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Research Article

Keywords:

Posted Date: August 24th, 2026

DOI: <https://doi.org/10.21203/rs.3.rs-10400682/v1>

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Additional Declarations: No competing interests reported.

ON HOLOMORPHIC RETRACTS OF POLYBALLS WITH STRICTLY CONVEX FACTORS AND APPLICATIONS

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ABSTRACT. The purpose of this article is to characterize (holomorphic) retracts through the center of polyballs as in the title i.e., products of strictly convex balls in complex Euclidean spaces. We obtain a characterization of all retracts through the center of such polyballs in terms of the linear retracts of their factors; indeed, we show that they are either biholomorphic to a linear retract of one of the factors of the polyball or to a product of such retracts. This suffices for an application to establish the essential uniqueness of product representations for the aforementioned polyballs; thereby showing that they cannot be biholomorphically equivalent to any other product domain (with irreducible factors) as soon as there is a mismatch in the number of factors or in the lists of their dimensions upto a permutation.

1. INTRODUCTION

Holomorphic retracts of domains i.e., images $Z = \rho(D)$ of idempotent holomorphic self-maps $\rho : D \rightarrow D$ where $D \subset_{\text{dom}} \mathbb{C}^N$ arise in various problems in complex analysis in dimensions $N > 1$; for instance, they arise naturally in the iterative dynamics of holomorphic maps on D . They are of interest in their own right as well. However, characterizing the non-trivial retracts ¹ is very challenging in general. Apart from the Euclidean ball $\mathbb{B} = \mathbb{B}^N$, the polydisc Δ^N or perhaps their products and certain other special domains such as the tetrablock in $\mathbb{C}^3$ attained only quite recently by Ghosh – Zwonek in [14], very little is known. Of course, we have the general fact that retracts Z of D are always closed connected complex submanifolds of D (well expositied by Abate in [7]). When the dimension d of Z is one of the extreme possibilities $d = 0$ or N , it is a point or all of D and in either case, we shall refer to Z as trivial retracts. By a retract then, we shall generally mean a non-trivial one. Certainly, a general submanifold of D is very far from being a retract, leading to the need for sharper results towards narrowing down the possibilities for Z . Naturally this requires imposing conditions on D ; in-fact, even after imposing conditions to the extent of taking D to be a special and relatively simple domain such as one of those mentioned above, this has been found to be non-trivial. To briefly explain what conditions have been imposed and with what effect, while simultaneously recalling a classical theorem in this regard, we first consider (convex) balls which generalize $\mathbb{B}$ in the following sense: balls with respect to some norm such that their boundaries are $\mathbb{C}$ -extremal. Then, the most general result about retracts through their center is the one due to Vesentini [9] which says that every such retract must be linear. At the other extreme is the case of the polydisc which is in a sense more interesting as it possesses, comparatively

2020 *Mathematics Subject Classification.* Primary: 32F45; Secondary: 32H15, 32H20, 13F20, 46B20.

Key words and phrases. Holomorphic Retracts, Balanced pseudoconvex domains, Schwarz lemma, Biholomorphic inequivalences, Kobayashi metric, One-complemented subspaces, Norm-one projections.

¹Henceforth, all retracts in this article are assumed to be holomorphic.

speaking, an apparently greater variety of retracts on the one hand but nevertheless allows for them to be characterized in simple terms on the other. Though product domains form the simplest of subdomains of $\mathbb{C}^N$, nothing much is known except perhaps for products of Euclidean balls and polydiscs; in-fact nothing much has been established for the case of $\mathbb{C}^N$ itself. While it is worthwhile to perform separate studies of retracts of particular domains as in the most recent article [14], another natural direction of research would be to consider classes of domains. In this direction, it is to be noted that at times certain particular results suffice for applications. Specifically, though we only determine retracts through the center of balls in a certain nice class defined by standard assumptions, we show that this itself can be useful. We state what this class is and indicate reasons for the choice below. Towards this, we view product domains as being the simplest class. However, we have to restrict much further, given that the case of $\mathbb{C}^N$ is very challenging as indicated above.

In this article, we consider domains decomposable into a product of balls with respect to some norms (which we shall refer to as polyballs) satisfying an extremality condition, much as the polydisc. While this theme of retracts of polyballs has indeed been considered earlier in [16], the class of polyballs actually dealt with therein was very restricted (only those whose factors are balls in the ℓ^2 -norm) and furthermore, it was only shown therein that all fixed point sets of holomorphic self-maps of such polyballs actually turn out to be retracts, leaving to be desired however, as to what the retracts of polyballs look like. In this regard, we have the following main result of this article which shows that retracts through the center of a product D of strictly convex balls $B_j \subset \mathbb{C}^{N_j}$ (with respect to some norms on $\mathbb{C}^{N_j}$) are all graphs over linear subspaces (indeed, linear retracts but we defer this for now), much as in the particular case of the polydisc due to by Heath – Suffridge [6]. Different proofs were given by several authors at various later points in time but unfortunately only for the polydisc, so the case of polyballs is very much unknown. Here and throughout, by a linear subspace of a balanced domain $D \subset \mathbb{C}^N$, we shall mean a subset of the form $D_L := D \cap L$ for some linear subspace L of $\mathbb{C}^N$.

When $\dim L = 1$ and D is convex which is the case for this article, D_L is a retract i.e., every one-dimensional subspace of D is a retract of D (as follows from the Hahn – Banach theorem). However, this becomes far from true as soon as $\dim L > 1$. While it is not difficult to ascertain that a holomorphic retract Z through the center of any polyball D is a graph of some holomorphic map over a linear subspace which itself is a retract of D , it is much more difficult to attain a description in terms of the *retracts of the factors* of D . The significance and need for such a description being that we would then immediately get the retracts of D , as soon as we have results about the factor domains of the product D . We begin by addressing this for the case of a product of a pair of strictly convex balls (possibly of different dimensions) in the following main theorem of this article; though it is a bit far from being as trivial as it may seem, the theorem can indeed be extended to the case of a product $D = B_1 \times \cdots \times B_n$ of any number of strictly convex balls and this is recorded in theorem 2.24. Specifically, we attain a complete characterization of all retracts Z through the center of such polyballs in terms of the linear retracts of their factors; the result can be paraphrased concisely as mentioned in the abstract. Our reason for restricting to the case of strictly convex factors is to showcase a natural class of polyballs where the occurrence

of non-linear retracts through the center is the least possible, arising precisely due to the operation of taking a product – this becomes necessarily false if the strict convexity condition on just one of the factors (of the polyball) is relaxed to allow a complex line segment in its boundary; we demonstrate this by showing the converse of Vesentini's theorem in proposition 2.12.

Theorem 1.1. *Let B_1, B_2 be a pair of strictly convex balls, possibly with respect to different norms in different complex Euclidean spaces. Then, every non-linear retract of $D = B_1 \times B_2$ through the origin is the graph, either of a B_2 -valued holomorphic map over a linear retract of B_1 or, of a B_1 -valued holomorphic map over a linear retract of B_2 . Further, every retract (whether linear or not) through the center of the product ball D , is either biholomorphic to a linear retract of one of the factors of the polyball or to a product of such retracts.*

More specifically and precisely, let $B_1 \subset \mathbb{C}^{N_1}$ and $B_2 \subset \mathbb{C}^{N_2}$ be as above, $D = B_1 \times B_2$. Let Z be any non-trivial retract of D , $L = T_0 Z$, the tangent space to Z at the origin, $D_L = L \cap D$ and $\partial D_L = L \cap \partial D$. Let π_1, π_2 denote the canonical linear projections from $\mathbb{C}^N = \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ onto $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Then, the following mutually exclusive and exhaustive cases, give a characterization of Z in-terms of where $L = T_0 Z$ intersects the boundary of D :

- (i) *if $\partial D_L \subset \partial B_1 \times \partial B_2$, then Z is a complex linear subspace of D which can be realized as the graph of a complex linear map over a retract of one of the factors B_1 or B_2 , taking values in the other factor.*
- (ii) *if $\partial D_L \subset \partial B_1 \times \overline{B_2}$ but $\partial D_L \not\subset \partial B_1 \times \partial B_2$, then $(B_1)_{L_1} := B_1 \cap L_1$ where $L_1 = \pi_1(L)$, is a retract of B_1 ; furthermore, Z is the graph of a B_2 -valued holomorphic map over this linear retract $(B_1)_{L_1}$.*
- (iii) *if $\partial D_L \subset \overline{B_1} \times \partial B_2$ but $\partial D_L \not\subset \partial B_1 \times \partial B_2$, then $(B_2)_{L_2} := B_2 \cap L_2$ where $L_2 = \pi_2(L)$, is a retract of B_2 ; furthermore, Z is the graph of a B_1 -valued holomorphic map over this linear retract $(B_2)_{L_2}$.*
- (iv) *if ∂D_L intersects both $\partial B_1 \times B_2$ and $B_1 \times \partial B_2$ – which are disjoint open pieces of the boundary of D , indeed connected components of $\partial D \setminus (\partial B_1 \times \partial B_2)$ – then Z is a linear retract of D of complex dimension ≥ 2 , which is neither contained in B_1 nor in B_2 . More precisely, Z can be precisely expressed as:*

$$Z = (L^{(1)} \oplus L^{(2)}) \cap D,$$

where $L^{(j)}$ is a graph of a linear map over a linear subspace $\Sigma_j \subset \mathbb{C}^{N_j}$ with $B_j \cap \Sigma_j$ being a linear retract of B_j (here $j = 1, 2$). Consequently, Z is a linear graph over the product of the linear retracts $(B_j)_{\Sigma_j} := B_j \cap \Sigma_j$ of the factors B_j ; thereby, $Z \simeq (B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$.

More precise information can be given about various cases in the above theorem. For instance, while case-(iv) only asserts existence of certain subspaces $L^{(j)}$'s and Σ_j 's with properties as stated therein, the proof is constructive and will tell how to attain them; we have preferred to defer mentioning about this to keep the statements relatively simple. However with that, it is easily checked that the converse of the above theorem is true. It is even easier to see that each of its four cases actually occurs, thereby that the characterization is complete. It must be noted that while the images of Z under the canonical

projections are retracts of the factors as stated in the theorem, this far from clear. To indicate why, let us only mention that the corresponding statement for retraction mappings is false. That is, it can very well happen that for some retraction maps $\rho : D \rightarrow Z$ that upon restriction to the factors B_1, B_2 , neither of the maps $\pi_1 \circ \rho$ nor $\pi_2 \circ \rho$ turn out to be retraction maps on B_1 or B_2 ; for a simple example, just consider the retraction on the bidisc given by $\rho(z, w) = (z + w, z + w)/2$. This is one among the difficulties involved in attaining the theorem, noting which would help in reading the proof. Next, it goes without saying that in order to apply this theorem to completely determine all the retracts even through the origin of a product as in the theorem (indeed, any polyball), one needs to know the retracts of the factors. For instance, owing to facts about norm one projections for l^p -norms (with $1 < p < \infty$) from the classical treatise [4] (see theorem 2.a.4 therein) on the geometry of Banach spaces, we have a precise knowledge of which linear subspaces of an l^p -ball turn out to be its linear retracts; they are also the only holomorphic retracts through the center, as follows from Vesentini's theorem. The above theorem 1.1 and its extension namely theorem 2.24 in the sequel, can then be applied to determine all retracts through the center, for a product of any number of l^p -balls (of possibly different dimensions and different values of p). As the special but important case of a product of Euclidean balls does not seem to have been recorded anywhere in the literature and we in-fact get *all* their retracts from theorem 2.24 owing to the action of the automorphism group being transitive, we record the result in the following corollary.

Corollary 1.2. *Let $D = \mathbb{B}_1 \times \mathbb{B}_2 \times \dots \times \mathbb{B}_m$ with $\mathbb{B}_j \subset \mathbb{C}^{N_j}$ for $1 \leq j \leq m$ being Euclidean balls (i.e., balls in the l^2 -norm). Then, every retract of D is biholomorphic to a Euclidean ball or to a (non-trivial) product of Euclidean balls.*

It must be noted that the foregoing theorem and its extension namely theorem 2.24, can be paraphrased as: every retract through the center of a polyball $D = B_1 \times \dots \times B_N$ as in these theorems is again equivalent to a product manifold, as soon as it is not biholomorphic to a linear retract of one of the factors (in which case it is irreducible). This can be used to give an application about establishing uniqueness of product representations of such polyballs D as above. Here and henceforth, by a product representation of a domain Ω , we mean an expression of the form $\Omega = \Omega_1 \times \dots \times \Omega_M$ where the Ω_j 's are *irreducible* domains (a domain is irreducible, if it is not *biholomorphic* to a product of two other domains)². Specifically, the aforementioned application is towards answering the following natural question. Can it happen that two essentially different pairs of (irreducible) domains upon taking products, lead to biholomorphic domains? For instance, is it somehow possible for $D_1 \times D_2$ to be biholomorphic to $\Omega_1 \times \Omega_2$ but the factors are not biholomorphic (upto a permutation). Or perhaps worse still, can it happen that a reducible domain has two different product representations wherein the number of factors are unequal? It will be shown below that as a consequence of the theorems on retracts attained, we can be assured that a polyball D as above is inequivalent to another product domain as soon as there is a mismatch in the number of factors or in the biholomorphism types of the factors. It then follows by known results that in-fact every biholomorphic mapping between a pair of possible product decompositions here, can be factorized as a product of biholomorphic mappings between the factors. Such a factorization has long been known for a biholomorphic (or

²Of-course, after repeated factorizations, we may write every domain in this form, perhaps with $n = 1$. The uniqueness of such representations is however far from obvious.

even a proper holomorphic) mapping between a pair of product domains with factors whose boundaries are devoid of analytic varieties, *provided it is already known that the number of factors of the source and the target of the mapping are the same* (see chapter 5 of the book [21]). Results along this line, particularly dropping the apriori requirement on the equality in the number of factors was attained by Ligočka in [22] and Tsyganov in [20]. While a certain generalization of their results to proper holomorphic mappings have also been obtained much more recently by Verma – Chakraborty in [19], it must be noted that in all these works, some smoothness assumption on all of the factors involved in both the source as well as the target domains of the mapping, is imposed. It is not just natural to examine the necessity of such assumptions (for results of the above kind to hold), it is actually useful to drop them and attain ‘complete’ results if possible, for applications to problems of biholomorphic inequivalences of the type mentioned above i.e., ensuring uniqueness of product representations. However, complete results for reducible domains even within certain class seem to be very scant. For instance, it does not seem to be known as to whether a product D of smoothly bounded domains D_j ’s for $j = 1, \dots, n$ for some $n > 1$, all satisfying condition R (of Bell), can be represented as: a product $G_1 \times \dots \times G_m$ with atleast one of the G_j ’s being biholomorphically different from the D_j ’s and wherein, *no* assumptions are made about the G_j ’s, except for being irreducible domains. The following corollary giving a complete answer for the class polyballs as already mentioned, is new to the best of our knowledge; or atleast its proof via retracts, which seems natural to us for the following reason. If $f : D = D_1 \times \dots \times D_n \rightarrow \Omega = \Omega_1 \times \dots \times \Omega_m$ is a biholomorphic map, it is in general not true that the $f(D_j)$ ’s turn out again to be the factors of Ω even if $m = n$ – a counterexample can be found in the simplest case of automorphisms of $\mathbb{C}^2$; a simple concrete one being $f(z, w) = (z + (w + z^2)^2, w + z^2)$. Moreover, there can be biholomorphic images of D which are not actual product domains. While the property of being factors does not remain invariant even if we restrict to biholomorphisms among product domains, the property of being a retract does. Of-course, this invariance is valid beyond the context of product domains but to use this, we need to be assured of the existence of non-trivial retracts (for atleast one out of a given pair of domains whose biholomorphic equivalence we wish to examine), which for product domains is guaranteed as the factors are always retracts.

