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Hygrothermoelastic responses in a fractal media by extending vector calculus to non-integer dimensions

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Abstract

This work attempts to generalize the concept of vector calculus to spaces that do not have integer dimensions, based on the ubiquity in nature of fractals. We present a new hygrothermal model of fractal media, based on a continuum model. Using analytic continuation, we obtain gradients, divergences, and scalar and vector Laplacian operators for first- and second-order differential vector operators in non-integer-based dimensional spaces. The model incorporates higher-order time frame derivatives and the Moore-Gibson-Thompson theory and redefines the derivatives that depend on forward memory in a sliding interval. The modified system of linearly coupled equations with sectional hygrothermal loads is then solved using the integral transformation method to find a precise solution to the problem. The Laplace transform approach is used to solve the governing equations, and a Fourier series expansion technique is used for numerical inversion. The computational outcomes and graphical illustrations demonstrate that the delay period and the relevance of different fractal dimensions are important parameters with significant impacts. The research also looks into the performance of various kernel functions; whereby nonlinear kernels perform better in this theoretical framework than linear ones. This hygrothermal model of fractal media with multi-kernel derivatives and relaxation phase lag delays is widely used in many different applications. It offers valuable information concerning how heat and moisture transport in various materials and devices.

Keywords: *Fractal media, non-integer dimensions, vector calculus, hygrothermal model, MGT theory, forward memory-dependent derivative, Laplace transforms*

Notations

D_h	thermal conductivity	D_m	moisture diffusion coefficient
E	Young's modulus	υ	Poisson's ratio
∇	gradient operator	C_p	specific heat parameter
γ	heat released from a unit mass	ϖ, ω	material constants
C	mass of moisture	T	conductive temperature
ρ_m	density of the material	t	Time variable
u	Displacement	T_0, C_0	surrounding conditions
∇^2	Laplacian operator	σ_{ij}	stress components
ε_{ij}	strain tensor	q_m	moisture flux vector
γ_1	temperature coefficient	γ_2	moisture coefficient
q_h	heat flux vector	λ, μ	Lame coefficients

1. Introduction

Fractals are natural phenomena that are complex structures that show self-similarity at various levels. This all-pervasive nature of fractals has led researchers to consider a wide range of mathematical structures to comprehend the nature of such phenomena better. In our work, we aim to generalize the concepts of vector calculus to spaces with non-integer dimensions and consequently generalize the applications of classical mathematical analysis. Fractals are complex patterns found in nature that are self-similar across varying scales. This pervasiveness has prompted scholars to explore different mathematical frameworks to understand these intricate patterns further. Fractals are objects in which self-similarity occurs, i.e., at different scales. This feature is referred to as scale invariance. Mandelbrot [1] coined the word fractal based on the Latin term, fractus, which translates to broken or fractured. Fractals are usually characterized by their fractal dimension, which is a measure of their complexity and may be a non-integer. This is unlike traditional Euclidean geometry, where the dimensions of shapes are whole numbers, e.g., a one-dimensional line, a two-dimensional square, or a three-dimensional cube. Remarkably, the groundbreaking work on fractals by Mandelbrot [2] revealed the existence of a fractal characteristic in natural objects such as coastlines, mountains, and clouds. The purpose of this study is to generalize traditional mathematical analysis by applying the idea of non-integer-dimensional spaces using vector calculus. This extension is inspired by the desire to

model and study intricate systems that are difficult to depict in integer-dimensional spaces. We can better simulate the intricate patterns and behaviour of natural fractal structures by using non-integer dimensions. The contributions of Tarasov [3], who made generalizations of the implementation of vector calculus in anisotropic fractal media in non-integer-dimensional spaces, are seminal. More recently, Tarasov [4] also generalized the concept of vector calculus to non-integer-dimensional space and used it in fractal media; we hope to develop this mathematical formalism. This generalization makes it possible to construct gradient, divergence, and Laplacian operators in non-integer dimensions, which is a powerful instrument to study fractal materials. With the following extension of vector calculus, we aim to create a better perception of fractal media and their characteristics. This study not only contributes to the field of mathematical physics, but also has practical consequences for materials science, engineering and environmental modelling.

Measurable metric sets with non-integer Hausdorff dimensions are known as fractals [5,6]. A fractal set is distinguished at all scales by its non-integer Hausdorff dimension. As the diameter gets closer to zero, the Hausdorff dimension—a local property that covers subsets of the points—quantifies the measure of a collection of dispersed points. This concept requires that the sets' diameters tend to zero. Typically, real materials have a distinctive minimum length scale R_0, illustrated by a particle's radius, such as that of an atom or molecule. The fractal structure in fractal materials is observable solely at scales where $R > R_0$, with R_0 being the distinctive scale of the particles. The Hausdorff dimension may be replaced with a non-integer mass dimension in real materials. Assuming uniform density, the mass dimension describes the relationship between a region's mass and volume. Often, the mass $M(W)$ of a material region W and its radius R have an asymptotic relationship. Fractal materials have a power-law mass distribution relationship $M(W) \sim R^D$. The non-integer mass dimension of a fractal material is represented by the parameter D, which is independent of the spherical packing type or the configuration of region W. One may think of fractal materials as media with a mass dimension that is not an integer. Even so, the geometrical and dynamic characteristics of fractal materials might not be adequately represented by the non-integer dimension; it makes significant inferences regarding their behavior. It facilitates the application of efficient models that consider non-integer dimensions.

There are numerous ways to model fractal materials, such as fractal analysis [7-12], in non-integer dimensional space using fractal analysis, fractional-integral

continuum models [13-16], fractional-differentiation models [17-21-], and integration and differentiation [22-24]. These approaches employ non-integer-order integrations and such ideas as dispersion of states and distribution functions, relating the relationship between the fractal materials' mass dimension and the order of fractional integrals. These integrals' kernels have a power-law density of states. Differential equations of the thermoelasticity and elasticity of fractal materials are solved using a fractional-integral continuum model. The major gap in the existing literature is the absence of detailed models that use vector calculus principles in non-integer dimensional spaces to examine the hygrothermoelastic response of fractal media. This paper is concerned with the hygrothermoelastic response of isotropic fractal materials defined in terms of non-integer dimensional space. Fractional-integral continuum models and continuum models with non-integer dimensional spaces is that the numerical factor of the density of states is determined by the volume equation of a non-integer-dimensional spherical space. The paper presents a framework of vector calculus in non-integer dimensional spaces, which is used to apply continuum models to the description of fractal materials. Applications for the non-integer dimensional space integration are numerous and include the dimensional regularization of ultraviolet divergences in quantum field theory, phase transitions in statistical physics, and the investigation of critical phenomena.

In recent decades, hygrothermoelasticity theories, which consider the finite speed of moisture and thermal signals, have received critical and diverse attention from researchers [25-31] due to their use in many other critical areas. The rekindled interest is as a result of several factors. To begin with, aerodynamic heating and moisture absorption of intense thermal and hygrothermal loads in aeronautical engineering can decrease the strength of aircraft structures by decreasing their elastic limits. Also, the exceptionally high temperatures and temperature gradients and humidity variations influence nuclear design and have their roots in nuclear reactors and operations. Likewise, undesired hygrothermal stress in modern propulsion systems (e.g., jet engines and rockets) is brought on by the high moisture and temperature levels of the combustion processes. Hygrothermal phenomena are faced by space vehicle and missile technologies, large steam turbine mechanics, and even shipbuilding technologies [32-33]. New experimental data on the finiteness of both the speed of heat and moisture propagation have resulted in some prominent studies using generalized hygrothermoelasticity theories [34-40]. During the last several years, the thermoelasticity community has been more interested in the Moore-Gibson-

Thompson (MGT) theory [41-43]. Nevertheless, the author did not manage to locate any articles that would specifically discuss hygrothermoelasticity under this theory. This concept highlights the significance of studying hygrothermoelastic responses in various engineering fields. It highlights the need to have sophisticated mathematical models to model behavior of materials under combined stresses of thermal and moisture.

