm{e}^{-i/x} \mathrm{e}^{-2\pi i n x}\, dx.$$

The phases are

$$\phi_n^+(x) = \frac{1}{x} - 2\pi n x, \qquad \phi_n^-(x) = -\frac{1}{x} - 2\pi n x.$$

For I_n^+ we have

$$(\phi_n^+)'(x) = -\frac{1}{x^2} - 2\pi n < 0,$$

so there is no stationary point. Thus, from Proposition 8 we obtain

$$I_n^+ = o(1).$$

Indeed this term is not responsible for the endpoint obstruction.

The second phase has a unique stationary point:

$$(\phi_n^-)'(x) = \frac{1}{x^2} - 2\pi n, \qquad x_n = (2\pi n)^{-1/2}.$$

At this point

$$\phi_n^-(x_n) = -\frac{1}{x_n} - 2\pi n x_n = -2\sqrt{2\pi n},$$

and

$$(\phi_n^-)''(x) = -\frac{2}{x^3}, \qquad (\phi_n^-)''(x_n) = -2(2\pi n)^{3/2}.$$

Applying Proposition 8 we get

$$\begin{aligned} I_n^- &= (x_n)^{-3/2} \mathrm{e}^{i\phi_n^-(x_n)} \mathrm{e}^{-i\pi/4} \left(\frac{2\pi}{2(2\pi n)^{3/2}}\right)^{1/2} + o(1) \\ &= (2\pi n)^{3/4} \mathrm{e}^{-2i\sqrt{2\pi n}} \mathrm{e}^{-i\pi/4} \frac{\sqrt{\pi}}{(2\pi n)^{3/4}} + o(1) \\ &= \sqrt{\pi}\, \mathrm{e}^{-i(2\sqrt{2\pi n} + \pi/4)} + o(1). \end{aligned}$$

Consequently

$$\begin{aligned} \widehat{f}(n) &= \frac{1}{2i}(I_n^+ - I_n^-) \\ &= -\frac{\sqrt{\pi}}{2i} \mathrm{e}^{-i(2\sqrt{2\pi n} + \pi/4)} + o(1) \\ &= \frac{\sqrt{\pi}}{2} \mathrm{e}^{-i(2\sqrt{2\pi n} - \pi/4)} + o(1). \end{aligned} \tag{34}$$

In particular the positive Fourier coefficients do not tend to zero in modulus:

$$|\widehat{f}(n)| \to \frac{\sqrt{\pi}}{2}.$$

This already indicates the sharpness of the threshold: for general parameters $a, b > 0$, the stationary point of the phase $-x^{-b} - 2\pi n x$ is

$$x_n = \left(\frac{b}{2\pi n}\right)^{1/(b+1)},$$

and the leading stationary phase size is proportional to

$$x_n^{-a} \left|(x^{-b})''|_{x=x_n}\right|^{-1/2} \asymp n^{\frac{a-(1+b/2)}{b+1}}.$$

Thus the contribution tends to zero if $a < 1 + b/2$, is of order one at $a = 1 + b/2$, and grows if $a > 1 + b/2$.

Next we check that the convergence of the Fourier series at $x_0 = 1/2$ failure. For a real-valued function, the Fourier coefficients satisfy

$$\widehat{f}(-n) = \overline{\widehat{f}(n)}.$$

Therefore the nth symmetric pair contributing to the Fourier series at $x_0 = 1/2$ is

$$\begin{aligned} T_n &:= \widehat{f}(n)\mathrm{e}^{2\pi i n(1/2)} + \widehat{f}(-n)\mathrm{e}^{-2\pi i n(1/2)} \\ &= (-1)^n \widehat{f}(n) + (-1)^n \widehat{f}(-n) \\ &= 2(-1)^n \operatorname{Re} \widehat{f}(n). \end{aligned}$$

By (34),

$$T_n = \sqrt{\pi}\, \cos\left(\pi n - 2\sqrt{2\pi n} + \frac{\pi}{4}\right) + o(1).$$