Corollary 1.3. *Let $D = B_1 \times \dots \times B_N$ be a polyball with the factors B_j ’s being strictly convex. Let $\Omega = \Omega_1 \times \dots \times \Omega_M$ where the Ω_j ’s are irreducible domains (with no regularity assumptions on their boundaries; in particular could be unbounded). If D, Ω are biholomorphically equivalent then, $M = N$ and the factors of Ω and D when suitably ordered, are biholomorphic. If Ω is balanced, then (upto a permutation), the Ω_j ’s are also strictly convex balls, indeed linearly equivalent to the B_j ’s.*

2. PROOFS OF ALL THE RESULTS

While the general strategy here is to follow the proof of theorem 1.5 of the earlier work [1], there are considerably many technical difficulties to implement that in order to attain all the assertions of theorem 1.1 and its extension 2.24), which we deal with in this section. Just to indicate what we mean by this: a key tool we use (that seems indispensable here) is the reverse Gauss map for strictly convex domains, which fails to be single valued as soon as the ball is not strictly convex; however, it is still possible to pull-back the Hausdorff measure of the boundary to the Euclidean sphere of the same dimension, with the strict

convexity condition ensuring that no great subsphere has positive mass (despite the well-known phenomenon of concentration of mass). The relevance of all this will become clear during the course of this section. To make the proof easier to follow, we have tried to break up the proof of the main theorems 1.1 and 2.24, into various lemmas. We begin by briefly recalling well-known notions and facts from convex analysis in the notations that we shall use and soon after, proceed forth to formulate preparatory lemmas as mentioned.

Definition 2.1. Let L be a linear subspace of $\mathbb{R}^N$. Let D be a domain in $\mathbb{R}^N$ and $p \in \partial D$. We say that L is a *supporting subspace* for ∂D at p if $(p + L) \cap D = \emptyset$. In such a case, L is called a *supporting subspace* for ∂D at p .

We also need a definition that is not standard, namely the following. A vector v is called a *supporting vector* for ∂D at p if $(p + \text{span}_{\mathbb{R}}(v))$ does not intersect D .

Remark 2.2. If D is a convex domain in $\mathbb{R}^N$ and $p \in \partial D$, then k is a support vector for ∂D at p if and only if k lies in a supporting hyperplane at p (as follows by the Hahn – Banach theorem). Equivalently, by taking the contrapositive, k is not a support vector for ∂D at p if and only if k does not lie in any of the supporting hyperplanes for ∂D at p . Let us rephrase this one last time in a form convenient for ready use later. Namely, in terms of outward normals, we have the following characterization: k is a non-supporting vector for ∂D at p with $p + tk \in D$ for all $t \in (-\epsilon, 0)$ for some $\epsilon > 0$ (depending on p and k), if and only if $\langle k, v_p \rangle > 0$ for every outward unit normal vector v_p at p .

Definition 2.3. (Gauss Map and Reverse Gauss map): Let K be a convex body in $\mathbb{R}^N$. The *Gauss map* of K denoted by ν_K , associates to each point $x \in \partial K$ the set of all outward unit normal vectors of K at x . The *reverse Gauss map* of K denoted by ν_K^{-1} , associates to each point $\omega \in S^{N-1}$, the set of all boundary points of K whose normal vector is ω .

Note that in general, the Gauss map and the reverse Gauss map are not single-valued. If ∂K is smooth then ν_K is single valued. If K is strictly convex then the reverse Gauss map is a single valued map. We recall the following well-known facts (see lemma 2.2.12 in [5]), which will be useful in the sequel.

Proposition 2.4. *Let K be a convex body in $\mathbb{R}^N$.*

- (1) *If K is smooth then ν_K is a continuous map from ∂K onto S^{N-1} ,*
- (2) *If K is strictly convex then ν_K^{-1} is a continuous (single-valued) map from S^{N-1} onto ∂K ,*
- (3) *If K is both smooth and strictly convex then ν_K is a homeomorphism from ∂K onto S^{N-1} and its inverse is ν_K^{-1} .*

A couple of other notions will be needed by less frequently. As there are directly available in the literature, for instance in [5], we skip recalling them here. We instead proceed to working out lemmas for which we could not find any references, starting with the following absolutely elementary lemma, all being quite important for the proof of the main theorems.

Lemma 2.5. *Let B be a ball in $\mathbb{R}^N$ with respect to some norm and k be a fixed vector in $\mathbb{R}^N$. Let L be a linear subspace of $\mathbb{R}^N$ and*

$$A := \{p \in \partial B_L : k \text{ is supporting for } \partial B \text{ at } p\}.$$

Then A is a closed subset of $\partial B_L := L \cap \partial B$.

Proof. To prove this lemma, let $p_n \in A$ with $p_n \rightarrow p \in \partial B_L$. It follows that k is supporting for ∂B at each $p_n \in \partial B_L$ and $p_n \rightarrow p$. Let $\|\cdot\|$ denotes the norm associated to B . Since k is supporting for ∂B at each p_n , $(p_n + \text{span}_{\mathbb{R}}(k)) \cap B = \emptyset$. This implies that $\|p_n + tk\| \geq 1$ for all $t \in \mathbb{R}$. Since $p_n \rightarrow p$, $\|p + tk\| \geq 1$ for all $t \in \mathbb{R}$. This shows that $(p + \text{span}_{\mathbb{R}}(k)) \cap B = \emptyset$, thereby $p \in A$. This proves that A is a closed subset of ∂B_L . $\square$

We cannot claim that A as in the above lemma is nowhere dense or small (as a subset of ∂B_L) in any sense; however, when $L = \mathbb{R}^N$, this is true as shown next.

Lemma 2.6. *Let B be a strictly convex ball in $\mathbb{R}^N$ with respect to some norm and k be a non-zero vector in $\mathbb{R}^N$. Let $A := \{q \in \partial B : k \text{ is a support vector for } \partial B \text{ at } q\}$. Then A is a (closed) nowhere dense subset of ∂B .*

Proof. We begin by noting for each $q \in \partial B$ that, any supporting vector for ∂B at q lies in a supporting hyperplane at q (this follows from the Hahn – Banach theorem; for a convenient version here, we suggest theorem 7 in chapter 1 of [17] for instance). Hence we obtain that

$$A = \{q \in \partial B : \text{there exists } n(q) \in N(q, \overline{B}) \text{ such that } \langle n(q), k \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ just denotes the usual dot product on $\mathbb{R}^N$ and $N(q, \overline{B})$ denotes the normal cone of $\overline{B}$ at q . Since B is strictly convex, the reverse Gauss map which we denote as usual by $\nu_{\overline{B}}^{-1}$, is a well-defined (single valued) continuous map from $\mathbb{S}^{N-1}$ onto ∂B . Now, we want to show that $A = \nu_{\overline{B}}^{-1}(\tilde{S})$, where $\tilde{S} := k^\perp \cap \mathbb{S}^{N-1}$. If $q \in A$, then there exists $n(q) \in N(q, \overline{B})$ such that $\langle n(q), k \rangle = 0$. Note that $n(q) \in \tilde{S}$. By the definition of $\nu_{\overline{B}}^{-1}$, $\nu_{\overline{B}}^{-1}(n(q))$ is the unique point in ∂B such that $n(q)$ is a normal vector for ∂B at $\nu_{\overline{B}}^{-1}(n(q))$. Since $n(q) \in N(q, \overline{B})$, we have $\nu_{\overline{B}}^{-1}(n(q)) = q \in \nu_{\overline{B}}^{-1}(\tilde{S})$.

We now prove the converse. To begin with, if $\eta \in \tilde{S}$ then $\langle \eta, k \rangle = 0$. Note that $\nu_{\overline{B}}^{-1}(\eta) \in \partial B \cap A$ because $\eta \in N(\nu_{\overline{B}}^{-1}(\eta), \overline{B})$. This proves that $A = \nu_{\overline{B}}^{-1}(\tilde{S})$. A notation that we shall briefly use here is $S_{\overline{B}}$, for the surface area measure of $\overline{B}$. This is the finite Borel measure on $\mathbb{S}^{N-1}$ defined for every Borel set $\omega \subset \mathbb{S}^{N-1}$ by $S_{\overline{B}}(\omega) = \mathcal{H}^{N-1}(\nu_{\overline{B}}^{-1}(\omega))$, where $\mathcal{H}^{N-1}$ denotes the $(N - 1)$ -dimensional Hausdorff measure. Since B is strictly convex, it follows from lemma 3.1 in [18] that surface area $S_{\overline{B}}$ does not have positive mass. This implies that $A := \nu_{\overline{B}}^{-1}(\tilde{S})$ is nowhere dense in ∂B . $\square$

We now turn from facts about convex bodies in real Euclidean spaces to those (facts about their retracts) in complex ones.

Theorem 2.7 (Vesentini). *Let D be a bounded balanced convex domain in $\mathbb{C}^N$ with $\mathbb{C}$ -extremal boundary. Then every retract of D passing through origin is a linear subspace of D .*

In view of the fact that for a given retract Z of a domain D , there may well be plenty of retraction maps with the same image Z , a basic question in statements such as the above is whether: it is somehow possible for a linear subspace of a domain D as in Vesentini's theorem, to be the image of a non-linear (holomorphic) retraction map, but not of a linear retraction (i.e., a norm-one projection). This is ruled out by the following lemma for (the larger class) of all possible balls irrespective of $\mathbb{C}$ -extremality of their boundaries, thereby

the polyballs of interest in this article; this will be used numerous times (tacitly) in the sequel.

Lemma 2.8. *Let D be a bounded balanced convex domain in $\mathbb{C}^N$, Z a retract passing through the origin with $\rho : D \rightarrow Z$, a retraction map. Let L denote the tangent space to Z at the origin and $\|\cdot\|$ the Minkowski functional of D (the convexity of D means that this functional gives a norm on $\mathbb{C}^N$). Then, $D\rho|_0$ is a norm one projection from $(\mathbb{C}^N, \|\cdot\|)$ onto $(L, \|\cdot\|)$, thus mapping D onto $D_L := D \cap L$; thereby, D_L is a linear retract of D .*

Proof. Since ρ is a retraction map on D , we have $\rho \circ \rho = \rho$. Differentiating the identity $\rho \circ \rho = \rho$ at the origin, we obtain that $D\rho|_0 \circ D\rho|_0 = D\rho|_0$. This shows that $D\rho|_0$ is a linear projection map from $\mathbb{C}^N$ onto L . Note that being a linear slice of the ball D by the subspace L , D_L is a ball with respect to some norm in L . As $\rho(0) = 0$, it follows from proposition C in [12] that $\|D\rho|_0\| \leq 1$; thereby $D\rho|_0$ maps D into D . As $D\rho|_0$ fixes L , $D\rho|_0$ is a norm-one projection, which maps D onto D_L . $\square$

The next proposition gives a geometric characterization of precisely when is a given linear subspace a retract of a ball with respect to a given norm. It is simple to prove but is a key observation for many of the proofs in the sequel.

Proposition 2.9. *Let D be a bounded balanced convex domain in $\mathbb{C}^N$. Let L be any linear subspace of $\mathbb{C}^N$. Then $D_L := D \cap L$ is a linear retract of D if and only if there exists a complementary subspace M such that for every $q \in \partial D_L$, $q + M$ does not intersect D .*

Proof. Assume that there exists a linear subspace M such that $\mathbb{C}^N = L \oplus M$ and for every $q \in \partial D_L$, $q + M$ does not intersect D . Consider the linear projection map ρ on $\mathbb{C}^N$ with $\text{image}(\rho) = L$ and $\ker(\rho) = M$. To prove D_L is a linear retract of D , it suffices to show that ρ maps D into D . Let z be an arbitrary point in D and $\|\cdot\|$ denotes the norm associated to D . Write $z = z_L + z_M$, where $z_L \in L$ and $z_M \in M$. Note that

$$(2.1) \quad \|\rho(z)\| = \|\rho(z_L + z_M)\| = \|z_L\|$$

Now, we claim that $\|z_L\| \leq \|z_L + z_M\|$. By the hypotheses, for every $q \in \partial D_L$, $q + M$ does not intersect D . As $\rho(z) = z_L \in L$, we have $(z_L/\|z_L\| + M)$ does not intersect D . This implies in particular that

$$\left\| \frac{z_L}{\|z_L\|} + \frac{z_M}{\|z_L\|} \right\| \geq 1.$$

This shows that $\|z\| = \|z_L + z_M\| \geq \|z_L\|$. Hence by (2.1) and the inequality $\|z_L\| \leq \|z\|$, it follows that ρ maps D into D .

Conversely assume that D_L is a linear retract of D . This implies that there exists a linear projection map T on $\mathbb{C}^N$ such that T maps D onto D_L . Let $M := \ker(T)$. Since T is a linear projection map from $\mathbb{C}^N$ onto L , we have $\mathbb{C}^N = L \oplus M$. Now, we have to show that for every $q \in \partial D_L$ such that $q + M$ does not intersect D . To prove this by contradiction, assume that there exists $q \in \partial D_L$ such that $q + M$ intersects D . This implies that there exists $v \in M$ such that $q + v \in D$. Hence, $T(q + v) = q \in \partial D$, which contradicts the fact that T maps D into D . $\square$

Determining retracts of balls and their products through their center (as in theorem 1.1, the main matter of this section), requires versions of the Schwarz lemma about them; one of the best possible versions specific to balls is the following result, which we shall refer to as Schwarz lemma due to Mazet [8].

Lemma 2.10. (*Mazet's Schwarz lemma*) *Let $(F, \|\cdot\|)$ be a complex Banach space, B_F its standard open unit ball centered at the origin $B_F = \{v \in F : \|v\| < 1\}$ and f a holomorphic curve in B_F i.e., a holomorphic map $f : \Delta \rightarrow B_F$. Suppose $\|df(0)\| = 1$ and that $df(0)$ is $\mathbb{R}$ -extremal in $\overline{B_F}$. Then, f must be a linear map: $f(z) = df(0)z$.*

We note that $\mathbb{R}$ -extremality just means strict convexity and the lemma fails if it is replaced by $\mathbb{C}$ -extremality as also shown in [8]. A quick illustration of the above is to confirm that strictly convex balls in $\mathbb{C}^N$ have holomorphically extreme boundaries i.e., do not admit any non-trivial analytic varieties in their boundaries.

