1 Supplementary Note: Fourier-Based Geometric and Topological Analysis of CRGs

Figure 1 — Single-frequency CRG

Figure 1 shows a closed resonant geometry (CRG) generated by single-frequency stimulation in a three-dimensional state space (x_1, x_2, x_3). The trajectories form a stable, periodic closed loop, indicating that even a simple sinusoidal input can induce a robust geometric attractor. This supports the RCNNG prediction that elementary perceptual states can be realized as resonance within single closed geometries.

1.1 Figure 2 — Multi-frequency Geometry (3D attractor + transients)

Figure 2 illustrates the effect of multi-frequency stimulation on the geometry of the system. The three-dimensional trajectories exhibit both transient excursions and a multi-frequency attractor, with richer oscillatory structure than in the single-frequency case. As additional frequency components are introduced, the geometry becomes more intricate and layered, consistent with RCNNG's claim that increasing frequency complexity expands the repertoire of available closed resonant geometries.

1.2 Figure 3 — Fourier Geometric Expansion

Figure 3 presents a three-panel geometric expansion:

- Panel A: Single-frequency CRG, showing a simple closed trajectory.
- Panel B: Multi-frequency geometry, with more complex oscillations.
- Panel C: Fourier-geometric expansion, combining multiple frequency components into a unified three-dimensional structure.

This progression demonstrates how Fourier components can be interpreted geometrically: single modes generate simple closed loops, while their superposition yields higher-order resonant geometries. This provides a concrete geometric realization of the RCNNG principle that perception arises from resonance in closed, frequency-structured geometries.

1.3 Figure 4 — Conceptual Fourier–Geometric Expansion (Three-panel schematic)

Figure 4 offers a conceptual schematic of the Fourier–geometric mapping:

- Panel A (Frequency Inputs): Sinusoidal signals of different frequencies.
- Panel B (Base CRGs): Circular closed geometries corresponding to individual frequency components.
- Panel C (Fourier-Geometric Expansion): A three-dimensional trajectory obtained by combining these base geometries.

This schematic clarifies the conceptual pipeline: frequency-domain inputs are mapped to base closed geometries, and their Fourier combination produces complex resonant structures. It thus bridges classical Fourier analysis with the geometric language of RCNNG.

1.4 Figure 5 — Persistent Homology of CRG Geometry

Figure 5 applies persistent homology to a CRG-related point cloud:

- Panel A (Stable Geometry): A point cloud in (x_1, x_2, x_3) representing a sampled resonant geometry.
- Panel B (Persistence Diagram): Birth–death pairs for H_0 and H_1 features, showing the scales at which connected components and loops appear and disappear.
- Panel C (Persistence Barcode): Persistence intervals for H_0 and H_1, highlighting long-lived topological features.

The presence of a persistent H_1 feature indicates a robust closed-loop structure, providing topological evidence that CRGs are stable and not merely transient artifacts. This aligns with RCNNG's interpretation of perceptual states as persistent resonant geometries with well-defined topological signatures.

1.5 Figure 6 — Single-frequency CRG

Figure 6 shows a single-frequency CRG rendered as a smooth, elliptical trajectory in three-dimensional space. This figure isolates the simplest resonant geometry, emphasizing its stability and periodicity. It serves as the foundational building block for the multi-frequency expansions shown in subsequent figures, illustrating the base case of RCNNG's geometric resonance mechanism.

1.6 Figure 7 — Multi-frequency CRG

Figure 7 depicts a multi-frequency CRG with multiple intertwined trajectories forming complex closed loops in (x_1, x_2, x_3). The geometry exhibits rich interference patterns and multi-layered structure, reflecting the interaction of several frequency components. This figure provides a fully developed example of multi-frequency resonance, demonstrating how RCNNG predicts the emergence of diverse, high-dimensional closed geometries as the substrate of complex perceptual states.

1.7 Summary of Supplementary Figures

Taken together, Figures 1–7 show that:

- Single-frequency stimulation yields simple, stable closed geometries (CRGs).
- Multi-frequency stimulation produces complex, multi-layered attractors and CRGs.
- Fourier-based combinations of base geometries generate higher-order resonant structures.
- Persistent homology confirms the stability and topological robustness of these geometries.

These supplementary analyses provide detailed geometric and topological support for the RCNNG framework, demonstrating that perception can be understood as resonance within a structured space of closed resonant geometries.