We now show that T_n does not tend to zero. Restrict to square indices $n = m^2$. Then

$$T_{m^2} = \sqrt{\pi}\, \cos\left(\pi m^2 - 2\sqrt{2\pi}\, m + \frac{\pi}{4}\right) + o(1).$$

Since

$$\pi m^2 \equiv \pi m \pmod{2\pi},$$

we may write the phase modulo 2π as

$$\pi m - 2\sqrt{2\pi}\, m + \frac{\pi}{4} = \beta m + \frac{\pi}{4}, \qquad \beta := \pi - 2\sqrt{2\pi}.$$

The number $\beta/(2\pi)$ is irrational, because otherwise $2\sqrt{2\pi}/\pi$ would be rational, which would force π to be rational. Hence the sequence

$$\left\{\frac{\beta m}{2\pi}\right\}_{m \geq 1}$$

is dense in $[0, 1]$. Therefore there is a subsequence m_j such that

$$\beta m_j + \frac{\pi}{4} \to 0 \pmod{2\pi}.$$

Along this subsequence,

$$\cos\left(\pi m_j^2 - 2\sqrt{2\pi}\, m_j + \frac{\pi}{4}\right) \to 1,$$

and consequently

$$T_{m_j^2} \to \sqrt{\pi}.$$

In particular, the individual symmetric Fourier increments at $x_0 = 1/2$ do not tend to zero.

But if the sequence of symmetric partial sums $S_N(f, 1/2)$ converged, then its successive increments would necessarily tend to zero:

$$S_N(f, 1/2) - S_{N-1}(f, 1/2) = T_N \longrightarrow 0.$$

Taking $N = n_j$, this contradicts the subsequence just found. Hence the Fourier series of the 1-periodic extension of

$$f(x) = x^{-3/2} \sin\frac{1}{x}$$

does not converge at $x_0 = 1/2$.

Remark 21 (Supracritical divergence). Let $b > 0$ and $a > 1 + b/2$. Let $f(x) = g(x)\cos(x^{-b} - \theta)$ on $(0,1)$ satisfy (i)–(iv) of Theorem 1 with $L \neq 0$. Then

$$|\hat{f}(n)| \;\geq\; c_{a,b,\theta,L}\, n^{\alpha/(b+1)}\,(1 + o(1)), \qquad n \to \infty,$$

with

$$c_{a,b,\theta,L} = |L| \sqrt{\frac{\pi\, b}{b+1}}\, (2\pi/b)^{(\alpha - 1/2)b/(b+1)},$$

where $\alpha = (a-1)/b$. In particular, $\lim_{n\to\infty} |\hat{f}(n)| = \infty$, the sequence of partial sums $S_n(f,\cdot)$ diverges at almost every point of $(0,1)$.

5.2 Localization principle may fail in Henstock-Kurzweil space

Next we give an example of a Henstock-Kurzweil integrable function for which the localization principle fails (see [6] and [28]).

Let $\alpha_0 = \frac{1}{4}$, $x_0 = \frac{3}{4}$, Let $0 < \varepsilon < 1/16$. Choose a smooth cut-off function $\eta \in \mathcal{C}^\infty([0,\varepsilon])$ such that

$$\eta(t) = 1 \quad \text{for } 0 \leq t \leq \varepsilon/2, \qquad \eta(t) = 0 \quad \text{for } 3\varepsilon/4 \leq t \leq \varepsilon.$$

Define

$$H(t) = \begin{cases} \eta(t) t^{1/2} \sin(1/t^2), & 0 < t < \varepsilon, \\ 0, & t = 0, \end{cases}$$

and extend H by zero outside $[0,\varepsilon]$. Now define G on $(0,1)$ by

$$G(x) = \begin{cases} H(x - \alpha_0), & \alpha_0 < x < \alpha_0 + \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, put

$$g(x) = G'(x)$$

at every point at which G is differentiable, and assign arbitrary values to g at the exceptional points. Explicitly, near α_0^+, where $x = \alpha_0 + t$, we have

$$g(\alpha_0 + t) = H'(t) = \frac{1}{2} t^{-1/2} \sin\frac{1}{t^2} - 2t^{-5/2}\cos\frac{1}{t^2}.$$