Corollary 2.11. *Let B be a ball in $\mathbb{C}^N$ with respect to some norm. If B is strictly convex then B is holomorphic extreme (i.e. for each $p \in \partial B$, there is no non-constant holomorphic mapping $\varphi : \Delta \rightarrow \overline{B}$ with $\varphi(0) = p$).*

Proof. To prove this, let $\varphi : \Delta \rightarrow \overline{B}$ be a holomorphic map with $\varphi(0) = p \in \partial B$. Consider another holomorphic map $\psi : \Delta \rightarrow \mathbb{C}^N$ defined by $\psi(t) = t\varphi(t)$. Note that ψ maps Δ into B and $\psi'(0) = p$ is $\mathbb{R}$ -extremal in $\overline{B}$. By Mazet's Schwarz lemma applied to ψ , we obtain that $\psi(t) = tp$. This implies that φ is a constant function and thereby B is holomorphic extreme. $\square$

The converse of Vesentini's theorem is also true and not difficult to prove, though it does not seem to have been noted anywhere once: if all retracts through the origin of a bounded balanced convex domain are linear, then the domain is $\mathbb{C}$ -extremal. This is done by the following proposition.

Proposition 2.12. *Let D be a bounded balanced convex domain in $\mathbb{C}^N$. Assume that there exists $p \in \partial D$ such that ∂D is not $\mathbb{C}$ -extremal at p . Then there exists a non-linear retract of D passing through the origin in the direction of p .*

Proof. As $p \in \partial D$ is not $\mathbb{C}$ -extremal, there exist $\epsilon_0 > 0$ and a non-zero $u \in \mathbb{C}^N$ such that $p + tu \in \partial D$ for all $t \in \mathbb{C}$ with $|t| < \epsilon_0$. After a rescaling of u , we may assume $\epsilon_0 = 1$. Consider the holomorphic map $\varphi : \Delta \rightarrow \mathbb{C}^N$ defined by $\varphi(t) = t(p + tu)$. Let $\|\cdot\|$ denotes the norm associated to D . Note that for all $t \in \Delta$, we have

$$\|\varphi(t)\| = |t|\|p + tu\| = |t| < 1.$$

Thus φ maps Δ into D . Let $L := \text{span}(p)$. Observe that $u \notin L$. Indeed, if $u = \lambda p$ for some $\lambda \in \mathbb{C}^*$ then for all $t \in \Delta$,

$$1 = \|p + tu\| = \|(1 + t\lambda)p\| = |1 + t\lambda|,$$

which is impossible. Hence $Z = \text{image}(\varphi)$ is a graph of the holomorphic map $\tilde{\varphi}(t) = t^2u$ over the linear retract D_L of D . Now, we want to show that Z is a retract of D . Since D_L is a linear retract of D , there exists a linear projection map P such that $P(D) = D_L$. By idempotency of P , we can write $\mathbb{C}^N = L \oplus M$, where $M = \ker(P)$ and $L = \text{span}(p)$. Consider the map $\psi : D \rightarrow D$ defined by $\psi(z) = P(z) + \tilde{\varphi}(P(z))$; this can be verified to be idempotent by a direct computation, thereby ψ is a holomorphic retraction from D onto Z . This proves that Z is a non-linear retract of D passing through the origin in the direction of p . $\square$

Remark 2.13. We may now confirm the remark in the introduction that the feature of non-linearity of retracts of polyballs arising from the operation of taking products (and thereby theorem 1.1), ceases to be true as soon as we allow the factors of the polyball to

have complex line segments in their boundaries. To see this, choose a ball B , which is not $\mathbb{C}$ -extremal. It follows as in the proof of proposition 2.12 that $Z_0 := \{\varphi(t) : t \in \Delta\}$ is a one-dimensional non-linear retract of B , where $\varphi(t) = tp + t^2u$. It then follows that $Z = \{(\varphi(t), \varphi(t)) : t \in \Delta\}$ is a one-dimensional non-linear retract of $B \times B$ passing through the origin. Note that $L := T_0Z = \{t(p, p) : t \in \mathbb{C}\}$ and thereby $\partial D_L \subset \partial B \times \partial B$.

Occasions arise where we have to view a direct sum of linear subspaces of $\mathbb{C}^N$ ‘externally’ in which case, we shall then refer to it as a product rather than sum. For instance, if $N_1, N_2, N \in \mathbb{N}$ with $N = N_1 + N_2$ we identify $\mathbb{C}^{N_1} \times \{0\}$ with $\mathbb{C}^{N_1}$ and write $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$ also as $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$. In short, we use ‘direct sum’ and ‘product’ interchangeably in the context of linear subspaces. We shall also need the following topological fact about certain linear subspaces of a polyball.

Lemma 2.14. *Let B_1, B_2 be bounded balanced convex domains in $\mathbb{C}^{N_1}, \mathbb{C}^{N_2}$ respectively. Let $D = B_1 \times B_2$ and L be a linear subspace of $\mathbb{C}^N$, where $N = N_1 + N_2$; let D_L denote as usual, the intercept $D \cap L$. If ∂D_L intersects $\partial B_1 \times B_2$ and $B_1 \times \partial B_2$ then ∂D_L intersects $\partial B_1 \times \partial B_2$.*

Proof. Assume that ∂D_L intersects $\partial B_1 \times B_2$ and $B_1 \times \partial B_2$. Choose $(p, w) \in \partial B_1 \times B_2$ and $(z, q) \in B_1 \times \partial B_2$. Being the boundary of a ball, ∂D_L is homeomorphic to the Euclidean sphere in L , thereby is path connected; therefore, one can choose a path $\gamma = (\gamma_1, \gamma_2)$ within ∂D_L joining (p, w) and (z, q) . It suffices to verify that this path intersects $\partial B_1 \times \partial B_2$. Towards this, let $\|\cdot\|_1, \|\cdot\|_2$ denote the norms associated to B_1, B_2 respectively and consider the continuous self-maps f, g of the unit interval $[0, 1]$ defined by $f(t) = \|\gamma_1(t)\|_1$ and $g(t) = \|\gamma_2(t)\|_2$. Note that $f(0) = 1, f(1) < 1$ and $g(0) < 1, g(1) = 1$. Just by the intermediate value property applied to the continuous function $g - f$ on $[0, 1]$, we are guaranteed $t_0 \in (0, 1)$ such that $\|\gamma_1(t_0)\|_1 = \|\gamma_2(t_0)\|_2$; as $\gamma(t) \in \partial D_L$ for all $t \in [0, 1]$, this common norm value must be 1. This means that $\gamma(t_0) \in \partial B_1 \times \partial B_2$, finishing the proof. $\square$

The above lemma cannot be generalized in too straightforward a manner for polyballs $B_1 \times \cdots \times B_N$ when the number of factors N exceeds two, since the ‘distinguished boundary piece’ $\partial B_1 \times \cdots \times \partial B_N$ taken in the boundary, is connected. Nevertheless of-course, an analogue holds and in order to formulate that we introduce some necessary definitions.

Definition 2.15. Let $D = \prod_{j=1}^N B_j$ be a polyball, where B_j is a ball in $\mathbb{C}^{n_j}$ with respect to some norm. For each $p \in \{1, 2, \dots, N\}$, denote by S_p , the set of all p -tuples $(j_1, j_2, \dots, j_p)$ where $\{j_1, j_2, \dots, j_p\} \subset \{1, 2, \dots, N\}$ with $j_1 < j_2 < \dots < j_p$. For each $\sigma = (j_1, \dots, j_p) \in S_p$, let $B^\sigma := B_{j_1} \times \dots \times B_{j_p}$, $(\partial B)^\sigma := \partial B_{j_1} \times \dots \times \partial B_{j_p}$, $\partial B_e := \partial B_1 \times \dots \times \partial B_N$ and π_σ denote the projection from $\mathbb{C}^m$ onto $\mathbb{C}^\sigma := \prod_{i=1}^p \mathbb{C}^{n_{j_i}}$, where $m = n_1 + \dots + n_N$. Let $\sigma_c \in S_{N-p}$ denotes the $(N-p)$ -tuple complementary to σ in the sense that $\sigma_c = (k_1, \dots, k_{N-p})$, where $\{k_1, \dots, k_{N-p}\} = \{1, \dots, N\} \setminus \{j_1, \dots, j_p\}$ with $k_1 < k_2 < \dots < k_{N-p}$.

With these notations, we have the following important definition. For each $i = 1, \dots, N$, the i -th facet of ∂D is the boundary-subset given by $\partial D_i := \partial B_i \times B^{(i)c}$. (For the definition of a facet, we refer the reader to Section 2.1 of Chapter 1 in [5].) The complement of the

union of all the facets is the boundary subset given by

$$\partial^{N-1}D := \bigcup_{k=2}^{N-1} \bigcup_{\tau \in S_k} (\partial B)^\tau \times \overline{B}^{\tau^c}.$$

Lemma 2.16. *Consider the polyball $D = \prod_{j=1}^N B_j$, where each B_j is a ball in $\mathbb{C}^{n_j}$ with respect to some norm. Let L be a linear subspace of $\mathbb{C}^m$, where $m = n_1 + \dots + n_N$. If ∂D_L intersects each of the facets ∂D_i for each $i = 1, \dots, N$, then ∂D_L intersects their complementary subset $\partial^{N-1}D$ as well.*

Proof. Since ∂D_L is connected, to prove this lemma, it suffices to show that $\partial D \setminus \partial^{N-1}D$ is disconnected. Note that ∂D_i 's are disjoint and

$$(2.2) \quad \partial D \setminus \partial^{N-1}D = \bigcup_{i=1}^N \partial D_i,$$

wherein $\partial D_i := \partial B_i \times B^{(i)c}$ which is a product of a topological sphere with balls. Hence each ∂D_i is a connected subset of $\partial D \setminus \partial^{N-1}D$. To prove $\partial D \setminus \partial^{N-1}D$ is disconnected, it is therefore enough to show that ∂D_i is a connected component of $\partial D \setminus \partial^{N-1}D$. To prove this by contradiction, assume that there exists a connected subset C_i of $\partial D \setminus \partial^{N-1}D$ that properly contains ∂D_i . This implies that C_i must contain a point of ∂D_{j_0} for some $j_0 \neq i$. It follows from (2.2) that

$$C_i = \bigcup_{j=1}^N (C_i \cap \partial D_j).$$

Since $C_i \cap \partial D_{j_0} \neq \emptyset$, C_i is the disjoint union of atleast two non-empty open subsets of C_i , contradicting the connectedness of C_i . $\square$

Lemma 2.17. *Let B_1, B_2 be bounded balanced convex domains in $\mathbb{C}^{N_1}, \mathbb{C}^{N_2}$ respectively. Let $D = B_1 \times B_2$ and Z be a non-trivial retract of D passing through the origin with $\rho : D \rightarrow Z$, a retraction map. Let $L = T_0 Z$, the tangent space to Z at the origin, $D_L = L \cap D$ and $\partial D_L = L \cap \partial D$. Let $M := \ker(D\rho|_0)$. Then the following statement holds.*

- (i) *Assume that B_1 and B_2 are strictly convex. If ∂D_L intersects $\partial B_1 \times B_2$, $B_1 \times \partial B_2$ or both (i.e. $\partial D_L \not\subseteq \partial B_1 \times \partial B_2$), then the kernel of $D\rho|_0$ is a product: $M = M_1 \times M_2$ where M_1, M_2 are the linear subspaces of $\mathbb{C}^{N_1}, \mathbb{C}^{N_2}$ respectively, given by $M_1 = \pi_1(M)$ and $M_2 = \pi_2(M)$.*
- (ii) *Suppose that $M = M_1 \times M_2$, where $M_1 = \pi_1(M)$ and $M_2 = \pi_2(M)$. Let $L^{(1)} := (\mathbb{C}^{N_1} \times M_2) \cap L$, $L^{(2)} := (M_1 \times \mathbb{C}^{N_2}) \cap L$, $\Sigma_1 := \pi_1(L^{(1)})$ and $\Sigma_2 := \pi_2(L^{(2)})$. Then the following holds:*
 - (a) $\Sigma_i \oplus M_i = \mathbb{C}^{N_i}$ for $i = 1, 2$ and $L = L^{(1)} \oplus L^{(2)}$.
 - (b) $(B_1)_{\Sigma_1} := \Sigma_1 \cap B_1, (B_2)_{\Sigma_2} := \Sigma_2 \cap B_2$ are linear retracts of B_1, B_2 respectively.
 - (c) $D_{L^{(i)}} := D \cap L^{(i)}$ is the graph of the linear map $(D\rho|_0 - I)$ over the linear retract $(B_i)_{\Sigma_i}$ for $i = 1, 2$ with $\partial D_{L^{(1)}} \subset \partial B_1 \times \overline{B_2}$ and $\partial D_{L^{(2)}} \subset \overline{B_1} \times \partial B_2$.
 - (d) D_L is the graph of the linear map $(D\rho|_0 - I)$ over $(B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$ and taking values in $(B_1)_{M_1} \times (B_2)_{M_2}$. Therefore, D_L is biholomorphic to product of the retracts $(B_1)_{\Sigma_1}$ of B_1 and $(B_2)_{\Sigma_2}$ of B_2 .

Proof. (i) Let $\pi^L := D\rho|_0$ and $M := \ker(\pi^L)$. We want to show that $M = M_1 \times M_2$, where $M_1 = \pi_1(M)$ and $M_2 = \pi_2(M)$. Since $M \subset M_1 \times M_2$, it suffices to show that $M_1 \times M_2 \subset M$. To prove this by contradiction, assume that there exists $(v_1, v_2) \in M_1 \times M_2$ which does not belong to M . Let $(\alpha, \beta) := \pi^L(v_1, v_2)$. As $(\alpha, \beta) \neq (0, 0)$, after scaling by a suitable scalar, we may assume that $(\alpha, \beta) \in \partial D_L$ and $(\alpha, \beta) = \pi^L(v_1, v_2)$. Now, we divide the proof into three cases depending on whether (α, β) belongs to one of the three pieces: $\partial B_1 \times B_2$, $B_1 \times \partial B_2$ and $\partial B_1 \times \partial B_2$.

Case-1: $(\alpha, \beta) \in \partial B_1 \times B_2$.

To deal with this case, first note that as $v_1 \in M_1 := \pi_1(M)$, there exists $w_2 \in M_2$ such that $(v_1, w_2) \in M$. Next, note that

$$(\alpha, \beta) = \pi^L(v_1, v_2) = \pi^L((v_1, w_2) + (0, v_2 - w_2)) = \pi^L(0, v_2 - w_2).$$

Let $v := v_2 - w_2 \in M_2$. We obtain that $\pi^L(0, v) = (\alpha, \beta)$. This implies that $(\alpha, \beta - v) \in M$. Observe that for all negative real numbers $s \in \mathbb{R}^-$ sufficiently close to 0, we have $(1+s)\alpha \in B_1$ and further as $\beta \in B_2$, we can be assured for the same set of values for s that $\beta + s(\beta - v) \in B_2$. Hence, the line

$$\ell := \{(\alpha, \beta) + s(\alpha, \beta - v) : s \in \mathbb{R}\}$$

intersects D . Note that $(\alpha, \beta) \in \partial D_L$ and $(\alpha, \beta - v) \in \ker(\pi^L)$. Therefore, $\ell \cap D \neq \emptyset$ contradicting the fact that π^L maps D into D and finishes the proof.

Case-2: $(\alpha, \beta) \in B_1 \times \partial B_2$. We skip the proof as it is similar to the previous case.

Case-3: $(\alpha, \beta) \in \partial B_1 \times \partial B_2$.