Fractional-order differential equations have recently seen important developments, which have opened up new research possibilities. Researchers have found that the present condition of heat flow or diffusion flux depends on its past condition during heat and mass transmission in colloids, porous media, biological materials, and other materials. Conventional integer-order derivatives do not reflect historical and path-dependent features of these processes. This limitation has led to the adoption of fractional-order calculus, which effectively addresses this issue. These theoretical developments have significantly enhanced our understanding and modeling capabilities for complex thermal and diffusion phenomena, particularly under extreme conditions. The Caputo fractional differential operator is well-defined as follows:

$$D_a^{\alpha} f(t) = \int_a^t K_{\alpha}(t-\xi) f^{(m)}(\xi) d\xi, \tag{1}$$

with kernel function $K_{\alpha}(t-\xi) = (t-\xi)^{m-\alpha-1} / \Gamma(m-\alpha)$ and $f^{(m)}(t)$ represents the usual m-derivative. Based on Eq. (1), Wang and Li [44] introduced the backward memory-dependent derivative (bMDD) as an alternative to fractional derivatives with the assumption that the temperature increases immediately after heat transfer. It is specified in the integral form of a conventional derivative, incorporating a kernel function over a sliding interval as

$$D_{\tau_i}^{m} f(t) = \frac{1}{\tau_i} \int_{t-\tau_i}^{t} K(t-\xi) f^{(m)}(\xi) d\xi, \tag{2}$$

where, $K(t-\xi)$ is the kernel function provided by and $\tau_i > 0$ is the delay hysteresis factor (also known as delay time).

$$K(t-\xi) = 1 - \frac{2f}{\tau_i}(t-\xi) + \frac{e^2}{\tau_i^2}(t-\xi)^2 = \begin{cases} 1; & \text{if } e=0, f=0 \\ 1-(t-\xi)/\tau_i; & \text{if } e=0, f=1/2 \\ \{1-[(t-\xi)/\tau_i]\}^2; & \text{if } e=1, f=1 \end{cases} \tag{3}$$

Here, it is called the *m*-order 'memory-dependent derivative' of f at τ_i relative to the time delay $\tau_i > 0$ and is *m*-times differentiable about τ_i and ξ. where is the delay time τ_i; e and f are constants, respectively. In addition, if K =1,

$$D_{\tau_i} f(t) = \frac{1}{\tau_i}\int_{t-\tau_i}^{t} f'(\xi)\mathrm{d}\xi = \frac{f(t)-f(t-\omega_i)}{\tau_i} \to f'(t) \quad \text{as } \tau_i \to 0\,. \tag{4}$$

Based on Eq. (2), Xue et al. [45] have enhanced the Cattaneo and Vernotte (CV) heat conduction model by incorporating the memory-dependent derivative (MDD), addressing the thermoelastic issue of a thermally insulated fracture that runs parallel to the boundary of a strip and is subjected to thermal impact stress. In their research Yu [46] introduces a novel Lord-Shulman (LS) model that integrates memory effects, recommending that in LS generalized thermoelasticity, memory-dependent derivatives (MDD) be used in place of fractional calculus. Memory dependency was included in the diffusion and heat transport equations by Gao and Ma [47]. Purkait et al. [48] used two-temperature theories based on the LS model to investigate the transient phenomena of a spherical shell.

Recently, Sun and Wang [49] endeavored to replace the backward memory-dependent derivative (bMDD) with the *m*-order forward memory-dependent derivative (fMDD), assuming that heat transmission is followed by a progressive process of temperature increase as

$$D_{\tau_i}^{m} f(t) = \frac{1}{\tau_i}\int_{t}^{t+\tau_i} K(t-\xi) f^{(m)}(\xi) d\xi\,, \tag{5}$$

with $K(t-\xi)$ being the kernel function given by

$$K(t-\xi) = \frac{2f}{\tau_i}(t-\xi) + \frac{e^2}{\tau_i^{\,2}}(t-\xi)^2 = \begin{cases} (t-\xi)/\tau_i; & \text{if } e=0, f=1/2 \\ (t-\xi)^2/\tau_i^{\,2}; & \text{if } e=1, f=1 \end{cases} \tag{6}$$

Given this context, the study of hygrothermoelasticity within the framework of backward or forward memory-dependent transportation theory is of significant importance. However, to the best of the author's knowledge, no study of a transient hygrothermoelastic field that incorporates a forward memory-dependent derivative has been done.

Generally, the dimensions of space and space-time, represented by *D*, are often assumed to be non-integers. The foundations of dimensional continuation methods and non-integer dimension spaces are found in quantum field theory and statistical

mechanics. The concept of non-integer dimensions, such as D=4−ϵ [50-52] and ϵ -expansion, is widely used in theories related to phase transitions and critical events in statistical physics. Rigorous research in this field has gained traction, leading to the extension of vector calculus to spaces with non-integer dimensions. Moreover, for these non-integer spaces, first- and second-order operations such as divergence, gradient, and scalar and vector Laplace operators have been developed. However, to the author's knowledge, there is no publication discussing the implications of generalized hygrothermoelasticity.

The paper presents a new theory of hygrothermoelasticity in fractal media. It employs a continuum model with dimensions that are not integers. Simplifications are made by considering scalar and vector functions that are rotationally covariant regardless of angles. The suggested operators make it possible to characterize fractal materials in different dimensions D. The Moore-Gibson-Thompson hypothesis is examined in relation to the hygrothermoelasticity issue in which forward memory-dependent derivatives are considered with the help of various kernel functions. The fractal medium has certain hygrothermal stresses and mechanical impacts on the boundaries. The solution of the governing equations is obtained through the Laplace transform method, and then a Fourier series expansion method is used to quantitatively invert the Laplace transform. The calculations and graphical displays indicate a substantial influence of different parameters, including delay length and different fractal dimensions. The impact of different kernel functions is also analyzed, demonstrating the superiority of nonlinear kernels over linear ones within this new theoretical framework.

Our hygrothermal framework for fractal media begins with mathematical modeling, which includes prerequisites for fractal media using a continuum framework for dimensions that are not integers and obtaining required operators such as gradient, divergence, and Laplacian. It then presents a higher-order hygrothermal fractal equation that includes the Moore-Gibson-Thompson theory and forward memory-dependent derivatives. The study assumes a hollow cylinder space for the problem statement and solves the hygrothermal equations in the Laplace domain using integral transformation techniques. Also, numerical inversion of the Laplace transform is addressed through the Fourier series expansion method. Results highlight the substantial impact of parameters like delay duration and fractal dimensions, demonstrating the superiority of nonlinear kernels. The study concludes by

emphasizing the model's applications in various fields, offering insights into heat and moisture transport behavior.

2. Mathematical modeling

2.1 Fractal media basics in hygrothermal framework

In a non-integer dimensional space, vector differential operators, we employ the equations for cylindrical coordinates in R^n for any arbitrary n. To a non-integer dimension D, define vector differential operators via expanding the dimension from the integer n to the non-integer D. Examine solely vector fields and scalar fields u that are angle-independent $\vec{u}(\vec{r}) = \vec{u}(r) = u_r \vec{e}_r$. In this context, $r = |\vec{r}|$ denotes the radial distance $\vec{e}_r = \vec{r}/r$ signifies the orthogonal local unit vector pointing in the direction of increasing r, and $u_r(r)$ indicates the radial component of $\vec{u}$. Therefore, either integration in a non-integer dimensional space combined with the applicable Gauss' theorem or continuation from integer n to any non-integer D may be used to provide explicit definitions of differential operators for non-integer dimensional spaces. For non-integer dimensional spaces, the differential vector operations, such as gradient, divergence, and scalar and vector Laplacian, may be expressed. For the sake of simplification, assume that the vector field $u = \vec{u}(\vec{r})$ is radially oriented and independent of angular variables. We can think of axially symmetric materials, where the fields $\varphi(\vec{r}) = \varphi(r)$ and also axially symmetric [53]. For cylindrical coordinates, with the Z-axis along the symmetry axis, the following vector differential operators in non-integer dimensional space are defined:

- In non-integer dimensional space, the divergence for the vector field $u = \vec{u}(\vec{r})$:

$$\mathrm{div}_r^D \vec{u} = \frac{\partial u_r}{\partial r} + \frac{D-2}{r} u_r . \tag{7}$$

- The gradient for the scalar field in non-integer dimensional space $\varphi = \varphi(r)$:

$$\mathrm{grad}_r^D \varphi = \frac{\partial \varphi}{\partial r} \vec{e}_r . \tag{8}$$

- The scalar Laplacian for the scalar field in non-integer dimensional space :

$${}^S\Delta_r^D \varphi = \mathrm{div}_r^D \mathrm{grad}_r^D \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D-2}{r} \frac{\partial \varphi}{\partial r} . \tag{9}$$

- The vector Laplacian for the vector field in non-integer dimensional space :

$$ {}^{\mathrm{V}}\Delta_{\mathrm{r}}^{D}\,\vec{u} = \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{D-2}{r}\,\frac{\partial u_r}{\partial r} - \frac{D-2}{r^2}\,u_r \right)\vec{e}_r \,. \tag{10} $$

Eqs. (7)-(8) with $D = 3$ give the well-known expressions for the gradient, divergence, scalar, and vector Laplacian operators [53].