Now set

$$f_1 = 0, \qquad f_2 = -g.$$

Let

$$I = \left(\frac{5}{8}, \frac{7}{8}\right).$$

Since $\operatorname{supp} g \subset (\alpha_0, \alpha_0 + \varepsilon)$ and $\alpha + \varepsilon < 5/16$, we have

$$g = 0 \quad \text{on } I.$$

Therefore

$$f_1 = f_2 \quad \text{on } I,$$

and in particular $x_0 = 3/4$ belongs to the interior of the interval where the two functions agree.

The function G is continuous on $[0,1]$ and differentiable except, at most, at finitely many points. Thus $g = G'$ is Henstock–Kurzweil integrable on $(0,1)$ in the usual sense given by the generalized fundamental theorem of calculus.

Moreover, for every integer $n \geq 0$, the functions

$$x \mapsto \cos(2\pi n x), \qquad x \mapsto \sin(2\pi n x)$$

are of bounded variation on $[0,1]$. By the multiplier theorem for the Henstock–Kurzweil integral, multiplication by a function of bounded variation preserves Henstock–Kurzweil integrability. Hence

$$g(x)\cos(2\pi n x), \quad g(x)\sin(2\pi n x) \in HK(0,1), \qquad n = 0, 1, 2, \ldots.$$

Therefore all trigonometric Fourier coefficients of g, and hence of f_1 and f_2, are well defined in the Henstock–Kurzweil sense.

On the other hand, $g \notin L^1(0,1)$. Indeed, the dominant term near α^+ is

$$-2t^{-5/2}\cos\frac{1}{t^2}, \qquad t = x - \alpha,$$

and

$$\int_0^\delta t^{-5/2}\left|\cos\frac{1}{t^2}\right| dt = \infty.$$

For instance, with the change of variable $u = 1/t^2$, this integral is comparable to

$$\int_{1/\delta^2}^\infty u^{-1/4}|\cos u|\, du,$$

which diverges.

We now show that the Fourier partial sums of g are unbounded at $x_0 = 3/4$. Since g is supported in $(\alpha, \alpha+\varepsilon)$, write $x = \alpha + t$ with $0 < t < \varepsilon$. Then

$$x_0 - x = x_0 - \alpha - t = \frac{1}{2} - t.$$

Put

$$\mu = (2N+1)\pi.$$

Using the period-one Dirichlet kernel,

$$D_N(u) = \frac{\sin((2N+1)\pi u)}{\sin(\pi u)},$$

we get

$$\begin{aligned} D_N\left(\frac{1}{2} - t\right) &= \frac{\sin\left((2N+1)\pi(1/2 - t)\right)}{\sin\left(\pi(1/2-t)\right)} \\ &= (-1)^N \frac{\cos(\mu t)}{\cos(\pi t)}. \end{aligned}$$

Thus, if

$$A(t) = \frac{1}{\cos(\pi t)},$$

then A is smooth on $[0,\varepsilon]$, $A(0) = 1$, and

$$S_N[g](x_0) = (-1)^N \int_0^\varepsilon H'(t)A(t)\cos(\mu t)\, dt. \tag{35}$$

Since $H(0) = H(\varepsilon) = 0$ and $A(t)\cos(\mu t)$ is of bounded variation, the Henstock–Kurzweil integration by parts formula gives

$$\begin{aligned} \int_0^\varepsilon H'(t)A(t)\cos(\mu t)\, dt &= -\int_0^\varepsilon H(t)\frac{d}{dt}\{A(t)\cos(\mu t)\}\, dt \\ &= \mu\int_0^\varepsilon H(t)A(t)\sin(\mu t)\, dt - \int_0^\varepsilon H(t)A'(t)\cos(\mu t)\, dt. \end{aligned} \tag{36}$$