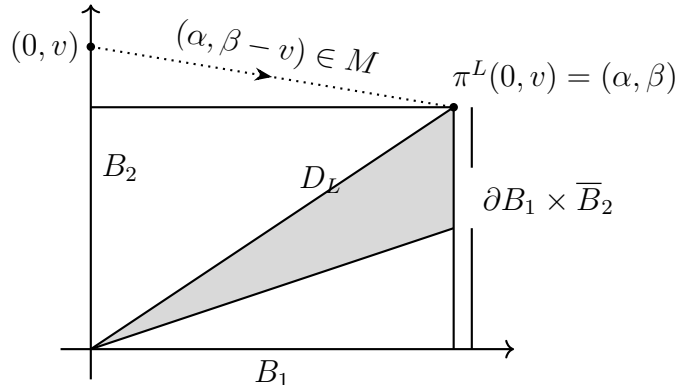
To deal with this case, first recall by hypothesis that $\partial D_L \subsetneq \partial B_1 \times \partial B_2$, thereby there are boundary points of D_L in $\partial(B_1 \times B_2) \setminus (\partial B_1 \times \partial B_2)$; accordingly, we divide into two subcases depending on whether ∂D_L intersects $\partial B_1 \times B_2$ or $B_1 \times \partial B_2$.

Subcase-1: ∂D_L intersects $\partial B_1 \times B_2$.

As in Case-1, there exists $v \in \mathbb{C}^{N_2}$ such that $\pi^L(0, v) = (\alpha, \beta)$. This implies that $(\alpha, \beta - v) \in M$. Note that for any $(q, w) \in \partial D_L \cap (\partial B_1 \times B_2)$, the line

$$(2.3) \quad \ell := \{(q, w) + s(\alpha, \beta - v) : s \in \mathbb{R}\}$$

does not intersect D .



Consider the open set $U = \mathbb{C}^{N_1} \times B_2$ in $\mathbb{C}^N$. Just by the fact that linear maps are open mappings onto *their range*, we have that $\pi_1(U \cap L)$ is an open subset of $L_1 := \pi_1(L)$. So, $\pi_1(U \cap L) = V \cap L_1$, where V is an open subset of $\mathbb{C}^{N_1}$. Note that

$$\pi_1(L \cap (\partial B_1 \times B_2)) = \pi_1(\partial D \cap L \cap U) \subset \pi_1(\partial D \cap U) \cap \pi_1(L \cap U) = \partial B_1 \cap V \cap L_1.$$

We want to now show that the containment in the above is not strict i.e., $\pi_1(L \cap (\partial B_1 \times B_2)) = \partial B_1 \cap V \cap L_1$. To prove this, let $(z, 0) \in \partial B_1 \cap \pi_1(L \cap U)$. As $(z, 0) \in \pi_1(L \cap U)$, there exists $(z, w) \in U_L := U \cap L$ such that $\pi_1(z, w) = (z, 0)$. Since $(z, w) \in U_L$, we have $w \in B_2$. Hence, we obtain that $(z, w) \in L \cap (\partial B_1 \times B_2)$. This shows that $\pi_1(L \cap (\partial B_1 \times B_2)) = \partial B_1 \cap V \cap L_1$.

Recalling from equation (2.3) that ℓ does not intersect D for any $(q, w) \in \partial D_L \cap (\partial B_1 \times B_2)$. Since $w \in B_2$, the line $\pi_1(\ell)$ does not intersect B_1 for every $q \in \partial B_1$ with the property that $(q, w) \in (\partial B_1 \times B_2) \cap \partial D_L$. Let $V' := \partial B_1 \cap V \cap L_1$, which is a non-empty open subset of $\partial(B_1)_{L_1}$. Since $\pi_1(L \cap (\partial B_1 \times B_2)) = \partial B_1 \cap V \cap L_1$, for every $q \in V'$, the line $\pi_1(\ell)$ (the line passing through q in the direction of α) does not intersect $(B_1)_{L_1}$. Hence, we obtain that for each $q \in V'$, α is a supporting vector for $\partial(B_1)_{L_1}$ at q . Note that $\alpha \neq 0$ because $\alpha \in \partial B_1$. Since $(B_1)_{L_1}$ is strictly convex, it follows from lemma 2.6 that $A := \{p \in \partial(B_1)_{L_1} : k \text{ is a support vector for } \partial(B_1)_{L_1} \text{ at } p\}$ is nowhere dense in $\partial(B_1)_{L_1}$. This contradicts the fact that V' is a non-empty open set in $\partial(B_1)_{L_1}$.

Subcase-2: ∂D_L intersects $B_1 \times \partial B_2$. We skip the proof as it is similar to the above subcase-1.

This completes the proof in all cases that the kernel must be a product: $M = M_1 \times M_2$, as asserted in part-(i) of the lemma.

Remark 2.18. The argument in Subcase-1 depends only on the strict convexity of B_1 , whereas that in Subcase-2 depends only on the strict convexity of B_2 . This observation will be used later in the proof of lemma 2.23.

We now proceed to prove part (ii) of the lemma which in turn has several parts, all involving the subspaces Σ_1, Σ_2 introduced in the statement of (ii). As the proofs are similar for Σ_1 and Σ_2 , we have only to detail the proof for Σ_1 .

Proof of (ii)(a): Observe that $L^{(1)} \cap L^{(2)} = \{0\}$ and $L^{(1)} + L^{(2)} \subset L$. Now, we want to show that $L = L^{(1)} \oplus L^{(2)}$. Note that

$$\dim(L^{(1)}) = \dim(\mathbb{C}^{N_1} \times M_2) + \dim(L) - \dim((\mathbb{C}^{N_1} \times M_2) + L).$$

Since $(\mathbb{C}^{N_1} \times M_2) + L = \mathbb{C}^N$ by the hypotheses that $M = M_1 \times M_2$, we have

$$(2.4) \quad \dim(L^{(1)}) = N_1 + \dim(M_2) + \dim(L) - N_1 - N_2 = \dim(M_2) + \dim(L) - N_2.$$

Similarly, we obtain that

$$(2.5) \quad \dim(L^{(2)}) = \dim(M_1) + \dim(L) - N_1.$$

Adding the equations (2.4) and (2.5), we have

$$\dim(L^{(1)}) + \dim(L^{(2)}) = \dim(M_1) + \dim(M_2) + 2\dim(L) - N.$$

Since $M = M_1 \times M_2$, we have $\dim(M) = \dim(M_1) + \dim(M_2) = N - \dim(L)$. Therefore, $\dim(L^{(1)}) + \dim(L^{(2)}) = \dim(L)$, thereby we obtain that $L^{(1)} \oplus L^{(2)} = L$.

To prove $\Sigma_1 \oplus M_1 = \mathbb{C}^{N_1}$, let $(z, 0) \in \Sigma_1$. By the definition of Σ_1 , there exists $m_2 \in M_2$ such that $(z, m_2) \in L$. If $(z, 0) \in M_1$ then by the hypotheses that $M = M_1 \times M_2$, we have $(z, m_2) \in L \cap M = \{0\}$. This shows that $\Sigma_1 \cap M_1 = \{0\}$. Let z be an arbitrary point in $\mathbb{C}^{N_1}$. Note that $\pi^L(z, 0) = (z + m_1, m_2)$ for some $(m_1, m_2) \in M$. Hence,

$$(z, 0) = (z + m_1, 0) + (-m_1, 0) = \pi_1(z + m_1, m_2) + (-m_1, 0) \in \Sigma_1 + M_1.$$

Therefore, $\Sigma_1 \oplus M_1 = \mathbb{C}^{N_1}$.

Proof of (ii)(b): Firstly note that π^L is injective on Σ_1 and Σ_2 because by (ii)(a) of this lemma, we have $\Sigma_1 \cap M_1 = \{0\} = \Sigma_2 \cap M_2$. Note that $\pi^L(\Sigma_1) = L^{(1)}$ and $(\pi_1)|_{L^{(1)}}$ is the inverse of $\pi^L|_{\Sigma_1} : \Sigma_1 \rightarrow L^{(1)}$ because if (z, m) is an arbitrary point in $L^{(1)}$, then it follows from the hypotheses $M = M_1 \times M_2$ that

$$(2.6) \quad \pi^L(z, 0) = \pi^L(z, 0) + \pi^L(0, m) = \pi^L(z, m) = (z, m).$$

Similarly, we obtain that $(\pi_2)|_{L^{(2)}}$ is the inverse of $\pi^L|_{\Sigma_2} : \Sigma_2 \rightarrow L^{(2)}$. Now, we have to show that $(B_1)_{\Sigma_1} := \Sigma_1 \cap B_1$ is a linear retract of B_1 . To prove this, consider the linear map $T : \mathbb{C}^{N_1} \rightarrow \mathbb{C}^{N_1}$ defined by $T(z) = \pi_1 \circ \pi^L(z)$. Note that T maps B_1 into $(B_1)_{\Sigma_1}$ and it follows from (2.6) that T fixes Σ_1 . This proves that $(B_1)_{\Sigma_1}$ is a retract of B_1 .

Proof of (ii)(c): As π_1 maps $L^{(1)}$ biholomorphically onto its range Σ_1 , it follows that the inverse $((\pi_1)|_{L^{(1)}})^{-1}$ can be used to parametrize $L^{(1)}$ as the graph of a linear map over the linear subspace Σ_1 as

$$L^{(1)} = \{(z, \varphi^{(1)}(z)) : z \in \Sigma_1\}.$$

By part (b) of this lemma, $((\pi_1)|_{L^{(1)}})^{-1} = \pi^L|_{\Sigma_1}$. Hence, $(z, \varphi^{(1)}(z)) = \pi^L(z, 0)$ for all $(z, 0) \in \Sigma_1$ and thereby $\text{image}(\varphi^{(1)}) \subset M_2$. Now, we want to show that $\partial D_{L^{(1)}} \subset \partial B_1 \times \overline{B_2}$. To prove this by contradiction, assume that there exists $(z, w) \in \partial D_{L^{(1)}} \cap (B_1 \times \partial B_2)$. As $(z, w) \in L^{(1)}$, we have $w \in M_2$. Hence, we obtain that

$$\pi^L(z, 0) = \pi^L(z, w) = (z, w) \in B_1 \times \partial B_2.$$

This contradicts the fact that π^L maps D into D . Therefore, $\partial D_{L^{(1)}} \subset \partial B_1 \times \overline{B_2}$. It follows that $D_{L^{(1)}}$ as the graph of a linear map over the linear retract $(B_1)_{\Sigma_1}$ as

$$D_{L^{(1)}} = \{(z, \varphi^{(1)}(z)) : z \in (B_1)_{\Sigma_1}\}.$$

Note that the linear map $z \mapsto \varphi^{(1)}(z)$ maps $(B_1)_{\Sigma_1}$ into $B_2 \cap M_2$. Similarly, we obtain that $L^{(2)}$ as the graph of a linear map over the linear subspace Σ_2 as

$$L^{(2)} = \{(\varphi^{(2)}(w), w) : w \in \Sigma_2\},$$

where $(\varphi^{(2)}(w), w) = \pi^L(0, w)$ for all $(0, w) \in \Sigma_2$ and $\text{image}(\varphi^{(2)}) \subset M_1$.

Proof of (ii)(d): Consider the linear map $T : \Sigma_1 \times \Sigma_2 \rightarrow M_1 \times M_2$ defined by $T(z, w) = (\varphi^{(2)}(w), \varphi^{(1)}(z))$. By (ii)(a), we have $\mathbb{C}^N = \Sigma \oplus M$ and $L = L^{(1)} \oplus L^{(2)}$, where $\Sigma = \Sigma_1 \oplus \Sigma_2$. Hence

$$L = \{(z, w) + T(z, w) : (z, w) \in \Sigma_1 \times \Sigma_2\}$$

is the graph of the linear map T over the linear subspace $\Sigma_1 \times \Sigma_2$. If $(z, w) \in (B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$ then, using the fact that π^L maps D into D , we obtain that

$$(2.7) \quad \pi^L(z, w) = \pi^L(z, 0) + \pi^L(0, w) = (z, w) + T(z, w) \in B_1 \times B_2.$$

This shows that $\{(z, w) + T(z, w) : (z, w) \in (B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}\} \subset D_L$. As we already noted that $(B_1)_{\Sigma_1}$ is a linear retract of B_1 by using the linear projection map $S(z) := \pi_1 \circ \pi^L(z)$. Since $\partial D_{L(1)} \subset \partial B_1 \times \overline{B_2}$, $S(z, 0) = 0$ implies that $z \in M_1$. Therefore, $\ker(S) = M_1$. Let $\|\cdot\|_i$ denotes the norm associated to B_i for $i = 1, 2$. Note that for every $(z, w) \in \Sigma_1 \times \Sigma_2$, we have

$$\|z\|_1 = \|S(z)\|_1 = \|S(z + \varphi^{(2)}(w))\|_1 \leq \|z + \varphi^{(2)}(w)\|_1.$$

Hence, we obtain that if $\|z + \varphi^{(2)}(w)\|_1 < 1$ then $\|z\|_1 < 1$ and thereby $z \in B_1$. Similarly, if $\|w + \varphi^{(1)}(z)\|_2 < 1$ then $w \in B_2$. This shows that if $(z + \varphi^{(2)}(w), w + \varphi^{(1)}(z)) \in D_L$ then $(z, w) \in (B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$. Thereby, $D_L = \{(z, w) + T(z, w) : (z, w) \in (B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}\}$, which is the graph of the linear map $T = \pi^L - I$ over $(B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$. From (2.7), we obtain that the image of $(B_1)_{\Sigma_1} \times (B_2)_{\Sigma_2}$ under the linear map T is contained in $(B_1)_{M_1} \times (B_2)_{M_2}$. $\square$

Lemma 2.19. *Let B_1, B_2 be bounded balanced convex domains in $\mathbb{C}^{N_1}, \mathbb{C}^{N_2}$ respectively. Let $D = B_1 \times B_2$ and $Z := \rho(D)$ be a non-trivial retract of D passing through the origin, where ρ is a retraction map from D onto Z . Assume that $\partial D_L \subset \partial B_1 \times \partial B_2$, where $L := T_0 Z$. Define $K_j := \ker(D\rho|_0) \cap \mathbb{C}^{N_j}$, $L_j = \pi_j(L)$ for $j = 1, 2$ and let $(p, q) \in \partial D_L$. Then either K_1 is supporting for ∂B_1 at $p \in \partial(B_1)_{L_1}$ or K_2 is supporting for ∂B_2 at $q \in \partial(B_2)_{L_2}$.*

Proof. To prove this lemma by contradiction, assume that there exist non-zero vectors $k_1 \in K_1$ and $k_2 \in K_2$ such that k_1 is not supporting for ∂B_1 at p and k_2 is not supporting for ∂B_2 at q . Since B_1, B_2 are convex, there exists $\epsilon > 0$ such that $p + tk'_1 \in B_1$ and $q + tk'_2 \in B_2$ for all $t \in (0, \epsilon)$, where $k'_1 = \pm k_1$ and $k'_2 = \pm k_2$. Hence $\{(p, q) + t(k'_1, k'_2) : t \in \mathbb{R}\}$ intersects D . This contradicts the fact that $K_1 \times K_2$ is contained in the kernel of a linear retraction map $D\rho|_0$. $\square$