2.2 Formulation of hygrothermal equation in a fractal media

The study of classical hygrothermoelastic theory [54

], considering the principles of Fourier's and Fick's law as $q_h(t) = -D_h \nabla T(t)$ and $q_m(t) = -D_m \nabla C(t)$. Tzou [55] provided two-phase-lag models that replace the classical Fourier law for the temperature gradient and the heat flow vector $q_h(t+\tau_q) = -D_h \nabla T(t+\tau_T)$ and $q_m(t+\tau_q) = -D_m \nabla C(t+\tau_T)$. The modified thermal conductivity law for the Green-Naghdi type III [56] model can be given as $q_h(t) = -D_h \nabla T(t) - D_h^* \nabla \upsilon(t)$ and $q_m(t) = -D_m \nabla C(t) - D_m^* \nabla \vartheta(t)$, in which $(D_h, D_m) > 0$ is the rate of conductivity and (υ, ϑ) is the displacement that satisfies $(\dot{\upsilon} = T, \dot{\vartheta} = C)$. By using the Moore-Gibson-Thompson (MGT) equation, Quintanilla [57] has made changes to the suggested MGT model by modulating the heat equation after incorporating the relaxation time into the GN-III [56] theory as

$$ \left(1 + \tau_h \frac{\partial}{\partial t}\right) q_h = -D_h \nabla T - D_h^* \nabla \upsilon, \tag{11} $$

$$ \left(1 + \tau_m \frac{\partial}{\partial t}\right) q_m = -D_m \nabla C - D_m^* \nabla \vartheta. \tag{12} $$

By taking the time derivative of Eqs. (11) and (12), and using $(\dot{\upsilon} = T, \dot{\vartheta} = C)$, one gets

$$ \left(1 + \tau_h \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} q_h = -\left(D_h \nabla \frac{\partial T}{\partial t} + D_h^* \nabla T \right), \tag{13} $$

$$ \left(1 + \tau_m \frac{\partial}{\partial t}\right) \frac{\partial}{\partial t} q_m = -\left(D_m \nabla \frac{\partial C}{\partial t} + D_m^* \nabla C \right). \tag{14} $$

Since both liquid and vapor are present, the moisture and heat transfer processes adhere to the principles of mass and energy conservation as follows:

$$ \nabla \cdot q_h = \rho_m \gamma \frac{\partial M}{\partial t} - \rho_m C_p \frac{\partial T}{\partial t}, \tag{15} $$

$$\nabla \cdot q_m = -\frac{\rho_m}{V}\frac{\partial M}{\partial t} - \frac{\partial C}{\partial t}. \tag{16}$$

where $M = \text{constant} + \varpi C - \omega T$, where $\partial M / \partial C = \varpi$ and $\partial M / \partial T = -\omega$ are material constants. Then, the amount of moisture in composite per unit mass of solid, *m*, can be expressed as $\rho_m m = VC + \rho_m M$, where V is the volume fraction of the voids, ρ_m is the density, $\rho_m = (1-V)\rho_s$, and ρ_S is the density of a solid without voids. Combining Eqs. (13) with (15) and Eqs. (14) with (16), then eliminating the quantity M, one obtains

$$\left(1+\tau_h \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\rho_m C_p \frac{\partial T}{\partial t} - \rho_m \gamma \frac{\partial M}{\partial t}\right) = \left(D_h \nabla^2 \frac{\partial T}{\partial t} + D_h^* \nabla^2 T\right), \tag{17}$$

$$\left(1+\tau_m \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\rho_m}{V}\frac{\partial M}{\partial t} + \frac{\partial C}{\partial t}\right) = \left(D_m \nabla^2 \frac{\partial C}{\partial t} + D_m^* \nabla^2 C\right). \tag{18}$$

It is observed that Eqs. (17) and (18) include the quantity M. Consequently, the heat transfer and moisture is interconnected, allowing for the elimination of the quantity $\partial M / \partial t$. This results in A linearly coupled system of equations:

$$\left(1+\tau_h \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t} - \eta \frac{\partial C}{\partial t}\right) = \nabla^2\left(L_h \frac{\partial T}{\partial t} + L_h^* T\right), \tag{19}$$

$$\left(1+\tau_m \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}\left(\frac{\partial C}{\partial t} - \lambda \frac{\partial T}{\partial t}\right) = \nabla^2\left(K_m \frac{\partial C}{\partial t} + K_m^* C\right), \tag{20}$$

in which

$$\begin{gathered}\eta = \frac{\gamma\varpi}{C_p - \gamma\omega}, L_h = \frac{D_h}{\rho_m (C_p - \gamma\omega)}, L_h^* = \frac{D_h^*}{\rho_m (C_p - \gamma\omega)}, \\ \lambda = \frac{\omega\rho_m}{\varpi\rho_m + V}, K_m = \frac{D_m V}{\varpi\rho_m + V}, K_m^* = \frac{D_m^* V}{\varpi\rho_m + V}.\end{gathered} \tag{21}$$

In the context of the Moore-Gibson-Thompson theory, by incorporating the forward memory-dependent derivative within a slipping interval as given in Eq. (5), the modified forms of Fourier's and Fick's laws can be expressed as follows:

$$\left(1+\tau_h D_{\tau_h}\right)\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial t} - \eta \frac{\partial C}{\partial t}\right) = \nabla^2\left(L_h \frac{\partial T}{\partial t} + L_h^* T\right), \tag{22}$$

$$\left(1+\tau_m D_{\tau_m}\right)\frac{\partial}{\partial t}\left(\frac{\partial C}{\partial t} - \lambda \frac{\partial T}{\partial t}\right) = \nabla^2\left(K_m \frac{\partial C}{\partial t} + K_m^* C\right). \tag{23}$$

By substituting Eq. (9) into Eqs. (22) and (23), the modified differential equations are derived as follows

$$(1+\tau_h D_{\tau_h})\frac{\partial^2}{\partial t^2}(T-\eta C)=\left(\frac{\partial^2}{\partial r^2}+\frac{D-2}{r}\frac{\partial}{\partial r}\right)\left(L_h\frac{\partial T}{\partial t}+L_h^* T\right), \tag{24}$$

$$(1+\tau_m D_{\tau_m})\frac{\partial^2}{\partial t^2}(C-\lambda T)=\left(\frac{\partial^2}{\partial r^2}+\frac{D-2}{r}\frac{\partial}{\partial r}\right)\left(K_m\frac{\partial C}{\partial t}+K_m^* C\right). \tag{25}$$

2.3 Fractal media basics in hygrothermoelastic field

The constitutive relations for hygrothermoelasticity in nonfractal material:

$$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}-K\alpha_t\gamma\,\delta_{ij}, \tag{26}$$

with

$$\gamma=\gamma_1(T-T_0)+\gamma_2(C-C_0). \tag{27}$$

The bulk modulus is related to the Lame coefficients [53] by

$$K=\lambda+(2\mu)/D. \tag{28}$$

Using Eqs. (26) and (28), one obtains the constitutive relations for fractal material as

$$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}-(D\lambda+2\mu)\alpha_t\gamma\delta_{ij}. \tag{29}$$

The hygrothermoelasticity equation for nonfractal material is given as

$$\lambda\,grad\,div\,\vec{u}+2\mu\Delta\vec{u}+\vec{f}-(3\lambda+2\mu)\alpha_t grad\,\gamma=\rho D_t^2\vec{u}, \tag{30}$$

where $\vec{f}$ is the vector field of the external force density.