The second term is bounded as $\mu \to \infty$, because $H \in L^1(0,\varepsilon)$ and A' is bounded. More precisely,

$$\left| \int_0^\varepsilon H(t)A'(t)\cos(\mu t)\,dt \right| \le \|A'\|_\infty \int_0^\varepsilon |H(t)|\,dt = O(1). \tag{37}$$

Hence the possible growth comes from

$$\mu \int_0^\varepsilon H(t)A(t)\sin(\mu t)\,dt = \mu \int_0^\varepsilon \eta(t)t^{1/2}A(t)\sin\frac{1}{t^2}\,\sin(\mu t)\,dt. \tag{38}$$

Since $\eta(t) = 1$ near 0 and $A(t) = 1 + O(t)$, the leading term is

$$\mu J(\mu), \qquad J(\mu) = \int_0^\varepsilon t^{1/2}\sin\frac{1}{t^2}\,\sin(\mu t)\,dt. \tag{39}$$

The error introduced by replacing $\eta(t)A(t)$ by 1 has lower order: the factor $A(t) - 1$ is $O(t)$, and therefore the same stationary phase calculation below has an amplitude with an additional factor t, which decreases the leading order.

We use

$$\sin\frac{1}{t^2}\sin(\mu t) = \frac{1}{2}\left\{\cos\left(\frac{1}{t^2} - \mu t\right) - \cos\left(\frac{1}{t^2} + \mu t\right)\right\}. \tag{40}$$

The phase

$$\phi_-(t) = \frac{1}{t^2} - \mu t$$

has no stationary point, since

$$\phi_-'(t) = -\frac{2}{t^3} - \mu < 0.$$

Thus this part contributes only a lower order term, by integration by parts or by the van der Corput lemma for monotone phases.

The relevant phase is

$$\phi_+(t) = \frac{1}{t^2} + \mu t.$$

It has one stationary point,

$$\phi_+'(t) = -\frac{2}{t^3} + \mu = 0, \qquad t_\mu = \left(\frac{2}{\mu}\right)^{1/3}.$$

Moreover,

$$\phi_+''(t) = \frac{6}{t^4}, \qquad \phi_+''(t_\mu) = 6\left(\frac{\mu}{2}\right)^{4/3},$$

and

$$\phi_+(t_\mu) = 3\cdot 2^{-2/3}\mu^{2/3}.$$

By Lemma 10,

$$\int_0^\varepsilon t^{1/2}\cos\left(\frac{1}{t^2} + \mu t\right)\,dt = C_0\mu^{-5/6}\cos\left(3\cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}\right) + o(\mu^{-5/6}),$$

as $\mu \to \infty$, with $C_0 \neq 0$.

Taking into account the factor $-1/2$ in the trigonometric identity (40), we obtain

$$J(\mu) = C\mu^{-5/6}\cos\left(3\cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}\right) + o(\mu^{-5/6}), \qquad C \neq 0.$$

Therefore

$$\mu J(\mu) = C\mu^{1/6}\cos\left(3\cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}\right) + o(\mu^{1/6}). \tag{41}$$

From (35)–(39) and (41), it follows

$$S_N[g](x_0) = C_1(-1)^N \mu^{1/6} \cos\left(3 \cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}\right) + O(1), \qquad C_1 \neq 0.$$

Since the phase

$$3 \cdot 2^{-2/3}\mu^{2/3} + \frac{\pi}{4}$$

defines an increasing unbounded sequence as $N \to \infty$, and the increments of this sequence tend to zero, it intersects infinitely many intervals on which the absolute value of the cosine is at least $1/2$. Hence there is a subsequence N_j such that

$$\left|\cos\left(3 \cdot 2^{-2/3}\mu_{N_j}^{2/3} + \frac{\pi}{4}\right)\right| \geq \frac{1}{2}.$$

It follows that

$$\lim_{j\to\infty} |S_{N_j}[g](x_0)| = \infty.$$

In particular, the partial sums of the Fourier series of g do not tend to 0 at x_0.