Proposition 2.20. *Let B_1, B_2 be bounded balanced convex domains in $\mathbb{C}^{N_1}, \mathbb{C}^{N_2}$ respectively. Let $D = B_1 \times B_2$ and $Z := \rho(D)$ be a non-trivial retract of D passing through the origin, where ρ is a retraction map from D onto Z . Assume that one of the balls, say B_1 , is strictly convex and that $\partial D_L \subset \partial B_1 \times \partial B_2$, where $L := T_0 Z$. Define $K_j := \ker(D\rho|_0) \cap \mathbb{C}^{N_j}$, $L_j = \pi_j(L)$ for $j = 1, 2$. Then either $L_1 \cap K_1 = \{0\}$ or $L_2 \cap K_2 = \{0\}$.*

Proof. To prove this proposition by contradiction, assume that there exist non-zero vectors k_1, k_2 such that $k_1 \in L_1 \cap K_1$ and $k_2 \in L_2 \cap K_2$. Then there exists $p_0 \in L_1$ and $q_0 \in L_2$ such that (p_0, k_2) and (k_1, q_0) belongs to L . After suitably dilating the vector (p_0, k_2) and (k_1, q_0) , we may assume that both (p_0, k_2) and (k_1, q_0) belongs to ∂D_L . It follows from lemma 2.19 that K_1 is supporting for ∂B_1 at p_0 and K_2 is supporting for ∂B_2 at q_0 . For each $j = 1, 2$, let

$$A_j(k_j) := \{p_j \in \partial(B_j)_{L_j} : k_j \text{ is supporting for } \partial B_j \text{ at } p_j\}.$$

and

$$\tilde{A}_j(k_j) := \{p_j \in \partial(B_j)_{L_j} : k_j \text{ is supporting for } \partial(B_j)_{L_j} \text{ at } p_j\}.$$

By lemma 2.5, $A_j(k_j)$ is a closed subset of $\partial(B_j)_{L_j}$ for $j = 1, 2$. Now, we want to show that $A_1(k_1)$ is a nowhere dense subset of $\partial(B_1)_{L_1}$. Note that $A_1(k_1) \subset \tilde{A}_1(k_1)$ because

every supporting vector for ∂B_1 at $p_1 \in \partial(B_1)_{L_1}$ is also a supporting vector for $\partial(B_1)_{L_1}$ at p_1 . Since $(B_1)_{L_1}$ is strictly convex, lemma 2.6 (upon identifying L_1 with $\mathbb{R}^{2d}$) implies that $\hat{A}_1(k_1)$ is nowhere dense in $\partial(B_1)_{L_1}$. Since $A_1(k_1) \subset \hat{A}_1(k_1)$, we obtain that $A_1(k_1)$ is nowhere dense in $\partial(B_1)_{L_1}$. For $j = 1, 2$, let

$$\hat{A}_j := \{p_j \in \partial(B_j)_{L_j} : K_j \text{ is supporting for } \partial B_j \text{ at } p_j\}$$

Since K_1 is supporting for ∂B_1 at p_0 and K_2 is supporting for ∂B_2 at q_0 , $\hat{A}_1$ and $\hat{A}_2$ are non-empty subsets of $\partial(B_1)_{L_1}$ and $\partial(B_2)_{L_2}$ respectively. Note that $\hat{A}_j = \bigcap_{k \in K_j} A_j(k)$ for $j = 1, 2$. Since $A_j(k)$ is a closed subset of $\partial(B_j)_{L_j}$ for each $k \in K_j$, $\hat{A}_j$ is a closed subset of $\partial(B_j)_{L_j}$. As $\hat{A}_1 \subset A_1(k_1)$, it follows from the fact that $A_1(k_1)$ is a nowhere dense subset of $\partial(B_1)_{L_1}$ that, $\hat{A}_1$ is also a nowhere dense subset of $\partial(B_1)_{L_1}$.

Since $\partial D_L \subset \partial B_1 \times \partial B_2$, π_j is an invertible linear map from L onto L_j for $j = 1, 2$. Hence, we can define an another invertible linear map $T : L_1 \rightarrow L_2$ by $T(z) = \pi_2 \circ ((\pi_1)|_L)^{-1}$. Note that T is invertible on L_1 and T maps $\partial(B_1)_{L_1}$ onto $\partial(B_2)_{L_2}$. Since $\hat{A}_1$ is nowhere dense in $\partial(B_1)_{L_1}$ and $\partial(B_1)_{L_1} = \hat{A}_1 \cup T^{-1}(\hat{A}_2)$, $T^{-1}(\hat{A}_2)$ is a dense subset of $\partial(B_1)_{L_1}$. As $\hat{A}_2$ is closed in $\partial(B_2)_{L_2}$, $T^{-1}(\hat{A}_2) = \partial(B_1)_{L_1}$. This implies that K_2 is supporting for ∂B_2 at every boundary point $q \in \partial(B_2)_{L_2}$. This leads to a contradiction: in particular, K_2 would support at k_2 , since $\partial D_L \subset \partial B_1 \times \partial B_2$. $\square$

We turn now to the proof of theorem 1.1. First, a few words about notations about retractions ρ specific to this theorem. When written out explicitly, ρ has $N_1 + N_2$ components: $(\rho_1, \rho_2, \dots, \rho_{N_1}, \rho_{N_1+1}, \dots, \rho_{N_1+N_2})$ where each of the ρ_j 's for $j = 1, \dots, N_1 + N_2$ can apriori depend on all of the variables $z_1, \dots, z_{N_1}, w_1, \dots, w_{N_2}$; so, we use $z = (z_1, \dots, z_{N_1})$ as our notation for the coordinates on $\mathbb{C}^{N_1} \supset B_1$ and $w = (w_1, \dots, w_{N_2})$ for that on $\mathbb{C}^{N_2} \supset B_2$. We denote the norms with respect to which B_1, B_2 are unit balls, by $\|\cdot\|_1, \|\cdot\|_2$ respectively.

Note that as our non-trivial retract Z is necessarily a non-compact subset of $B_1 \times B_2$, it will therefore meet the boundary of D i.e., $\bar{Z} \cap \partial(B_1 \times B_2) \neq \emptyset$ where it may noted that

$$\partial(B_1 \times B_2) = (\partial B_1 \times B_2) \cup (B_1 \times \partial B_2) \cup (\partial B_1 \times \partial B_2).$$

However, rather than this disjoint decomposition of the boundary, we shall for the most part work with a slightly different decomposition in which the pieces are not disjoint but are closed (in $\mathbb{C}^{N_1+N_2}$); essentially,

$$(2.8) \quad \partial(B_1 \times B_2) = (\partial B_1 \times \bar{B}_2) \cup (\bar{B}_1 \times \partial B_2).$$

Moreover, we shall break up our discussion depending on where L (rather than Z) meets the boundary of $D := B_1 \times B_2$ as in cases (i) – (iv) in the statement of theorem 1.1. In each of these cases, we will need to use the fact that $\rho(D_L) = Z$; obviously $\rho(D_L) \subset Z$, so the equality is what needs justification. This is not difficult, if we invoke the identity principle for complex analytic sets (as in section 5.6 of the first chapter of the book [10] by Chirka, for instance), once we have ascertained beforehand that $\rho(D_L)$ is a submanifold, which in turn follows in each of the aforementioned cases as we show that $\rho(D_L)$ is a graph over D_L . As we shall use this argument multiple times (tacitly) in the sequel, we are mentioning it here. Borrowing an idea from the proof of the first main theorem of the

work [11] by Abate – Vigue, we consider the map $\pi^L \circ \rho$, which we observe to be a self-map of the ball D_L fixing the origin. Its derivative at the origin is computed by chain-rule as

$$D(\pi^L \circ \rho)|_0 = \pi^L \circ \pi^L = \pi^L,$$

which when restricted to L is just the identity map. A well-known uniqueness principle due to H. Cartan forces $\pi^L \circ \rho$ itself to be the identity map finishing the proof that $\rho(D_L)$ is a graph over D_L ; thereby due to the aforementioned identity principle that $\rho(D_L) = Z$. With this preparation, we can now deal with the four cases to finish the proof.

Case-(i): $\partial D_L \subset \partial B_1 \times \partial B_2$. In this case, both the projections $\pi_1 : D_L \rightarrow B_1$ and $\pi_2 : D_L \rightarrow B_2$ are proper. Indeed, if $\{(z^k, w^k) : k \in \mathbb{N}\}$ is a sequence in D_L accumulating only on its boundary, then by definition of the case-(i) that we are in now, $z^k \rightarrow \partial B_1$ and $w^k \rightarrow \partial B_2$ i.e., $\pi_1(z^k, w^k) \rightarrow \partial B_1$ and $\pi_2(z^k, w^k) \rightarrow \partial B_2$.

Next, note that $\dim(D_L) = \dim(L) \leq \min\{N_1, N_2\} = N_1$ since proper holomorphic maps cannot decrease dimension and π_1 is a proper holomorphic map carrying D_L into the N_1 -dimensional B_1 . Infact, as D_L is just a linear subvariety of D , the properness of π_1 means also that the fibres of π_1 are actually singletons; so, π_1 maps D_L biholomorphically (indeed linearly) onto its range in B_1 and the inverse of this map can be used to parametrize D_L as the graph of a linear map over the linear subspace $\pi_1(D_L) \subset B_1$ as:

$$(2.9) \quad D_L = \{(z, \beta_1(z), \dots, \beta_{N_2}(z)) : z \in \pi_1(D_L)\}.$$

for some $\mathbb{C}$ -linear functionals $\beta_1, \dots, \beta_{N_2}$ on the complex linear subspace $\pi_1(L) \subset \mathbb{C}^{N_1}$. To tie this up with our goal of describing Z , we shall now show that $Z = D_L$! Indeed to this end, pick any point $(z^0, w^0) \in \partial D_L \subset \partial B_1 \times \partial B_2$ and consider the complex line segment l through it i.e., l is the image of the map $\varphi(t)$ defined for $t \in \Delta$, by

$$\varphi(t) = (z^0 t, w^0 t) = (z^0, w^0)t = t(z^0, w^0),$$

where we have written out $\varphi(t)$ in various notations only to indicate once of the alternative usages (that may be more natural in certain contexts to come later). Next compose this parametrization of l with our retraction ρ :

$$t \longmapsto (\rho_1(tz^0, tw^0), \dots, \rho_{N_1+N_2}(tz^0, tw^0)).$$

This is of course, the restriction of the retraction ρ to l and we shall now analyse it componentwise. More precisely, let

$$g_1(t) = (\pi_1 \circ \rho \circ \varphi)(t) = (\rho_1(tz^0, tw^0), \dots, \rho_{N_1}(tz^0, tw^0))$$

and

$$g_2(t) = (\pi_2 \circ \rho \circ \varphi)(t) = (\rho_{N_1+1}(tz^0, tw^0), \dots, \rho_{N_1+N_2}(tz^0, tw^0)).$$

So g_1, g_2 are holomorphic maps from the unit disc into the balls B_1, B_2 respectively, mapping $0 \in \Delta$ to the origin in the respective balls. Since B_1, B_2 are strictly convex, we may apply the Schwarz lemma of Mazet recalled above as lemma 2.10, provided only that we verify that the derivatives of these maps at the origin satisfy the extremality condition as required. To do this, let us begin with the map g_1 whose derivative we compute using the chain-rule as:

$$dg_1(0) = d\pi_1 \circ d\rho|_{\varphi(0)} \circ d\varphi(0).$$

Then, using the facts that $d\pi_1 \equiv \pi_1$ (as the projection π_1 is just linear), φ maps $0 \in \Delta$ to the origin in B_1 which in turn is fixed by ρ , and that $d\rho|_0$ is just the identity map on T_0Z and that $(z^0, w^0) = d\varphi(0) \in T_0Z$, we get $dg_1(0) = z^0 \in \partial B_1$ which is a $\mathbb{R}$ -extremal boundary point by hypothesis. The aforementioned theorem applied to g_1 shows that that $\pi_1 \circ \rho(tz^0, tw^0) = tz^0$, which means that $\rho^1 := \pi_1 \circ \rho$ acts like the projection onto the z -axes on D_L :

$$\rho^1|_{D_L}(z, w) = z.$$

The same considerations then applied to the other component g_2 gives likewise for all $t \in \Delta$ that $\rho^2(tz^0, tw^0) = tw^0$ where $\rho^2 = \pi_2 \circ \rho$. In other words, ρ^2 also acts as a projection, this time onto the w -space:

$$\rho^2|_{D_L}(z, w) = w.$$

Putting this together with the similar conclusion made about ρ^1 , this just means that ρ acts as the identity mapping on l . But this argument may be carried out for *every* one-dimensional linear subspace of D_L , as we are in the case that ∂D_L is completely contained within $\partial B_1 \times \partial B_2$. It therefore follows that ρ must act as the identity mapping on D_L and thereby $D_L \subset Z$. By the identity principle for complex analytic sets (as mentioned earlier), we obtain that $D_L = Z$. Thus, Z in the present case (Case-(i)) must be of the form

$$Z = \{(z, \beta_1(z), \dots, \beta_{N_2}(z)) : z \in \pi_1(D_L)\}.$$

Note that the boundary of $Z = D_L$ (what we have denoted by ∂D_L), lies in $\partial B_1 \times \partial B_2$ means for the functionals β_j 's appearing in the above description of Z , is that the linear map represented by their collective (whose graph is Z), namely

$$(2.10) \quad z \longmapsto (\beta_1(z), \dots, \beta_{N_2}(z))$$

maps $B_1^L := \pi_1(D_L)$ into B_2 .

We may phrase the last observation as saying that the map (2.10) is an isometry from the Banach space $(\pi_1(L), \|\cdot\|_1)$ into the Banach space $(\mathbb{C}^{N_2}, \|\cdot\|_2)$ and therefore maps B_1^L which is the unit ball in $\pi_1(L)$ with respect to the norm $\|\cdot\|_1$ isometrically into B_2 ; the mapping is surjective if $N_1 = N_2$ and $\pi_1(L) = \mathbb{C}^N$ where $N = N_1 = N_2$. Hence, we see that this places strong restrictions about when this Case-(i) can happen. For instance, this case cannot occur if $N_1 = N_2 = N > 1$ and $B_1 = \Delta^N$ is the standard unit polydisc and $B_2 = \mathbb{B}$, the standard ball with respect to the l^2 -norm in $\mathbb{C}^N$ and $\dim(D_L) = N$; or more generally, case-(i) cannot arise as soon as $\pi_1(D_L)$ is not biholomorphic to any of the linear sections of B_2 . But then again, one must not be amiss to note that case-(i) does happen at least once regardless of what particular balls B_1, B_2 are, namely when $\dim(D_L) = 1$; indeed, the complex line segment spanned by every point in $\partial B_1 \times \partial B_2$ is a retract.

Finally, we want to show that D_L is a graph of a linear map over a linear *retract* of B_1 or B_2 . We already noted that D_L is a graph of some linear map over $(B_1)_{L_1}$ or $(B_2)_{L_2}$. So, we have only to show that either $(B_1)_{L_1}$ is a linear retract of B_1 or $(B_2)_{L_2}$ is a linear retract of B_2 (or both). By the above proposition 2.20, either $L_1 \cap K_1 = \{0\}$ or $L_2 \cap K_2 = \{0\}$ (possibly both). Now, we can divide the proof into two subcases: (a) $L_1 \cap K_1 \neq \{0\}$ or

$L_2 \cap K_2 \neq \{0\}$ and (b) $L_1 \cap K_1 = \{0\}$ and $L_2 \cap K_2 = \{0\}$.