Regarding the case $\vec{u}=\vec{u}(r,t)$ and $\vec{f}=f(r,t)$, Eq. (28) gives

$$(\lambda+2\mu)\Delta\vec{u}+\vec{f}-(3\lambda+2\mu)\alpha_t grad\,\gamma=\rho D_t^2\vec{u}. \tag{31}$$

The equation with $\vec{f}=f_r(r,t)\vec{e}_r$ and $\vec{u}=u_r(r,t)\vec{e}_r$ for fractal material:

$$(\lambda+2\mu)^V\Delta_r^D\,\vec{u}+\vec{f}-(D\lambda+2\mu)\alpha_t grad_r^D\,\gamma=\rho D_t^2\vec{u}. \tag{32}$$

The strain tensor $\varepsilon_{ij}(r)$ which may be expressed as

$$\begin{aligned}\varepsilon_{rr}&=(\vec{e}_r,grad_r^D u_r)=\frac{\partial u_r}{\partial r},\\ \varepsilon_{\theta\theta}&=div_r^D\vec{u}-(\vec{e}_r,grad_r^D u_r)=\frac{D-2}{r}u_r,\end{aligned} \tag{33}$$

as well as the invariant is considered

$$e=\varepsilon_{kk}=div_r^D\vec{u}=\frac{\partial u_r}{\partial r}+\frac{D-2}{r}u_r. \tag{34}$$

Using Eq. (8), we can rewrite Eq. (32) as

$$\frac{\partial^2 u_r(r,t)}{\partial r^2}+\frac{D-2}{r}\frac{\partial u_r}{\partial r}-\frac{D-2}{r^2}u_r(r,t)+\frac{1}{(\lambda+2\mu)}f_r(r,t)$$
$$-\frac{\alpha(D\lambda+2\mu)}{\lambda+2\mu}\frac{\partial T(r,t)}{\partial r}=\frac{\rho}{\lambda+2\mu}D_t^2 u_r(r,t). \tag{35}$$

Using Eq. (29), the constitutive relations are reformulated for fractal materials:

$$\sigma_{rr}=(\lambda+2\mu)\frac{\partial u_r}{\partial r}+\lambda\frac{D-2}{r}u_r-(D\lambda+2\mu)\alpha_t\gamma,$$
$$\sigma_{\theta\theta}=\lambda\frac{\partial u_r}{\partial r}+(\lambda+2\mu)\frac{D-2}{r}u_r-(D\lambda+2\mu)\alpha_t\gamma, \tag{36}$$

and the motion equation when there is no body force as

$$(\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}+\frac{D-2}{r}u_r\right)-(D\lambda+2\mu)\alpha_t\frac{\partial \gamma}{\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2}. \tag{37}$$

3. Statement of problem

The hollow cylinder occupies the space $D\subset\mathbf{R}^3$ defined by *B*, as illustrated in Figure 1. The region $D=\{(r,\theta,z)\subset\mathbf{R}^3\,|\,r_1\le r\le r_2, 0<\theta\le 2\pi, -\infty<z<+\infty\}$ represents the space occupied by the hollow cylinder, with the radial coordinate ranging from $r_1\le r\le r_2$, and the axial coordinate *z* extending infinitely in both directions along the length of the cylinder from $-\infty$ to ∞.

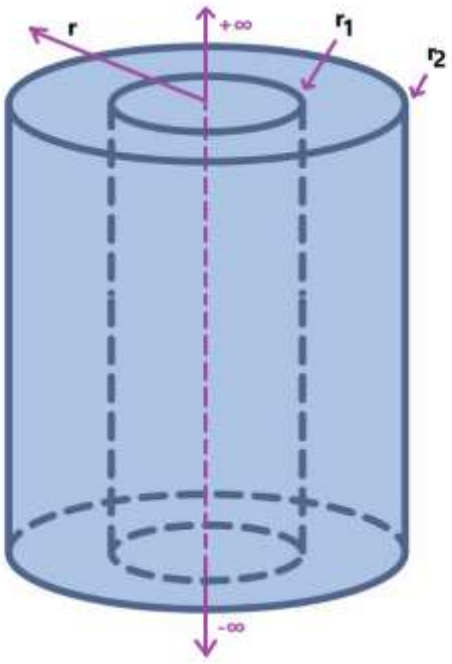


Figure 1. Schematic portrayal depicting the cylindrical geometry

We introduce the suitable nondimensional variables that are

$$\begin{aligned}\widehat{\Upsilon} &= \frac{\Upsilon}{k}\sqrt{\frac{\lambda+2\mu}{\rho}},\ \widehat{t} = \frac{t}{k}\left(\frac{\lambda+2\mu}{\rho}\right), \widehat{\tau}_i = \frac{\widehat{\tau}_i}{k}\left(\frac{\lambda+2\mu}{\rho}\right), \widehat{\sigma} = \frac{\sigma}{\mu},\\ \widehat{\theta} &= \frac{T}{T_0}, \widehat{\varphi} = \frac{C}{C_0}, \alpha^2 = \left(\frac{\lambda+2\mu}{\mu}\right), i = Th, Cm,\ \aleph = \frac{K_m^* \rho}{\lambda+2\mu},\ \wp = \frac{L_h^* \rho}{\lambda+2\mu},\end{aligned} \tag{38}$$

and henceforth, we omit the prime hat for the sake of simplicity.

The equations for heat transfer and moisture are represented in non-dimensional form:

$$(1+\tau_h D_{\tau_h})\frac{\partial^2}{\partial t^2}(\theta-\eta\varphi) = \left(\frac{\partial^2}{\partial r^2}+\frac{D-2}{r}\frac{\partial}{\partial r}\right)\left(L_h\frac{\partial\theta}{\partial t}+\wp\theta\right), \tag{39}$$

$$(1+\tau_m D_{\tau_m})\frac{\partial^2}{\partial t^2}(\varphi-\lambda\theta) = \left(\frac{\partial^2}{\partial r^2}+\frac{D-2}{r}\frac{\partial}{\partial r}\right)\left(K_m\frac{\partial\varphi}{\partial t}+\aleph\varphi\right), \tag{40}$$

and the constitutive relations in fractal materials are

$$\begin{aligned}\sigma_{rr} &= \alpha^2\frac{\partial u_r}{\partial r}+(\alpha^2-2)\frac{D-2}{r}u_r-\alpha_1\gamma,\\ \sigma_{\theta\theta} &= (\alpha^2-2)\frac{\partial u_r}{\partial r}+\alpha^2\frac{D-2}{r}u_r-\alpha_1\gamma,\end{aligned} \tag{41}$$

$$\nabla^2 u_r-\alpha_2\frac{\partial\gamma}{\partial r} = \frac{\partial^2 u_r}{\partial t^2}. \tag{42}$$

The boundary conditions are represented as

$$\theta(r,0) = \frac{\partial\theta(r,0)}{\partial t} = 0, \varphi(r,0) = \frac{\partial\varphi(r,0)}{\partial t} = 0, \tag{43}$$

$$\theta(r_1,t) = 0, \varphi(r_1,t) = 0. \tag{44}$$

The boundaries of a hollow cylinder are traction-free; physically, there is no mechanical load applied to the inner and outer boundary surfaces

$$\theta(r_2,t) = g_1(t), \varphi(r_2,t) = g_2(t), \sigma_{rr}(r_1,t) = \sigma_{rr}(r_2,t) = 0\ , \tag{45}$$

Where,

$$\alpha_1 = \frac{(D\lambda+2\mu)\alpha_t}{\mu}, \alpha_2 = \frac{(D\lambda+2\mu)\alpha_t}{\lambda+2\mu},$$

$$g_i(t) = \sin(\omega_i t), \omega_i > 0, i = 1,2 \tag{46}$$

4. Solution of the hygrothermal in the Laplace domain

Utilizing the Laplace transformation stated as

$$\bar{f}(\Upsilon,s) = \mathrm{L}[f(\Upsilon,t)] = \int_0^\infty e^{-st} f(\Upsilon,t)dt, \mathrm{Re}(s) > 0. \tag{47}$$

By utilizing the convolution theorem, the memory derivative of higher order D_{τ_i} is subsequently determined by applying the Laplace transform, fulfilling the property

$$\mathrm{L}[\tau_i D^p_{\tau_i} f(r,t)] = \mathrm{L}\left[\int_t^{t+\omega} K(t-\xi) f^p(r,\xi) d\xi\right] = s^{p+1} G(s,\tau_i)\mathrm{L}[f(r,t)], \tag{48}$$

with kernel function [55] as

$$K(t-\xi) = \frac{2f}{\tau}(t-\xi) + \frac{e^2}{\tau^2}(t-\xi)^p = \begin{cases} \dfrac{(t-\xi)}{\tau}, & \text{if } e=0, f=\dfrac{1}{2}, p=0 \\ \left(\dfrac{t-\xi}{\tau}\right)^2, & \text{if } e=1, f=1, p=1 \end{cases}, \tag{49}$$

where

$$G(\tau,s) = \begin{cases} \dfrac{1-e^{-s\tau}}{s\tau}; & K(t-\xi) = \dfrac{t-\xi}{\tau} \\ \dfrac{2}{s\tau} + \dfrac{2(1-e^{-s\tau})}{s^2\tau^2}; & K(t-\xi) = \left(\dfrac{t-\xi}{\tau}\right)^2 \end{cases}, \tag{50}$$