Recall that

$$f_1 = 0, \qquad f_2 = -g.$$

Since $g = 0$ on $I = (5/8, 7/8)$, we have

$$f_1 = f_2 \quad \text{on } I.$$

However,

$$S_N[f_1](x_0) - S_N[f_2](x_0) = S_N[g](x_0),$$

and the right-hand side is unbounded along a subsequence. Therefore

$$S_N[f_1](x_0) - S_N[f_2](x_0)$$

does not tend to 0. Thus the Fourier series of f_1 and f_2 are not equiconvergent at x_0, and a fortiori they are not uniformly equiconvergent on any interval $I' \Subset I$ containing x_0.

This proves that the localization principle may fail outside the L^1 setting, even though all Fourier coefficients are well defined as Henstock–Kurzweil integrals.

5.3 Fourier coefficient asymptotics

Next we give the asymptotic behaviour of the Fourier coefficients for a subclass of functions which we have considered.

Proposition 22 (Fourier coefficient asymptotics)**.** *Let $a > 0$, $b > 0$, and let*

$$f(x) = g(x) \sin\frac{1}{x^b}, \qquad 0 < x < 1.$$

Assume that

$$h(x) := x^a g(x) \longrightarrow L \neq 0 \qquad (x \to 0+),$$

that h is of bounded variation on $(0, \eta_0)$ for some $\eta_0 > 0$, and that

$$g \in BV([\eta, 1]) \qquad \text{for every } \eta \in (0, 1).$$

Assume also that

$$a < 1 + \frac{b}{2},$$

Then, as $n \to +\infty$,

$$\widehat{f}(n) \sim C_{a,b,L}(2\pi n)^{-\alpha/(b+1)} \exp\left(-i(b+1)b^{-b/(b+1)}(2\pi n)^{b/(b+1)} + \frac{i\pi}{4}\right)$$

where $\alpha = (1 + b/2) - a$,

$$C_{a,b,L} = \frac{L}{2}\left(\frac{2\pi}{b(b+1)}\right)^{1/2} b^{\frac{\alpha}{b+1}}.$$

Proof. Choose $\chi \in \mathcal{C}^\infty([0,1])$ such that $\chi = 1$ on $(0, \eta_0/3]$ and $\chi = 0$ on $[2\eta_0/3, 1]$. Decompose

$$f = f_0 + f_1, \qquad f_0(x) = \chi(x)g(x)\sin\frac{1}{x^b}, \qquad f_1(x) = (1-\chi(x))g(x)\sin\frac{1}{x^b}.$$

The function f_0 contains the whole oscillatory singularity at 0, whereas f_1 is supported in $[\eta_0/3, 1]$.

For f_0, the stationary phase computation in the previous section applies. The only stationary point of the relevant phase

$$\Phi_\lambda(x) = x^{-b} + \lambda x, \qquad \lambda = 2\pi n,$$

is

$$x_\lambda = \left(\frac{b}{\lambda}\right)^{1/(b+1)},$$

and $x_\lambda \to 0$ as $\lambda \to +\infty$. Since $\chi = 1$ near 0 and $h(x) \to L$, the amplitude at the stationary point is asymptotic to

$$g(x_\lambda) = x_\lambda^{-a} h(x_\lambda) \sim L x_\lambda^{-a}.$$

The part with phase $x^{-b} - \lambda x$ is non-stationary and is of smaller order. Lemma 9 gives

$$\widehat{f_0}(n) \sim C_{a,b,L}(2\pi n)^{-\alpha/(b+1)} \exp\left(-i(b+1)b^{-b/(b+1)}(2\pi n)^{b/(b+1)} + \frac{i\pi}{4}\right).$$

where $\alpha = (1 + b/2) - a$. Thus it remains to prove that $\widehat{f_1}(n) = o(n^{-\alpha/(b+1)})$.