Subcase-(a): $L_1 \cap K_1 \neq \{0\}$ or $L_2 \cap K_2 \neq \{0\}$. Without loss of generality, we may assume that $L_1 \cap K_1 = \{0\}$ and $L_2 \cap K_2 \neq \{0\}$. By the dimension formula for the intersection of a pair of subspaces, we have

$$(2.11) \quad \dim(L_1 + K_1) = \dim(L_1) + \dim(K_1) - \dim(L_1 \cap K_1) = d + \dim(K_1).$$

One more application of the same formula, this time phrased for a sum of subspaces, we have

$$\dim(K_1) = \dim(M \cap \mathbb{C}^{N_1}) = \dim(M) + N_1 - \dim(M + \mathbb{C}^{N_1}).$$

Since $\dim(M + \mathbb{C}^{N_1}) \leq N$ and $\dim(M) = N - d$, we have $\dim(K_1) \geq N_1 - d$. This shows that $d + \dim(K_1) \geq N_1$. From (2.11), we obtain that $\dim(L_1 + K_1) \geq N_1$. Since $L_1 + K_1 \subset \mathbb{C}^{N_1}$, we have $L_1 \oplus K_1 = \mathbb{C}^{N_1}$. For $j = 1, 2$, let

$$\widehat{A}_j := \{p_j \in \partial(B_j)_{L_j} : K_j \text{ is supporting for } \partial B_j \text{ at } p_j\}$$

By using the arguments in the proof of the proposition 2.20, it follows that $\widehat{A}_2$ is a nowhere dense subset of $\partial(B_2)_{L_2}$ and thereby we get that K_1 is supporting for ∂B_1 at every $q \in \partial(B_1)_{L_1}$. It now follows by proposition 2.9 that $(B_1)_{L_1}$ is a linear retract of B_1 which suffices to yield the assertion of theorem 1.1.

Subcase-(b): This is the case where $L_1 \cap K_1 = \{0\}$ as well as $L_2 \cap K_2 = \{0\}$. We claim that K_j is a supporting subspace for ∂B_j at all points in $\partial(B_j)_{L_j}$ for both $j = 1, 2$; it suffices to prove for $j = 1$. To prove this claim by contradiction, assume that there exists a point $p \in \partial(B_1)_{L_1}$ and a (non-zero) $k_1 \in K_1$ such that k_1 is not a supporting vector for ∂B_1 at $p \in \partial(B_1)_{L_1}$. This means (after replacing k_1 by its negative if needed), that:

$$p + \alpha k_1 \in B_1 \text{ for all sufficiently small } \alpha < 0.$$

Since $p \in \partial(B_1)_{L_1}$, we have a (uniquely determined) point $q \in \partial(B_2)_{L_2}$ such that $(p, q) \in \partial D_L \subset \partial B_1 \times \partial B_2$. Let $K^{gr} := (L_1 \times L_2) \cap M$. Note that $K^{gr} \cap (K_1 \oplus K_2) = \{0\}$ because by assumption in this subcase that $L_1 \cap K_1 = \{0\}$ and $L_2 \cap K_2 = \{0\}$. Since $L \oplus M = \mathbb{C}^N$, we have

$$\dim(K^{gr}) = \dim(L_1 \times L_2) + \dim(M) - \dim((L_1 \times L_2) + M) = 2d + N - d - N = d.$$

Hence, we have

$$\dim(K_1) + \dim(K_2) + \dim(K^{gr}) = (N_1 - d) + (N_2 - d) + d = N - d.$$

Therefore, we obtain that $M = K_1 \oplus K_2 \oplus K^{gr}$. Note that π_1, π_2 are injective on the subspace K^{gr} because $L_1 \cap K_1 = \{0\}$ and $L_2 \cap K_2 = \{0\}$. So, K^{gr} is indeed a linear graph over each of L_1 and L_2 . Hence there exists $\tilde{k}_1 \in \pi_1(K^{gr}) = L_1$ such that $(\tilde{k}_1, q) \in K^{gr}$. Next, we consider the vector given by

$$(k_1 + \lambda \tilde{k}_1, \lambda q) = (k_1, 0) + \lambda(\tilde{k}_1, q),$$

where λ is a positive scalar yet to be chosen. Note that $(k_1 + \lambda \tilde{k}_1, \lambda q) \in M$. As $(p, q) \in \partial D_L$, the line

$$(2.12) \quad \ell := \{(p, q) + t(k_1 + \lambda \tilde{k}_1, \lambda q) : t \in \mathbb{R}\}$$

does not intersect the polyball $D = B_1 \times B_2$. Write

$$(p, q) + t(k_1 + \lambda \tilde{k}_1, \lambda q) = (p + t(k_1 + \lambda \tilde{k}_1), (1 + \lambda t)q).$$

Now, note that for all sufficiently small $t < 0$,

$$(1 + \lambda t)q \in B_2 \text{ if and only if } \|(1 + \lambda t)q\|_2 < 1.$$

Let $\mathcal{F}_p$ denote the family of functions on $\mathbb{R}^{2N}$ of the form $f_p(x) = \langle x, v_p \rangle$, where v_p is one of the outward unit normal vectors at p . Note that $|f_p(x)| \leq \|x\|_{\ell^2} \|v_p\|_{\ell^2}$, where $\|\cdot\|_{\ell^2}$ denotes the ℓ^2 norm on $\mathbb{R}^{2N}$ (which is identified with $\mathbb{C}^N$ as real vector spaces as usual). Since the normal cone $N(p, \partial B_1)$ is a closed set, the set of all outward unit normal vectors at p forms a compact set. Hence there exists $\kappa > 0$ such that $\|v_p\|_{\ell^2} \leq \kappa$ and thereby every element in $\mathcal{F}_p$ is a Lipschitz function with Lipschitz constant not more than κ . Therefore, we conclude that the family $\mathcal{F}_p$ forms an equicontinuous family of functions. This implies that there exists a neighbourhood U around k_1 such that $f_p(x) > 0$ for all $x \in U$ and for all $f_p \in \mathcal{F}_p$. Choose $\lambda > 0$ such that $k_1 + \lambda \tilde{k}_1 \in U$. Note that $f_p(k_1 + \lambda \tilde{k}_1) > 0$ for all $f_p \in \mathcal{F}_p$. It follows from remark 2.2 that $k_1 + \lambda \tilde{k}_1$ is not a supporting vector at p with $p + t(k_1 + \lambda \tilde{k}_1) \in B_1$ for all sufficiently small $t < 0$. This contradicts the fact that ℓ in (2.12) does not intersect D , which finishes the proof of the claim that K_1 is a supporting subspace for ∂B_1 at all points in $\partial(B_1)_{L_1}$. As argued for subcase-(a), $L_1 + K_1 = \mathbb{C}^{N_1}$ owing to the assumption that $L_1 \cap K_1 = \{0\}$. It now follows by proposition 2.9 that $(B_1)_{L_1}$ is a linear retract of B_1 .

Remark 2.21. In the above case-(i), we can find a retraction map $\hat{\rho}$ on $\mathbb{C}^N$ such that $M := \ker(\rho)$ is a non-trivial product of linear subspaces. To see this, without loss of generality, we assume that $L_1 \cap K_1 = \{0\}$. By proposition 2.9, we obtain that there exists a linear projection map $\tilde{\rho}$ on $\mathbb{C}^{N_1}$ such that $\tilde{\rho}(B_1) = (B_1)_{L_1}$. Consider the linear map $\hat{\rho}$ on $\mathbb{C}^N$ defined by $\hat{\rho}(z, w) = (\pi_1)_L^{-1} \circ \tilde{\rho}(z, 0)$. Note that $\hat{\rho}$ is a retraction map on D with $\ker(\hat{\rho}) = K_1 \times \mathbb{C}^{N_2}$.

Case-(ii): $\partial D_L \subset \partial B_1 \times \overline{B_2}$ but $\partial D_L \not\subset \partial B_1 \times \partial B_2$. In this case, note that the projection $\pi_1 : D_L \rightarrow B_1$ is proper and again the general fact that proper holomorphic maps cannot decrease dimension gives $\dim D_L \leq N_1$. Noting that the arguments in case-(i) leading to (2.9), relied only on the properness of π_1 and not that of π_2 , we see that the same arguments apply in this case as well to show again that D_L must be of the form

$$D_L = \{(z, \beta_1(z), \dots, \beta_{N_2}(z)) : z \in \pi_1(D_L)\},$$

for some $\mathbb{C}$ -linear functionals $\beta_1, \dots, \beta_{N_2}$ on the complex linear subspace $\pi_1(L) \subset \mathbb{C}^{N_1}$; again, the fact that $\overline{D_L}$ lies in $\overline{B_1} \times \overline{B_2}$ implies that the linear transformation given by

$$T_\beta(z) = (\beta_1(z), \dots, \beta_{N_2}(z))$$

maps $B_1^L := \pi_1(D_L)$ into B_2 ; this may be phrased as: T is an isometry from the Banach space $(\pi_1(L), \|\cdot\|_1)$ into $(\mathbb{C}^{N_2}, \|\cdot\|_2)$. Picking a point $(z^0, w^0) \in \partial D_L$, considering the complex line segment joining the origin to this point and the restriction of the retraction of ρ to this segment, we get by the arguments as in case-(i) by applying the Schwarz lemma of Mazet that $\rho^1|_{D_L}(z, w) = z$ where $\rho^1 = \pi_1 \circ \rho$. As points of D_L are of the form $(z, T_\beta(z))$ for $z \in \pi_1(D_L)$, we have for all $z \in \pi_1(D_L)$ that

$$(2.13) \quad \rho(z, T_\beta(z)) = (z, \rho^2(z, T_\beta(z)))$$

where $\rho^2 = \pi_2 \circ \rho$; thereby, it follows that $\rho(D_L)$ is of the form

$$\rho(D_L) = \{(z, \rho^2(z, T_\beta(z))) : z \in \pi_1(D_L)\}.$$

This means that $\rho(D_L)$ is the graph of a $\mathbb{C}^{N_2}$ -valued holomorphic function over $B_1^L := \pi_1(D_L)$, more precisely the graph of the B_2 -valued holomorphic map given by

$$z \rightarrow (\rho_{N_1+1}(z, T_\beta(z)), \dots, \rho_{N_1+N_2}(z, T_\beta(z))).$$

for w varying through B_1^L , the unit ‘ball’ of $\pi_1(L)$ with respect to the functional μ_1 . Note that $\pi_1(L) \cap B_1$ is a linear retract of B_1 using lemma 2.17 (ii).

Case-(iii): $\partial D_L \subset \bar{B}_1 \times \partial B_2$ but $\partial D_L \not\subset \partial B_1 \times \partial B_2$. The proof for this case is essentially like Case-(ii), and hence we skip the details.

Case-(iv): This is the final case when ∂D_L intersects both $\Gamma_1 := \partial B_1 \times B_2$ as well as $\Gamma_2 := B_1 \times \partial B_2$. While this does not preclude the possibility that L does not intersect $\Gamma_0 := \partial B_1 \times \partial B_2$, certainly any one dimensional complex linear subspace l which intersects Γ_j for $j = 1$ or 2 , intersects the boundary of D only in Γ_j (note firstly that the Γ_j ’s are disjoint). That is to say, if $D_l = l \cap D$ for some one-dimensional subspace l intersects Γ_j (for $j = 1$ or 2) then ∂D_l is fully contained in Γ_j (for $j = 1$ or 2), as soon as it contains a point from Γ_j . Indeed, if $(z^0, w^0) \in \Gamma_j$ then the complex line l is the two dimensional real linear subspace spanned by (z^0, w^0) and $i \cdot (z^0, w^0) = (iz^0, iw^0)$. Noting that z^0 (resp. w^0) lies in the interior of $\bar{B}_1$ (resp. $\bar{B}_2$) if and only if iz^0 (resp. iw^0) does, verifies the claim that ∂D_l is fully contained within Γ_j (for $j = 1$ or 2) as soon as it intersects Γ_j . As L intersects both Γ_j for $j = 1, 2$ we have in particular that $L = T_0 Z$ cannot equal l and $\dim(D_L) \geq 2$. This also means that for each of Γ_1, Γ_2 there are complex lines in L which intersect it. Moreover, if l is a complex line in D_L which intersects Γ_j (for $j = 1$ or 2) then lines obtained by perturbing the coefficients of linear equations which define L , slightly, also intersect the open piece Γ_j (for $j = 1$ or 2) of the boundary of D and by our forgoing remarks, the boundary of the intercept of each such line (near D_L) with D is fully contained in Γ_j . Observe that the union of these lines contains an open subset U_L of D_L . Next, we apply the Schwarz lemma arguments in Case-(ii) (resp. Case-(iii)) to each line that intersects Γ_1 (resp. Γ_2). As there are lines in D_L which intersect Γ_1 as well as (other) lines in D_L which intersect Γ_2 , we have both the conclusions of these cases about the major pair of components of ρ namely, $\rho^j := \pi_j \circ \rho$ for $j = 1, 2$, which are that ρ^1 just returns the first set of variables on D_L : $\rho^1|_{D_L}(z, w) = z$ and likewise for the second component $\rho^2|_{D_L}(z, w) = w$. Putting together, we therefore see that ρ acts just as the identity map, first atleast on U_L , and thereby on L as well, by the identity principle applied to both the holomorphic maps ρ^1, ρ^2 . As $\dim(D_L) = \dim(Z)$ obviously, this means that $D_L = Z$. Therefore, we conclude in this case that Z is a linear subspace of D (of complex dimension ≥ 2), which is neither contained in B_1 nor B_2 . More importantly, note that by lemma 2.17 (ii), Z can then be written as a direct sum of graphs of linear maps over linear retracts of each of the factors taking values in the other:

$$Z = (L^{(1)} \oplus L^{(2)}) \cap D,$$

where $L^{(j)}$ is a graph over the linear subspace $\Sigma^{(j)} \subset \mathbb{C}^{N_j}$ with $B_j \cap \Sigma^{(j)}$ being a linear retract of B_j for $j = 1, 2$. Again by lemma 2.17 (ii), Z can also be realized as the graph

of the linear map $(D\rho|_0 - I)$ over the product of linear retracts $(B_j)_{\Sigma_j}$ of B_j and taking values in $(B_1)_{M_1} \times (B_2)_{M_2}$. Consequently, Z is biholomorphic to product of the retracts $(B_1)_{\Sigma_1}$ of B_1 and $(B_2)_{\Sigma_2}$ of B_2 . $\square$

Remark 2.22. Note that in case (i) and (iv), Z is the graph of the linear map $D\rho|_0 - I$ over a product of linear retracts of the factor domains B_j , with values in $(B_1)_{M_1} \times (B_2)_{M_2}$, where ρ is a retraction map from D onto Z and $M := \ker(D\rho|_0)$. In cases (ii) and (iii), Z is the graph of the holomorphic map $\rho - \pi_j$ over a linear retract of B_j , also with values in $(B_1)_{M_1} \times (B_2)_{M_2}$, where π_j denotes the standard projection from $\mathbb{C}^N$ onto $\mathbb{C}^{N_j}$.