Applying Laplace transforms to Eqs. (38)-(44), one obtains

$$(1+G(\tau_h,s))s^2(\bar{\theta} - \eta\bar{\varphi}) = \nabla^2\bar{\theta}\left(L_h s + \wp\right), \tag{51}$$

$$(1+G(\tau_m,s))s^2(\bar{\varphi} - \lambda\bar{\theta}) = \nabla^2\bar{\varphi}\left(K_m s + \aleph\right), \tag{52}$$

$$\nabla^2\bar{u}_r - s^2\bar{u}_r = \alpha_2\bar{\gamma}, \tag{53}$$

$$\begin{aligned} \bar{\sigma}_{rr} &= \alpha^2\frac{\partial\bar{u}_r}{\partial r} + (\alpha^2-2)\frac{D-2}{r}\bar{u}_r - \alpha_1\bar{\gamma}, \\ \bar{\sigma}_{\theta\theta} &= (\alpha^2-2)\frac{\partial\bar{u}_r}{\partial r} + \alpha^2\frac{D-2}{r}\bar{u}_r - \alpha_1\bar{\gamma}. \end{aligned} \tag{54}$$

Introducing potential function Ψ, which is defined as

$$\bar{u}_r = \frac{\partial\bar{\Psi}}{\partial r}. \tag{55}$$

Employing Eq. (43) in Eq. (46), one can obtain,

$$\nabla^2\bar{\Psi} - s^2\bar{\Psi} = \alpha_2\bar{\gamma}, \tag{56}$$

$$\theta(r_1,s) = 0, \varphi(r_1,s) = 0, \tag{57}$$

$$\theta(r_2,s) = g_1(s), \varphi(r_2,s) = g_2(s), \tag{58}$$

$$g_i(s) = \omega_i / (s^2 + {\omega_i}^2). \tag{59}$$

Simultaneously eliminating $\bar{\varphi}, \bar{\Psi}$ from Eqs. (51), (52), and (53), ones obtain

$$[(K_m s-\aleph)+(L_h s-\wp)]\nabla^6\bar{\theta}-[(L_h s-\wp)(K_m s-\aleph)s^2-s^2\ell_m(L_h s-\wp)$$
$$-s^2\ell_h(K_m s-\aleph)]\nabla^4\bar{\theta}+[s^4\ell_h\ell_m(-1+\eta\lambda)-(L_h s-\wp)s^4\ell_m- \quad (60)$$
$$s^4\ell_h(K_m s-\aleph)]\nabla^2\bar{\theta}-s^6\ell_h\ell_m(-1+\eta\lambda)\bar{\theta}=0.$$

The characteristic form of equation (60) for order sixth is

$$[\nabla^6-H_1\nabla^4+H_2\nabla^2-H_3]\bar{\theta}=0. \quad (61)$$

where

$$H_1=\frac{(L_h s-\wp)(K_m s-\aleph)s^2-s^2\ell_m(L_h s-\wp)-s^2\ell_h(K_m s-\aleph)}{(K_m s-\aleph)+(L_h s-\wp)},$$
$$H_2=\frac{s^4\ell_h\ell_m(-1+\eta\lambda)-(L_h s-\wp)s^4\ell_m-s^4\ell_h(K_m s-\aleph)}{(K_m s-\aleph)+(L_h s-\wp)}, \quad (62)$$
$$H_3=\frac{s^6\ell_h\ell_m(-1+\eta\lambda)}{(K_m s-\aleph)+(L_h s-\wp)}.$$

The equation (61) is factorized as

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)(\nabla^2-m_3^2)\bar{\theta}=0. \quad (63)$$

Where m_1, m_2, m_3 are the roots of positive real numbers, then the roots of the characteristic equation (61) can be written as

$$m_1^2=\frac{1}{3}(2\sin q+H_1),$$
$$m_2^2=-\frac{n}{3}(\sqrt{3}\cos q+\sin q)+\frac{H_1}{3}, \quad (64)$$
$$m_3^2=-\frac{n}{3}(\sqrt{3}\cos q-\sin q)+\frac{H_1}{3}.$$

where

$$n=-\sqrt{H_1^2-3H_2},\ q=\frac{1}{3}\sin^{-1}p,\ p=-\frac{2H_1^3-9H_1H_2+27H_3}{3n^3}. \quad (65)$$

The solution of equation (61) is written as

$$(\bar{\theta},\bar{\varphi},\bar{\Psi})=\sum_{i=1}^{3}\{D_i,D_i',D''\}I_0(m_i r)+\{E_i,E_i',E''\}K_0(m_i r), \quad (66)$$

where $I(.)$ and $K(.)$ are the modified Bessel functions of the first and second kinds and D_i, E_i , D_i', D_i'' , E_i', E_i'' are the parameters depending on s can be obtained as

$$D_i'=\frac{\left[(1+\tau_h D_{\tau_h})s^2-m_i^2L_h s-m_i^2\wp)\right]}{(1+\tau_h D_{\tau_h})s^2\eta}D_i, \quad (67)$$

$$E_i' = \frac{\left[(1+\tau_h D_{\tau_h})s^2 - m_i^2 L_h s - m_i^2 \wp\right]}{(1+\tau_h D_{\tau_h})s^2\eta} E_i, \tag{68}$$

$$D_i'' = \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta} D_i, \tag{69}$$

$$E_i'' = \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta} E_i. \tag{70}$$

Substituting Eqs. (64)-(67) in (62) and (63), one obtains

$$\bar{\varphi} = \sum_{i=1}^{3} \frac{\left[(1+\tau_h D_{\tau_h})s^2 - m_i^2 L_h s - m_i^2 \wp)\right]}{(1+\tau_h D_{\tau_h})s^2\eta}\left[D_i I_0(m_i r) + E_i K_0(m_i r)\right], \tag{71}$$

$$\bar{\Psi} = \sum_{i=1}^{3} \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta}\left[D_i I_0(m_i r) + E_i K_0(m_i r)\right]. \tag{72}$$

On differentiating equation (72) displacement u_r, one obtains

$$\bar{u}_r = \sum_{i=1}^{3} \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta}\left[D_i I_1(m_i r) + E_i K_1(m_i r)\right]. \tag{73}$$

Using Eqs. (72), (73), one obtains stresses

$$\begin{aligned}\bar{\sigma}_{rr} = &\sum_{i=1}^{3} \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta} \\ &\times\left\{\begin{array}{l}\left(\alpha^2 I_0(m_i r) - \frac{m_i}{r} I_1(m_i r) + (\alpha^2 - 2)\frac{D-2}{r} I_1(m_i r)\right) D_i + \\ \left(\alpha^2 K_0(m_i r) - \frac{m_i}{r} K_1(m_i r) + (\alpha^2 - 2)\frac{D-2}{r} K_1(m_i r)\right) E_i\end{array}\right\} - \alpha_1\bar{\gamma},\end{aligned} \tag{74}$$

$$\begin{aligned}\bar{\sigma}_{\theta\theta} = &\sum_{i=1}^{3} \frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta) - m_i^2(L_h s+\wp)\right)\right]}{(m_i^2 - s^2)(1+\tau_h D_{\tau_h})s^2\eta} \\ &\times\left\{\begin{array}{l}\left((\alpha^2 - 2) I_0(m_i r) - \frac{m_i}{r} I_1(m_i r) + \alpha^2\frac{D-2}{r} I_1(m_i r)\right) D_i + \\ \left((\alpha^2 - 2) K_0(m_i r) - \frac{m_i}{r} K_1(m_i r) + \alpha^2\frac{D-2}{r} K_1(m_i r)\right) E_i\end{array}\right\} - \alpha_1\bar{\gamma},\end{aligned} \tag{75}$$