Because $g \in BV([\eta, 1])$ for every $\eta > 0$, in particular

$$g \in BV([\eta_0/3, 1]).$$

The factor $x \mapsto \sin(x^{-b})$ is $\mathcal{C}^1$ on $[\eta_0/3, 1]$, hence it is of bounded variation there. The product of two bounded-variation functions on a compact interval is again of bounded variation, and multiplication by the smooth cut-off $1 - \chi$ preserves bounded variation. Therefore

$$f_1 \in BV([0,1]),$$

where f_1 is understood to be zero on $[0, \eta_0/3]$.

A standard integration-by-parts estimate for Fourier coefficients of a bounded-variation function gives

$$\widehat{f_1}(n) = O\left(\frac{1}{n}\right).$$

Indeed, if $F = f_1$ and $\lambda = 2\pi n$, then the Riemann–Stieltjes integration-by-parts formula yields

$$\int_0^1 F(x)\mathrm{e}^{-i\lambda x}\,dx = \left[-\frac{F(x)\mathrm{e}^{-i\lambda x}}{i\lambda}\right]_0^1 + \frac{1}{i\lambda}\int_{[0,1]} \mathrm{e}^{-i\lambda x}\,dF(x),$$

and hence

$$\left|\widehat{f_1}(n)\right| \le \frac{|F(1)| + |F(0)| + \mathrm{Var}_{[0,1]}(F)}{2\pi n} = O\left(\frac{1}{n}\right).$$

Finally, since

$$\rho = \frac{b+2-2a}{2(b+1)} > 0$$

and $a > 0$, we also have

$$\rho < \frac{b+2}{2(b+1)} < 1.$$

Therefore

$$n^{-1} = o(n^{-\rho}).$$

It follows that

$$\widehat{f}_1(n) = O(n^{-1}) = o(n^{-\rho}),$$

so the regular part is negligible compared with the stationary contribution of the singularity. Combining this with the asymptotic formula for $\widehat{f}_0(n)$ gives the asserted global asymptotic formula for $\widehat{f}(n)$. □

Concluding remarks

The results of this paper have been presented for a model situation in which the non-Lebesgue behaviour is concentrated at the origin. This choice simplifies the notation, but it is not essential to the method. The arguments are local with respect to the point at which convergence is tested, and the contribution of the singular region is controlled independently by the oscillatory estimates proved above.

The same conclusions therefore extend to functions with finitely many singular points, provided that each singularity is of the same admissible oscillatory type as the one considered here. More precisely, if the singular set is finite, then it divides the period into finitely many intervals. On each of these intervals one may impose the corresponding local hypotheses appearing in the preceding theorems, such as bounded variation, a Dini-type condition, or the hypotheses needed for localization. The proof can then be applied separately on each interval. The singular contributions are controlled by the same van der Corput–type and Abel–Stieltjes estimates, while the parts away from the singular set are treated by the Riemann–Lebesgue lemma and the usual local criteria for Fourier convergence.

In this sense, the convergence theorems obtained above are not tied to the presence of a single singularity at the origin. They remain valid, with the same local conclusions, for functions having finitely many admissible oscillatory singularities. At every point x_0 which is not one of the singular points and where the corresponding local bounded-variation or Dini-type hypothesis holds, one has

$$\lim_{n\to\infty} S_n(f, x_0) = \frac{f(x_0+) + f(x_0-)}{2},$$

and, in particular, the limit is $f(x_0)$ whenever f is continuous at x_0.

References

[1] R. G. Bartle, *A modern theory of integration*, Grad. Stud. Math. **32**, AMS, Providence, RI, 2001.

[2] R. P. Boas, Jr., *Integrability theorems for trigonometric transforms*, Ergeb. Math. Grenzgeb., Band **3**8, Springer-Verlag New York, Inc., New York, 1967.

[3] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math., **116** (1966), 135–157.

[4] A. Denjoy, *Leçons sur le calcul des coefficients d'une série trigonométrique enseignées à l'Université Harvard*, Gauthier–Villars, Paris, 1941–1949.

[5] A. Denjoy, *Leçons sur le calcul des coefficients d'une série trigonométrique enseignées à l'Université Harvard. Quatrième partie: Les totalisations. Solution du problème de Fourier*, Gauthier–Villars, Paris, 1949.