Lemma 2.23. *Let $B_1, B_2, \dots, B_N$ be a collection of bounded balanced strictly convex domains, possibly in different complex Euclidean spaces (say, $B_j \subset \mathbb{C}^{n_j}$ for $j = 1, \dots, N$). Let Z be a retract of D passing through the origin with $\rho : D \rightarrow Z$, a retraction map. Let $L = T_0 Z$, $D_L = L \cap D$ and $\partial D_L = L \cap \partial D$. Recall the notation introduced in definition 2.15, write $D = B^\tau \times B_N$, where $\tau = (1, \dots, N-1)$. Assume that ∂D_L intersects $\partial B^\tau \times B_N$, $B^\tau \times \partial B_N$ or both. Then $M := \ker(D\rho|_0) = M_\tau \times M_N$, where $M_\tau := \pi_\tau(M)$ and $M_N := \pi_N(M)$. Moreover, D_L is the graph of the linear map $(D\rho|_0 - I)$ over the product of linear retracts $(B^\tau)_{\Sigma_\tau}$ of B^τ and $(B_N)_{\Sigma_N}$ of B_N , taking values in $(B^\tau)_{M_\tau} \times (B_N)_{M_N}$, where Σ_τ and Σ_N are linear subspaces of L_τ and L_N respectively.*

Proof. Let $\pi^L := D\rho|_0$. By the hypotheses that ∂D_L intersects $\partial B^\tau \times B_N$, $B^\tau \times \partial B_N$ or both, we divide the proof into the following three cases:

- (1) $\partial D_L \subset \partial B^\tau \times \overline{B}_N$ and ∂D_L intersects $\partial B^\tau \times B_N$.
- (2) $\partial D_L \subset \overline{B}^\tau \times \partial B_N$ and ∂D_L intersects $B^\tau \times \partial B_N$.
- (3) ∂D_L intersects both $\partial B^\tau \times B_N$ and $B^\tau \times \partial B_N$.

Since B_N is strictly convex, the same argument as in the proof of lemma 2.17(i), applied to Cases (2) and (3) (see remark 2.18), shows that $M = M_\tau \times M_N$. So, we only have to consider Case-(1). To prove $M = M_\tau \times M_N$ for this case by contradiction, assume that there exists $(v', w) \in M_\tau \times M_N$ does not belongs to M . Let $(\alpha, \beta) := \pi^L(v', w)$. As $(\alpha, \beta) \neq (0, 0)$, after scaling by a suitable scalar, we may assume that $(\alpha, \beta) \in \partial D_L$. For $v' \in M_\tau$, there exists $v_N \in M_N$ such that $(v', v_N) \in M$. Note that

$$(\alpha, \beta) = \pi^L(v', w) = \pi^L(v', v_N) + \pi^L(0, w - v_N).$$

We obtain that $\pi^L(0, w') = (\alpha, \beta)$, where $w' := w - v_N$. This implies that $\tilde{\alpha} := (\alpha, \beta - w') \in M$. Since $\partial D_L \subset \partial B^\tau \times \overline{B}_N$, it follows that $(\alpha, \beta) \in \partial B^\tau \times \overline{B}_N$. If $(\alpha, \beta) \in \partial B^\tau \times B_N$, then applying the same argument as in Case-1 of the proof of lemma 2.17(i), we obtain a contradiction.

Assume that $(\alpha, \beta) \in \partial B^\tau \times \partial B_N$. Keeping the notation of Definition 2.15, let σ be a maximal subset of $\{1, \dots, N-1\}$ with respect to inclusion such that $\alpha_\sigma := \pi_\sigma(\alpha) \in \partial(B^\sigma)_{L_\sigma} \subset (\partial B)^\sigma$, where $L_\sigma := \pi_\sigma(L)$. Since $\partial(B^\sigma)_{L_\sigma} \subset (\partial B)^\sigma$, it follows that for each $j \in \sigma$, π_j is a proper map from $(B^\sigma)_{L_\sigma}$ onto $(B_j)_{L_j}$. Moreover, π_j is a linear isometry from L_σ onto L_j . Since $(B_j)_{L_j}$ is strictly convex for each $j \in \sigma$, $(B^\sigma)_{L_\sigma}$ is strictly convex. Note that for any $(q_\sigma, w_{\sigma^c}) \in \partial D_L \cap (\partial(B^\sigma) \times B^{\sigma^c})$,

$$(2.14) \quad \ell := \{(q_\sigma, w_{\sigma^c}) + s(\tilde{\alpha}_\sigma, \tilde{\alpha}_{\sigma^c}) : s \in \mathbb{R}\}$$

does not intersect D , where $(\tilde{\alpha}_\sigma, \tilde{\alpha}_{\sigma^c}) = \tilde{\alpha} \in M$. Consider the open set $U = \mathbb{C}^\sigma \times B^{\sigma^c}$ in $\mathbb{C}^m$, where $m = n_1 + \dots + n_N$. Note that $\pi_\sigma(U \cap L) = V' \cap L_\sigma$, where V' is an open

subset of $\mathbb{C}^\sigma$. Therefore, we obtain that

$$\pi_\sigma(L \cap (\partial(B^\sigma) \times B^{\sigma^c})) = \partial(B^\sigma) \cap V' \cap L_\sigma.$$

Let $V := \partial(B^\sigma) \cap V' \cap L_\sigma$. Note that V is an open subset of $\partial(B^\sigma)_{L_\sigma}$. Recalling from (2.14) that ℓ does not intersect D for any $(q_\sigma, w_{\sigma^c}) \in \partial D_L \cap (\partial(B^\sigma) \times B^{\sigma^c})$. Since $w_\sigma \in B^{\sigma^c}$, $\pi_\sigma(\ell)$ does not intersect B^σ for every $q_\sigma \in \partial(B^\sigma)$ with $(q_\sigma, w_{\sigma^c}) \in \partial D_L \cap (\partial(B^\sigma) \times B^{\sigma^c})$. This implies that for every $q_\sigma \in V$, $\tilde{\alpha}_\sigma$ is supporting for $\partial(B^\sigma)_{L_\sigma}$ at q_σ . Since $(B^\sigma)_{L_\sigma}$ is strictly convex, an argument analogous to that used in Case 3 of the proof of Lemma 2.17(i) yields a contradiction. This shows that $M = M_\tau \times M_N$, allowing us to use lemma 2.17(ii). We then get that D_L is the graph of the linear map $(D\rho|_0 - I)$ over the product of linear retracts $(B^\tau)_{\Sigma_\tau}$ of B^τ and $(B_N)_{\Sigma_N}$ of B_N , taking values in $(B^\tau)_{M_\tau} \times (B_N)_{M_N}$, where Σ_τ and Σ_N are linear subspaces of L_τ and L_N respectively. $\square$

Theorem 2.24. *Let $B_1, B_2, \dots, B_N$ be a collection of bounded balanced strictly convex domains, possibly in different complex Euclidean spaces (say, $B_j \subset \mathbb{C}^{n_j}$ for $j = 1, \dots, N$). Then, every retract Z through the origin of $D = B_1 \times B_2 \times \dots \times B_N$ is a holomorphic graph over a (possibly trivial) product of linear retracts of the factors B_j . Moreover, in the following cases, Z is forced to be linear; specifically,*

- (i) *if $\partial D_L \subset \partial B_e := \partial B_1 \times \dots \times \partial B_N$ then Z is a complex linear subspace of D and can be realized as a graph of some linear map over a linear retract of one of the factors B_j ;*
- (ii) *on the other hand, if ∂D_L intersects the facets $\partial B_i \times B^{i^c}$ for each $i = 1, \dots, N$ then Z is a complex linear subspace of D which can be realized as a graph of some linear map over a product of linear retracts of B_j .*

Proof. Let ρ be a retraction map from D onto Z and set $\pi^L := D\rho|_0$, which we recall is a linear retraction map by lemma 2.8. Define $M := \ker(\pi^L)$. Write $D = B^\tau \times B_N$, where $\tau = \{1, \dots, N-1\}$. Let $L_\tau := \pi_\tau(L)$ and $L_N := \pi_N(L)$. We can divide the proof into following cases:

Case-(1): $\partial D_L \subset \partial B_e := \partial B_1 \times \dots \times \partial B_N$.

In this case, note that π_j is a proper map from D_L onto B_j for each $j = 1, \dots, N$. Moreover, for each j , π_j is an invertible linear map from L onto L_j . By arguments similar to those in the proof for case-(i) in theorem 1.1, we obtain that Z is a linear subspace of D . It then follows that Z can be realized as the graph of the linear map $\pi^L - I$ over linear retract of one of the factors B_j and taking values in M .

Case-(2): ∂D_L intersects $\partial B^\tau \times B_N$, $B^\tau \times \partial B_N$ or both.

We want to show that Z is a graph of some holomorphic map over a product of linear retracts of each factor domains B_j , taking values in M . To prove this by induction on k , the number of factor balls B_j . The case $k = 2$ follows from theorem 1.1. Assume that this result is true for $k < N$. By the above lemma 2.23, we have $M = M_\tau \times M_N$, where $M_\tau := \pi_\tau(M)$ and $M_N := \pi_N(M)$. By lemma 2.17 (ii), we obtain that D_L is the graph of the linear map $T := \pi^L - I$ over the product of linear retracts $(B^\tau)_{\Sigma_\tau}$ of B^τ and $(B_N)_{\Sigma_N}$ of B_N , taking values in $(B^\tau)_{M_\tau} \times (B_N)_{M_N}$, where Σ_τ and Σ_N are linear subspaces of L_τ

and L_N respectively. This implies that

$$D_L = \{(z, w) + T(z, w) : (z, w) \in (B^\tau)_{\Sigma_\tau} \times (B_N)_{\Sigma_N}\}.$$

Owing to general arguments mentioned at the outset of the proof of theorem 1.1, we obtain that $\pi^L \circ \rho|_{D_L}$ is the identity map on D_L ; thereby, that $Z = \rho(D_L)$ is a graph over D_L . Hence, we obtain that

$$(2.15) \quad Z = \rho(D_L) = \{(z, w) + \varphi(z, w) : (z, w) \in (B^\tau)_{\Sigma_\tau} \times (B_N)_{\Sigma_N}\}.$$

Therefore, Z is the graph of the holomorphic map $\varphi = (\rho \circ \pi^L) - I$ over the product of linear retracts $(B^\tau)_{\Sigma_\tau}$ of B^τ and $(B_N)_{\Sigma_N}$ of B_N , taking values in $(B^\tau)_{M_\tau} \times (B_N)_{M_N}$. By the induction hypotheses, $(B^\tau)_{\Sigma_\tau}$ is the graph of some linear map ψ over the product of linear retracts $(B_j)_{\Sigma'_j}$ of the factor domains B_j , taking values in M_τ . Hence, we obtain that $\mathbb{C}^\tau = M_\tau \oplus \Sigma'$ and

$$(2.16) \quad (B^\tau)_{\Sigma_\tau} = \{z' + \psi(z') : z' \in (B^\tau)_{\Sigma'}\},$$

where $\Sigma' = \bigoplus_{j \in \tau} \Sigma'_j$ and $(B^\tau)_{\Sigma'} := \prod_{j \in \tau} (B_j)_{\Sigma'_j}$. From (2.16), Z in (2.15) can be rewritten as

$$Z = \{(z', w) + \Phi(z', w) : (z', w) \in (B^\tau)_{\Sigma'} \times (B_N)_{\Sigma_N}\},$$

where $\Phi(z', w) = \psi(z') + \varphi(z' + \psi(z'), w)$. Let $\tilde{\Sigma} := \Sigma' \oplus \Sigma_N$. Note that $\tilde{\Sigma} \cap M = \{0\}$ because if $(z, w) \in (\tilde{\Sigma}) \cap M$ then $z \in M_\tau \cap \Sigma' = \{0\}$. Just by the dimension formula for sum of subspaces, we have

$$\dim(\tilde{\Sigma} + M) = \dim(\Sigma') + \dim(\Sigma_N) + \dim(M) = \dim(\Sigma_\tau) + \dim(\Sigma_N) + \dim(M) = \dim(\mathbb{C}^m);$$

This ensures that $\mathbb{C}^m = \tilde{\Sigma} \oplus M$. Noting that $\text{image}(\Phi) \subset M$, we conclude therefore that Z is the graph of the holomorphic map Φ over the product of linear retracts of (some of) the factor domains B_j 's and taking values in M .

Case-(3): $\partial D_L \subset \partial B^\tau \times \partial B_N$.

In this case, note that π_N is a proper map from D_L onto $(B_N)_{L_N}$ and π_τ is a proper map from D_L onto $(B^\tau)_{L_\tau}$. Since π_τ and π_N are invertible maps from L onto L_τ and L_N respectively, it follows that Z can be realized as the graph of some holomorphic map φ over the linear subspace $(B^\tau)_{L_\tau}$ of B^τ . Likewise, Z also can be realized as the graph of some linear map ϕ over the linear subspace $(B_N)_{L_N}$ of B_N . Since B_N is strictly convex, it follows from proposition 2.20 that $L_\tau \cap K_\tau = \{0\}$ or $L_N \cap K_N = \{0\}$, where $K_\tau := M \cap \mathbb{C}^\tau$ and $K_N = M \cap \mathbb{C}^N$. By proposition 2.9, we obtain that either $(B^\tau)_{L_\tau}$ is a retract of B^τ or $(B_N)_{L_N}$ is a retract of B_N or both are retracts of respective balls. If $(B^\tau)_{L_\tau}$ is a retract of B^τ , then by the induction hypothesis, $(B^\tau)_{L_\tau}$ is a graph of some linear map ψ over the product of linear retracts $(B_j)_{\Sigma'_j}$ of the factor domains B_j and taking values in K_τ . Note that Z is a graph of some holomorphic map φ over the linear retract $(B^\tau)_{L_\tau}$ of B^τ . As argued in the previous case, it follows that Z is the graph of the holomorphic map $\tilde{\varphi}$ over a product of linear retracts of (some of) the factor domains B_j 's and taking values in M ; specifically, $\tilde{\varphi}(z) := \psi(z) + \varphi(z + \psi(z))$.

Case-(4): ∂D_L intersects $\partial B_i \times B^{i_c}$ for each $i = 1, \dots, N$, where $(i)_c$ denotes $(N - 1)$ -tuple complementary to (i) .

In this case, note that for each $j = 1, \dots, N$, π_j is a proper map from D_L onto B_j and thereby π_j is an invertible linear map from L onto L_j . Now, we apply the similar arguments for case-(iv) in the proof of 1.1, we obtain that Z is a linear subspace of D . It follows from the above lemma 2.23 that Z can be realized as the graph of the linear map $(\pi^L - I)$ over a linear retract of one of B_j . $\square$

We now use the above theorem to derive corollary 1.3. To do so, it would be useful to make the following observations about the content of the above theorem 2.24.

Remark 2.25. Let L be any linear subspace such that $(B_j)_L = B_j \cap L$ is a retract of the factor ball B_j . Then being a linear slice of the strictly convex B_j and as strictly convex domains are always irreducible, we have in particular that $(B_j)_L$ is an irreducible retract of D as well. Let Z be any non-trivial irreducible retract of the polyball D as in theorem 2.24 i.e., Z is not biholomorphic to a product of manifolds of dimensions ≥ 1 . Then, Z is a graph of a holomorphic map over one of the linear retracts $(B_j)_L = B_j \cap L$ of one of the factor balls B_j , where both j as well as the linear subspace L depend on Z . We therefore get that Z is biholomorphic to $(B_j)_L$, a strictly convex ball, thereby that these are precisely all the irreducible retracts of the polyball D .