To evaluate the unknown parameter D_i, E_i for (i=1,2,3), using the boundary condition in Laplace domain with Eqs. (56)-(59), one obtains

$$\begin{aligned}&\sum_{i=1}^{3} \frac{\left[(1+\tau_h D_{\tau_h})s^2 - m_i^2 L_h s - m_i^2 \wp)\right]}{(1+\tau_h D_{\tau_h})s^2\eta}, \\ &\times\left[D_i I_0(m_i r_1) + E_i K_0(m_i r_1)\right] = 0\end{aligned} \tag{76}$$

$$\sum_{i=1}^{3}\frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta)-m_i^2(L_h s+\wp)\right)\right]}{(m_i^2-s^2)(1+\tau_h D_{\tau_h})s^2\eta} \times[D_i I_0(m_i r_1)+E_i K_0(m_i r_1)]=0, \tag{77}$$

$$\sum_{i=1}^{3}\frac{\left[(1+\tau_h D_{\tau_h})s^2-m_i^2 L_h s-m_i^2\wp)\right]}{(1+\tau_h D_{\tau_h})s^2\eta} \times[D_i I_0(m_i r_2)+E_i K_0(m_i r_2)]=g_1(s), \tag{78}$$

$$\sum_{i=1}^{3}\frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta)-m_i^2(L_h s+\wp)\right)\right]}{(m_i^2-s^2)(1+\tau_h D_{\tau_h})s^2\eta} \times[D_i I_0(m_i r_2)+E_i K_0(m_i r_2)]=g_2(s), \tag{79}$$

$$\sum_{i=1}^{3}\frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta)-m_i^2(L_h s+\wp)\right)\right]}{(m_i^2-s^2)(1+\tau_h D_{\tau_h})s^2\eta} \times\left\{\begin{array}{l}\left(\alpha^2 I_0(m_i r)-\frac{m_i}{r}I_1(m_i r_1)+(\alpha^2-2)\frac{D-2}{r}I_1(m_i r_1)\right)D_i+\\ \left(\alpha^2 K_0(m_i r)-\frac{m_i}{r}K_1(m_i r_1)+(\alpha^2-2)\frac{D-2}{r}K_1(m_i r_1)\right)E_i\end{array}\right\}-\alpha_1\bar{\gamma}=0, \tag{80}$$

$$\sum_{i=1}^{3}\frac{\alpha_2\left[\left((1+\tau_h D_{\tau_h})s^2(1+\eta)-m_i^2(L_h s+\wp)\right)\right]}{(m_i^2-s^2)(1+\tau_h D_{\tau_h})s^2\eta} \times\left\{\begin{array}{l}\left(\alpha^2 I_0(m_i r)-\frac{m_i}{r}I_1(m_i r_2)+(\alpha^2-2)\frac{D-2}{r}I_1(m_i r_2)\right)D_i+\\ \left(\alpha^2 K_0(m_i r)-\frac{m_i}{r}K_1(m_i r_2)+(\alpha^2-2)\frac{D-2}{r}K_1(m_i r_2)\right)E_i\end{array}\right\}-\alpha_1\bar{\gamma}=0. \tag{81}$$

On solving the linear system of the Eqs. (76) – (81), we obtain the parameters D_i, E_i (i=1,2,3). This provides the parameter solution in the Laplace transform domain.

5. Fourier series method for the numerical inversion

The Laplace inverse transform is expressed as

$$h(x,t)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\exp(st)\,\bar{h}(x,s)ds, \tag{82}$$

with d is a stochastic real number that exceeds the singularities of the actual components

of $\bar{h}(x,s)$. Let $s=d+i\,w$, then the previous integral form takes

$$h(x,t)=\frac{\exp(dt)}{2\pi i}\int_{-\infty}^{\infty}e^{itw}\bar{h}(x,d+iw)dw. \tag{83}$$

Intensifying the approximate formula for the function $p(x,t)=e^{-dt}h(x,t)$ in a Fourier series with the interval [*0, 2T*]. [58]

$$h(x,t)=h_{\infty}(x,t)+E_D, \tag{84}$$

where

$$h_{\infty}(x,t)=\frac{1}{2}C_0+\sum_{k=1}^{\infty}C_k \text{ for } 0\le t\le 2T, \tag{85}$$

and

$$C_k=\frac{\exp(dt)}{T}\left\{\exp\left[ik\pi t/T\,\overline{f}\left(x,d+i\frac{k\pi t}{T}\right)\right]\right\}. \tag{86}$$

By selecting d large enough, one can make the discretization error arbitrarily low [38]. Since N is a finite number of terms that can be obtained by adding the infinite sequences in Eq. (85), the projected sum of $h(x,t)$

$$h_N(x,t)=\frac{1}{2}C_0+\sum_{k=1}^{N}C_k \text{ for } 0\le t\le 2T. \tag{87}$$

Utilizing the previously mentioned formula $h(x,t)$, to generate the overall approximation error, we add E_D a truncation error that must be added to the discretization error. Initially, decrease the discretization error using the "Korrektur" method to lower the overall error. The convergence is then accelerated using the ε-algorithm [59]. Using the formula below to evaluate $h(x,t)$:

$$h(x,t)=h_{\infty}(x,t)-\exp(-2dT)\,h_{\infty}(x,2T+t)+E'_D, \tag{88}$$

with the discretization error $|E'_D|\square\ |E_D|$.

Therefore, the approximate value of $h(x,t)$ yields

$$h_{N_K}(x,t)=h_N(x,t)-\exp(-2dT)h_{N'}(x,2T+t), \tag{89}$$

where an integer is N' as $N'>N$. Also, in order to accelerate the convergence of the series in Eq. (87), we define the ε-algorithm. Let $S_m=\sum_{k=1}^{m}C_k$ be the sequence of partial sums of the series in (87) and $N=2q+1$, here q is a natural number. The ε-sequence is well defined by $\varepsilon_{0,m}=0,\ldots\varepsilon_{1,m}=S_m$ and $\varepsilon_{p+1,m}=\varepsilon_{p-1,m+1}+1/(\varepsilon_{p,m+1}-\varepsilon_{p,m}), p=1,2,3,\ldots.$ It can be shown [59] that the sequence $\varepsilon_{1,1},\varepsilon_{3,1},\varepsilon_{5,1},\ldots\varepsilon_{N,1}$ converges to $h(x,t)+E_D-C_0/2$ quicker than the sequence of partial sums $s_m\,(m=1,2,3,..)$. The actual procedure used to invert the

Laplace transform consists of using Eq. (89) together with the ε-algorithm. The values of d and T are chosen according to the criterion outlined in [59].

6. Numerical Results, Discussion, and Remarks

In this section, we present the properties of temperature response and moisture response of fractal materials using an approach to generalized vector calculus for spaces with non-integer dimensions. It frames hygrothermoelastic equations and analyses the impact of thermal time-delay factors τ_q, moisture time-delay factor τ_m, future kernel functions $K(t-\xi)$ on temperature, moisture, displacement, and stress. Numerical calculations were performed using a fiber-reinforced composite hollow cylinder composed of Thornel T300 graphite fibers and Narmco 5208 epoxy resin. One of the composite's features is that it can do numerical simulations:

$$\gamma_1 = 31.3\times10^{-6}\ \text{cm/cm}\times\text{K},\ \gamma_2 = 2.68\times10^{-3}\ \text{cm/cm}(wt\%H_2O),$$

$$L = 6.90\times10^{-6}\ \text{cm}^2/\text{hr},\ D = 6.90\times10^{-7}\ \text{cm}^2/\text{hr},\ k = 386\,\text{kg m K}^{-1}\text{s}^{-3},$$

$$\alpha_t = 1.78\times10^{-5}\ \text{K}^{-1},\ \ \rho = 895\,\text{kg m}^{-3},\ \ \mu = 3.86\times10^{10}\ \text{kg m}^{-1}\text{s}^{-2},$$

$$\lambda = 7.76\times10^{10}\ \text{kg m}^{-1}\text{s}^{2},\ \eta_c = 0.122\,\text{kg/m}^3\text{K},\ \lambda_c = 2.053\,\text{m}^3\text{K/kg}.$$

6.1 Effect of kernel function response along the radius

Figures 1 and 2 illustrate the radial temperature and moisture distributions at different time intervals. The kernel function $\kappa(t-\xi) = \left((t-\xi)/\tau_i\right)^2$ with $e=1, f=1, b=0.5, \xi=0.01$ is indicated by the dashed lines, while the kernel function $\kappa(t-\xi) = \left((t-\xi)/\tau_i\right)$ is shown by the solid lines with $e=0, f=0, b=0.5, \xi=0.01$. Several factors affect how heat is distributed within a system. These include time delays, rate of moisture change, memory-dependent derivatives, and the temperature conditions at the system's boundaries. Increasing moisture adds heat while decreasing moisture removes it. Typically, the core of the system maintains a relatively stable temperature, while the temperature rises as one moves outward towards the edge, as illustrated in Figure 1. Our analysis indicates a direct relationship between temperature and phase delay in a fractal medium with a future memory kernel function. Also, the magnitude of the nonfractal curves scenario exhibits higher temperatures than the fractal scenario along the radius with a harmonically varying heat source. This temperature difference is more pronounced closer to the center and

diminishes towards the outer edge. Figure 2 shows the distribution of moisture curves along the radius. This distribution of curves is impacted by temperature rate change and a harmonically varying heat source with a future memory kernel function. In the early stages, at the surface, the temperature fluctuates more quickly, and the moisture content is comparatively low. As time progresses, the moisture spreads inward from the boundary, and its concentration diminishes, ultimately becoming negligible.