[6] J. Duoandikoetxea, *Fourier Analysis*, Grad. Stud. Math., **29**, AMS, Providence, RI, 2001.

[7] R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Grad. Stud. Math., **4**, AMS, Providence, RI, 1994.

[8] L. Grafakos, *Classical Fourier analysis*, Grad. Texts in Math., **249**, Springer New York, 2008.

[9] R. Henstock, *The General Theory of Integration*, Oxford Mathematical Monographs, 1961.

[10] E. W. Hobson, On the Second Mean-Value Theorem of the Integral Calculus, Proc. London Math. Soc. (2), **7** (1909), 14–23.

[11] R. A. Hunt, On the convergence of Fourier series, Orthogonal Expansions and their Continuous Analogues, Southern Illinois University Press, (1968), 235–255.

[12] D. Khadjiev, The widest continuous integral, J. Math. Anal. Appl., **326** (2007), no. 2, 1101–1115.

[13] A. Kolmogoroff, Une série de Fourier–Lebesgue divergente presque partout, Fund. Math., **4** (1923), 324–328.

[14] J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, Cz. Math. J., **7** (1957), no. 3, 418–449.

[15] F. J. Mendoza Torres, M. G. Morales Macías, J. A. Escamilla Reyna, J. H. Arredondo Ruiz, Several aspects around the Riemann-Lebesgue lemma, J. Adv. Res. Pure Math. **5** (2013), no. 3, 33–46.

[16] J. Milnor, *Morse Theory*, Ann. of Math. Stud., vol. **51**, Princeton University Press, Princeton, NJ, 1963.

[17] P. Mikusiński, K. Ostaszewski, The space of Henstock integrable functions. II, New integrals (Coleraine, 1988), Lecture Notes in Math., 1419 (1990) 136–149.

[18] T. Y. Lee, *Henstock-Kurzweil integration on Euclidean spaces*, Series in Real Analysis, 12, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.

[19] M. G. Plotnikov, Recovery of the coefficients of multiple Haar and Walsh series, Real Anal. Exchange **33** (2007/2008), no. 2, 291–308.

[20] W. Rudin, *Functional Analysis*, Internat. Ser. Pure Appl. Math., vol. 8, 2nd ed., McGraw-Hill, Inc., 1991.

[21] V. A. Skvortsov and P. Sworowski, Comparison of some trigonometric integrals II, Eur. J. Math., **5** (2019), no. 1, 629–640.

[22] V. A. Skvortsov and F. Tulone, Multidimensional dyadic Kurzweil–Henstock and Perron-type integrals in the theory of Haar and Walsh series, J. Math. Anal. Appl., **421** (2015), no. 2, 1470–1496.

[23] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, 1993.

[24] C. Swartz, *Introduction to gauge integrals*, World Scientific Publishing Co., Singapore, 2001.

[25] E. Talvila, Fourier series with the continuous primitive integral, J. Fourier Anal. Appl., **18** (2012), no. 1, 27–44.

[26] E. Talvila, The Fourier transform with Henstock-Kurzweil and continuous primitive integrals, Poincare J. Anal. Appl., **12** (2025), no. 1, 17–37.

[27] J. van der Corput, Zahlentheoretische Abschätzungen, Math. Ann. **84** (1921) 53-79.

[28] A. Zygmund, *Trigonometric Series*, 3rd ed., Cambridge University Press, 2002.

M. Bello-Hernández, Corresponding author. Departamento de Matemáticas y Computación, Universidad de La Rioja. c/ Madre de Dios, 53, 26006 Logroño, La Rioja, Spain. Research supported in part by the grant PID2022–138342NB–I00 AEI, from Spanish Government.
mbello@unirioja.es

P. Sáenz-López-Guerra, Departamento de Matemáticas y Computación, Universidad de La Rioja. c/ Madre de Dios, 53, 26006 Logroño, La Rioja, Spain.
pablo.saenz-lopez@alum.unirioja.es