Lemma 2.26. *Let D be a balanced pseudoconvex domain in $\mathbb{C}^N$. If D is biholomorphic to product domain $\Omega_1 \times \dots \times \Omega_M$ then there exist irreducible balanced domains $\Omega'_j \subset \mathbb{C}^{n_j}$ for $j = 1, \dots, r$, with $r \geq M$ such that D is biholomorphic to the product $\Omega'_1 \times \dots \times \Omega'_r$.*

Proof. Let F be a biholomorphic map from $\Omega := \Omega_1 \times \dots \times \Omega_M$ onto D and $q = (q_1, \dots, q_M) := F^{-1}(0)$. Let I_Ω and I_D denote the Kobayashi indicatrices of Ω at q and D at 0 respectively. By proposition 3.5.3 in [15], $I_D = D$. Hence, the derivative $T := DF|_0$ is a linear isomorphism that maps I_Ω onto D . By proposition 3.7.1 (b) in [15], K_Ω has the product property. This means that if $v = (v_1, \dots, v_M) \in \mathbb{C}^N$ with $v_j \in \mathbb{C}^{n_j}$ then

$$K_\Omega(q, v) = \max\{K_{\Omega_1}(q_1, v_1), \dots, K_{\Omega_M}(q_M, v_M)\}.$$

This shows that $I_\Omega = \prod_{j=1}^M I_{\Omega_j}$, where I_{Ω_j} denotes the indicatrix of Ω_j at q_j . Therefore, we conclude that D is biholomorphic to product of balanced domains I_{Ω_j} for $j = 1, \dots, M$. We now claim that I_{Ω_j} is pseudoconvex for each j . The pseudoconvexity of I_Ω follows from the fact that D is pseudoconvex and that linearly equivalent to I_Ω . Hence, the Minkowski functional h of I_Ω is plurisubharmonic. This implies in particular that h restricted to I_{Ω_j} is plurisubharmonic. By remark 2.2.1 (o) in [15], h has the product property. Therefore, the Minkowski functional of I_{Ω_j} is precisely $h|_{I_{\Omega_j}}$, which is plurisubharmonic. This shows that I_{Ω_j} is a balanced pseudoconvex domain. If I_{Ω_j} is not irreducible for some j , then it is biholomorphic to a product of irreducible domains. Applying the above argument to the balanced pseudoconvex domain I_{Ω_j} , we conclude that I_{Ω_j} is biholomorphic to a product of balanced domains. Continuing this process for each reducible domain in the decomposition of I_Ω , we obtain the desired result. $\square$

Proof of Corollary 1.3: First we consider the case that the polyball D as in the assertion is biholomorphic to $\Omega := \Omega_1 \times \dots \times \Omega_M$ with Ω being a balanced product domain (wherein the factors as always are assumed to be irreducible). Being biholomorphic to the bounded D , Ω is hyperbolic which is then forced to be bounded by Kodama's theorem ([23]); of-course Ω has got to be pseudoconvex as well. Actually, we have here that D must

be convex: by a theorem of Kaup – Upmeyer, a biholomorphism F^0 from Ω onto D preserving the origin exists; as the Kobayashi indicatrices at the origin in D, Ω are just copies of D, Ω respectively (see [13]), the derivative $T := DF^0|_0$ then in-fact renders a linear equivalence between D and Ω which yields that Ω is a polyball. Nevertheless, the remaining assertions as in the statement of the corollary being proven are not immediate at this juncture; towards that we first introduce some notations: $j_i := \dim(\Omega_i)$ for each $i = 1, \dots, M$ and for each $i = 1, \dots, N$, let $k_i := \dim(B_i)$. Further let $J := j_1 + \dots + j_M$ and $K := k_1 + \dots + k_N$. As $\Omega \simeq D$ via T , it follows that T is a linear isomorphism from $\mathbb{C}^J$ onto $\mathbb{C}^K$, thereby $J = K$. After permuting the factor balls B_i 's and Ω_i 's (if needed), we may assume that $j_1 \leq \dots \leq j_M$ and $k_1 \leq \dots \leq k_N$. As the factors Ω_i are retracts of Ω and biholomorphisms preserve the property of being retracts as well as irreducibility, it follows that $T(\Omega_i)$ is an *irreducible* retract of D passing through the origin for each $i = 1, \dots, M$. By theorem 2.24 applied to D , every irreducible retract of D passing through the origin is a graph over (thereby biholomorphic to) a linear retract of one of the factors of D , thereby to a strictly convex domain in particular. Thus, this holds for the irreducible retracts $T(\Omega_i)$ for each i ; as T is linear, the Ω_i 's are now assured to be strictly convex. Therefore, we may now apply theorem 2.24 to Ω , which guarantees that Ω_M is a retract of Ω of the highest possible dimension *amongst* the *irreducible* retracts of Ω . We now claim that $T(\Omega_i)$ is biholomorphic to one of the factors of D , not a possibly proper linear subspace of a factor. To establish this, first observe that biholomorphic mappings preserve all inequalities between the dimensions of submanifolds; so, in particular, we have

$$\dim(T(\Omega_1)) \leq \dim(T(\Omega_2)) \leq \dots \leq \dim(T(\Omega_M))$$

and that $T(\Omega_M)$ is a retract of D of the highest possible dimension *amongst* the irreducible retracts of D . By remark 2.25 above, it follows that $T(\Omega_M)$ is biholomorphic to one of the factors of D of the highest dimension, thereby after a permutation if needed, we have $T(\Omega_M) = B_N$. It then follows that T is a linear isomorphism from $\{0\} \times \mathbb{C}^{j_M}$ onto $\{0\} \times \mathbb{C}^{k_N}$; thereby, that

$$\ker(' \pi) \cap (\mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_{N-1}}) = \{0\},$$

where $' \pi$ denotes the projection of $\mathbb{C}^K$ onto $\mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_{N-1}}$ (and $\{0\}$, the origin in the appropriate complex Euclidean space). It follows that $'T := ' \pi \circ T$ gives a linear isomorphism of $\mathbb{C}^{j_1} \times \dots \times \mathbb{C}^{j_{M-1}}$ onto $\mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_{N-1}}$, which carries $\Omega_1 \times \dots \times \Omega_{M-1}$ precisely onto $B_1 \times \dots \times B_{N-1}$. Both of these again being polyballs with strictly convex factors, we may apply the above arguments iteratively to get that $\Omega_{M-1}, \dots, \Omega_2, \Omega_1$ are respectively biholomorphic (indeed, linearly equivalent) to $B_{N-1}, \dots, B_2, B_1$ and $M = N$. This finishes the proof in case Ω is balanced.

Next we consider the case that $\Omega \subset \mathbb{C}^J$ is not necessarily a balanced domain. By the hypotheses that $\Omega = \Omega_1 \times \dots \times \Omega_M$ is biholomorphic to polyball D , it follows from lemma 2.26 that there exist irreducible *balanced* domains Ω'_j for $j = 1, \dots, r$ with $r \geq M$ such that D is biholomorphic to $\Omega'_1 \times \dots \times \Omega'_r$. We may now apply the result for the balanced case as already prove above, to the *balanced* biholomorphic copy of D given by $\Omega' := \Omega'_1 \times \dots \times \Omega'_r$; this gives: $N = r \geq M$.

Next, let F be any biholomorphic map from Ω onto D and $q = (q_1, \dots, q_M) := F^{-1}(0)$. For each $i = 1, \dots, M$, consider the retract $Z_i := \{(q_1, \dots, q_{i-1}, w_i, q_{i+1}, \dots, q_M) : w_i \in \Omega_i\}$ passing through q ; this is of-course just an affine copy of Ω_i , thereby an irreducible retract of Ω . Its biholomorphic image, $\tilde{Z}_i := F(Z_i)$ is thereby an irreducible retract of D passing through the origin and can be realized as a graph of some holomorphic map over a linear retract of one of the factors of D by remark 2.25. Hence, for each $i = 1, \dots, M$, there exists $\phi(i) \in \{1, \dots, N\}$ such that we have in particular that: $\tilde{F}_i(Z_i)$ is contained in $B_{\phi(i)}$, where $\tilde{F}_i := \pi_{\phi(i)} \circ F$ and $\pi_{\phi(i)}$ denotes the standard projection from $\mathbb{C}^K$ onto $\mathbb{C}^{\phi(i)}$. This induces a mapping between the index sets namely, $\phi : \{1, \dots, M\} \rightarrow \{1, \dots, N\}$. We want to show that this map is onto. Let $S := \text{image}(\phi) \subset \{1, \dots, N\}$. Note that

$$(2.17) \quad J = j_1 + \dots + j_M = \sum_{s \in S} \sum_{i \in \phi^{-1}(s)} j_i.$$

Fix any $s \in S$. Observe that for each $i \in \phi^{-1}(s)$, $F_s(Z_i) \subset B_s$, where $F_s = \pi_s \circ F$ and π_s denotes the standard projection from $\mathbb{C}^K$ onto $\mathbb{C}^{k_s}$. Note that for each $i \in \phi^{-1}(s)$, π_s is injective on $F(Z_i)$, thereby F_s is a biholomorphic map from Z_i onto $F_s(Z_i) \subset B_s$. Hence, the derivative $D(F_s)|_q$ is a injective linear map from $\mathbb{C}^{j_i}$ onto some subspace of $\mathbb{C}^{k_s}$; indeed, $D(F_s)|_q$ is injective on $\oplus_{i \in \phi^{-1}(s)} \mathbb{C}^{j_i}$. Therefore, we have firstly that

$$(2.18) \quad V := \sum_{i \in \phi^{-1}(s)} D(F_s)|_q(\mathbb{C}^{j_i}) \text{ is contained in } \mathbb{C}^{k_s}.$$

and (just by the fact the $\mathbb{C}^{j_i}$'s i.e., their standard copies in $\mathbb{C}^J$ intersect only at the origin, it follows that the same holds for their images under $D(F_s)|_q$):

$$\bigcap_{i \in \phi^{-1}(s)} D(F_s)|_q(\mathbb{C}^{j_i}) = \{0\}.$$

Therefore, we obtain that the sum defining the linear subspace V in (2.18) becomes a direct sum and thereby

$$\dim(V) = \sum_{i \in \phi^{-1}(s)} j_i \leq \dim(\mathbb{C}^{k_s}) = k_s.$$

Now, if the map ϕ were not onto, then it would follow from (2.17) that $J \leq \sum_{s \in S} k_s < K$, contradicting our earlier observation that $J = K$. This finishes the verification that ϕ is onto; thereby $M \geq N$. As the reverse inequality $M \leq N$ was already attained, we are now assured that $M = N$. After permuting balls B_i (if needed), we may assume that $\phi(i) = i$. As the number of (irreducible) factors defining the products D, Ω are same, we may now apply proposition 1 in chapter 5 of Narasimhan's book [21] to get that Ω_i is biholomorphic to B_i for every $i = 1, \dots, N$. $\square$

Remark 2.27. The main part of the proof above consists in establishing equality of the number of factors $M = N$. With that, it actually follows from proposition 1 at page-71 of Narasimhan's book [21] that *every* biholomorphic mapping F between Ω and D , is a product map: $F = F_1 \times F_2 \times \dots \times F_N$ where $F_i : \Omega_i \rightarrow B_i$ is a biholomorphic map for each $i = 1, \dots, N$.

Let us just conclude by mentioning that parts of theorem 1.1 and its extension, theorem 2.24, can be rephrased as a result for one-complemented subspaces which is of much interest in the geometry of Banach spaces or convex analysis (for instance, see [2] for a long survey in this regard). Owing to this (and as a result of this kind does not seem to have been recorded), we believe it would be helpful to record this rephrasing below.

Corollary 2.28. *Let $(X_j, \|\cdot\|_j)$ for $j = 1, \dots, N$, be strictly convex finite dimensional (complex) Banach spaces. Let X be the ℓ^∞ -sum of these spaces i.e., $X = \bigoplus_{j=1}^N X_j$ (the external direct sum of the X_j 's) endowed with norm $\|(x_1, \dots, x_N)\|_\infty = \max\{\|x_j\|_j : 1 \leq j \leq N\}$. A linear subspace L of X is a one-complemented subspace of X if and only if L is a graph of some linear map over $\bigoplus_{j=1}^N \Sigma_j$, where Σ_j is a one-complemented subspace of X_j for $j = 1, \dots, N$. Thereby, L is isometrically isomorphic to $\bigoplus_{j=1}^N \Sigma_j$.*

Proof. Owing to the finite dimensionality hypotheses we may as well assume that as complex vector spaces $X_j = \mathbb{C}^{n_j}$ and $X = \mathbb{C}^m$, where $m = n_1 + \dots + n_N$ (with the norms induced by the ones coming from the hypothesis). For each $j = 1, \dots, N$, let B_j denote the unit ball with respect to $\|\cdot\|_j$. The strict convexity of the $(X_j, \|\cdot\|_j)$'s means that the balls B_j are strictly convex. Now, the unit ball in $\mathbb{C}^m$ with respect to the ℓ_∞ -sum norm as in the theorem, is $D := \prod_{j=1}^N B_j$. Since L is a one-complemented subspace of $(\mathbb{C}^m, \|\cdot\|_\infty)$, it follows that D_L is a linear retract of D . We now apply the above theorem 2.24 to get that D_L is a graph of some linear map T over the product of the linear retracts $(B_j)_{\Sigma_j}$ of B_j , where Σ_j is a linear subspace of $L_j := \pi_j(L)$. Because $(B_j)_{\Sigma_j}$ is a linear retract of B_j , Σ_j is a one-complemented subspace of $\mathbb{C}^{n_j}$. Let $\Sigma := \bigoplus_{j=1}^N \Sigma_j$ and $B_\Sigma := \prod_{j=1}^N (B_j)_{\Sigma_j}$. Note that L is the graph of the linear map T over Σ , i.e.,

$$L = \{z + T(z) : z \in \Sigma\}.$$

Now, consider the invertible linear map $S : \Sigma \rightarrow L$ given by $S(z) = z + T(z)$. Note that unit balls in Σ and L , are B_Σ and D_L respectively. Since D_L is a graph of the linear map T over B_Σ , S maps B_Σ onto D_L . It follows from the invertibility of S that S maps $\partial(B_\Sigma)$ onto ∂D_L . This shows that L is isometrically isomorphic to Σ . $\square$

Acknowledgments: We thank Profs. Zwonek, Gautam Bharali, C. R. Jayanarayanan, Rohith Varma, J. Jaikrishnan and Kaushal Verma. The second named author would like to acknowledge the support of the CSIR-JRF fellowship provided by the Council of Scientific and Industrial Research, India.

DECLARATIONS

- **Availability of data and materials:** Not Applicable (there are no data associated with this article).
- **Competing interests:** The authors declare that they have no competing interests.
- **Funding:** The first named author is supported by a grant from the Anusandhan National Research Foundation (ANRF), India with id: ANRF/ARGM/2025/000658/MTR. The second named author would like to acknowledge the support of the CSIR-JRF fellowship provided by the Council of Scientific and Industrial Research, India.

- **Authors' contributions:** Both authors equally contributed to the contents and to the writing of the paper.
- **Authors' information:** The first named author is an Assistant Professor while the second named author is a PhD student, at the Indian Institute of Technology – Palakkad, India.

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