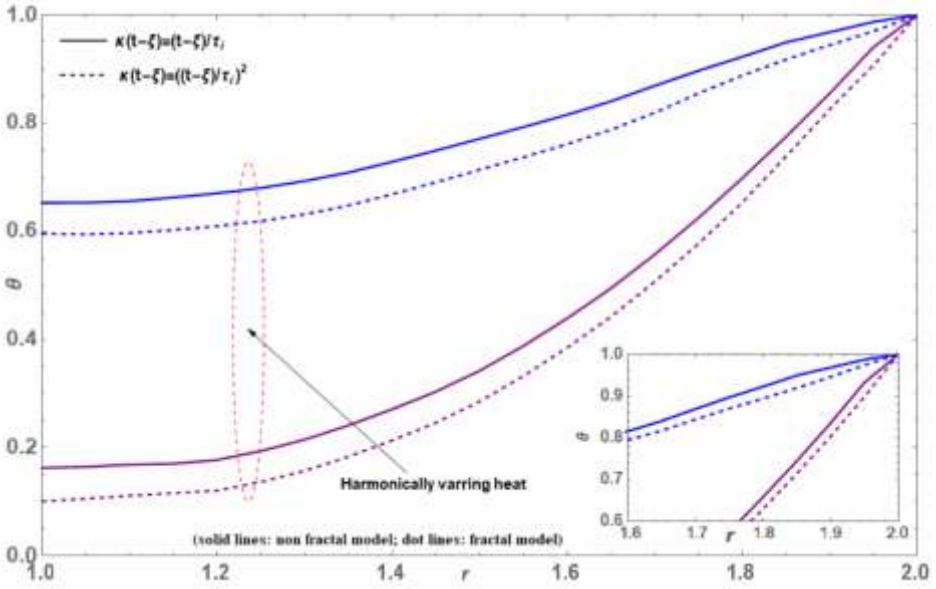


Figure 1. Temperature profile along r for different kernels

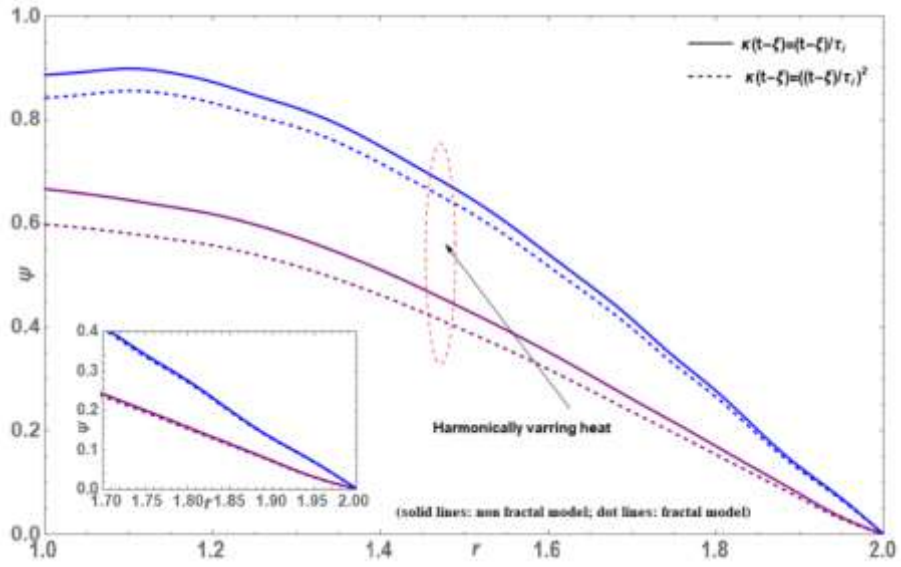


Figure 2. Moisture profile along for different kernels

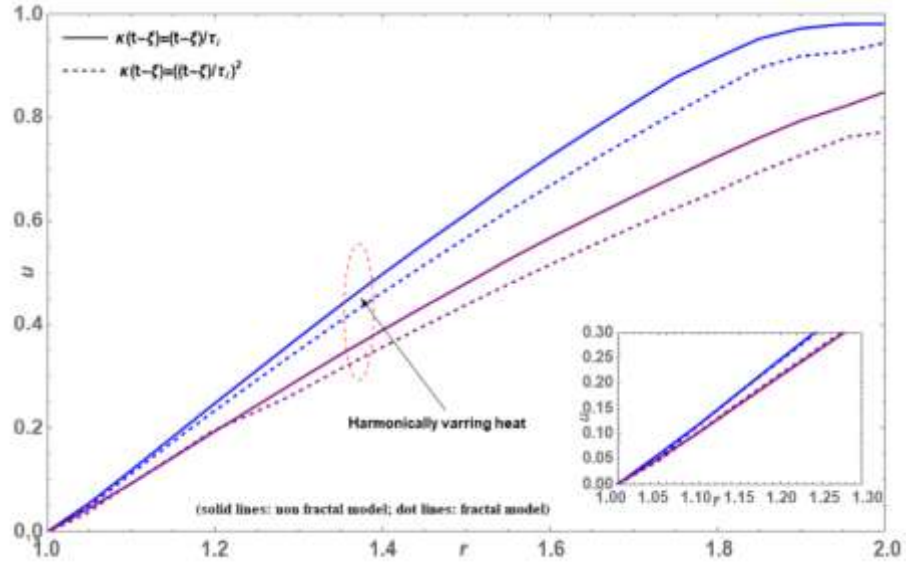


Figure 3. Displacement profile along r for different kernels

Figure 3 shows nondimensional displacement curve variation along the radius within a fractal medium for future nonlinear kernel functions. Harmonically varying heat sources affect the displacement curves. At first, the fractal case exhibits a low-magnitude displacement trend, then expands outwards for different periods. The future kernel function and fractal dimension influence the displacement curves distribution and magnitude.

Figures 4 and 5 show the distribution of nondimensional radial and hoop stress along the radius for various future kernels in the fractal media. For various time slots, in the steady state, the radial stress finally decreases to zero as it transitions from tensile to compressive. Restricted thermal expansion keeps the stresses continuously compressive. The fractal media and harmonically varying heat source have an impact on this distribution. The fractal curves are lower in magnitude than nonfractal curves; also, the fractal structure can result in changes and concentrations of stress that are not observed in traditional materials. After a period, the peak compressive stress's magnitude increases over time, but it may do so at a different rate or follow a distinct pattern because of the fractal nature of the material.

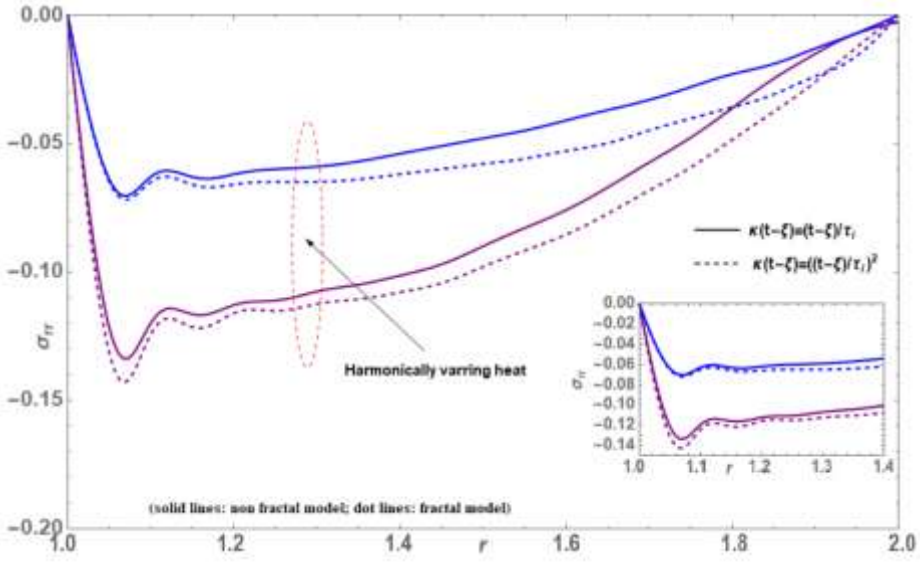


Figure 4. σ_{rr} *along* r *for different kernels*

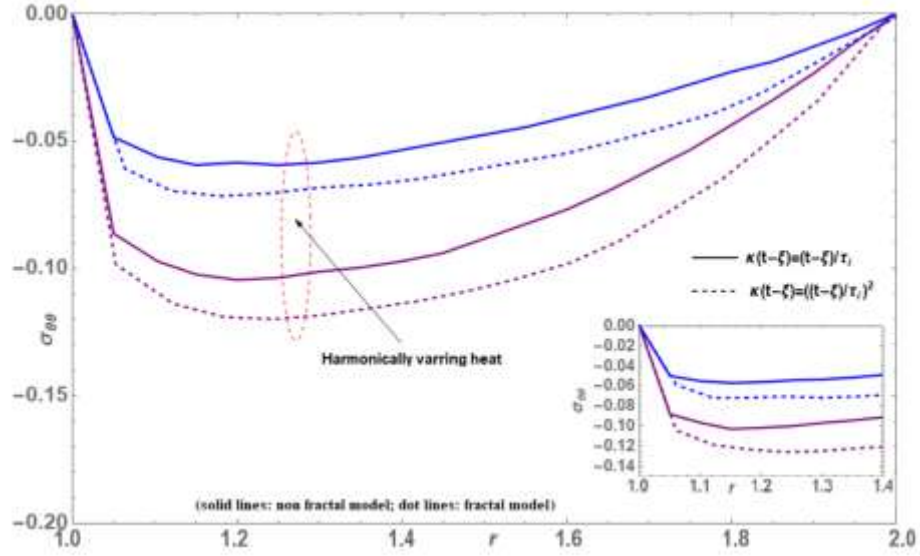


Figure 5. $\sigma_{\theta\theta}$ *along* r *for different kernels*

6.2 Histories of hygrothermal coupling over time

Figures 6 and 7 visually depict the temperature and humidity change over time at different radial points. These figures illustrate the interplay between temperature, humidity, and time for nonlinear future kernels within fractal media. Temperature changes are affected by both humidity levels and how heat propagates through the material. Increasing humidity adds heat to the system while decreasing humidity absorbs heat. Furthermore, the amount of moisture present influences how quickly heat moves. This relationship is clearly illustrated in Figure 6.

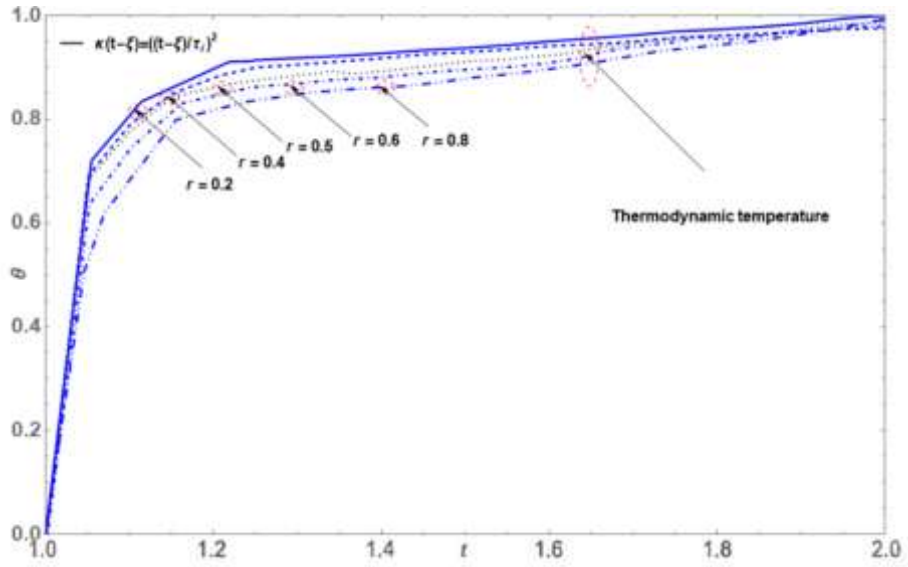


Figure 6. Temperature distribution in a cylinder at different ratios

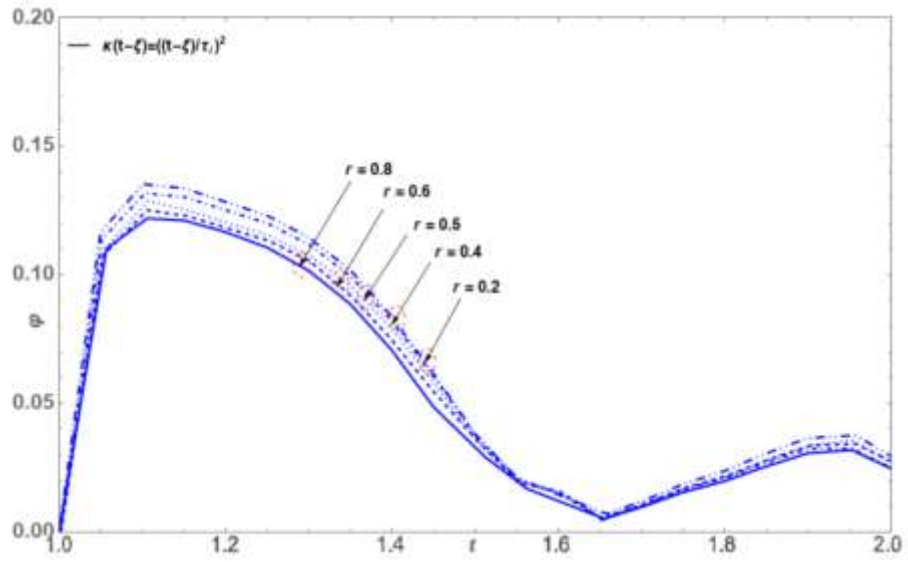


Figure 7. Moisture distributions at time t=0.1 with different ratios

Figure 7 shows that humidity initially reaches a maximum value, then gradually decreases and eventually stabilizes. This behaviour is likely caused by a sudden shift in temperature and the movement of water molecules. The timing of these peak values varies, which can be explained by the different times it takes for the combined heat and moisture front to reach different locations. Initially, the changes in humidity are primarily driven by how quickly water molecules spread through the material.

7. Conclusion

The study effectively employs a generalized vector calculus for non-integer dimensions to study the hygrothermal behavior of fractal materials. It presents these materials as continua within non-integer dimensional spaces, formulating and deriving hygrothermoelasticity equations. The study also investigates thermoelastic and moisture interaction, analyzing the impact of moisture and temperature loading on the boundary plane. The introduction of a hygrothermoelastic model, which integrates a future memory-dependent heat transport law with phase lag, enhances the understanding of complex interactions within fractal materials. The study emphasizes the importance of including future memory concepts with harmonically varying heat sources for a comprehensive understanding of thermoelastic and moisture responses in fractal materials. This study focuses on the thermoelastic and moisture properties of fractal materials. The findings can guide nanostructure design and processing, potentially leading to more advanced and optimized materials for various applications. The results of this work have practical implications for engineering design, particularly in situations necessitating precise control over thermal and mechanical properties. This includes applications in aerospace, automotive, and civil engineering where material performance under varying thermal conditions is critical. Also, the findings are relevant to the biomedical field, informing the design of implants and prosthetics. In these applications, carefully controlling thermal and mechanical properties is essential for ensuring biocompatibility and proper function within the body. Still, fractal materials with carefully designed thermoelastic properties hold promise for improving the efficiency and performance of energy storage and conversion technologies. Future studies should develop more comprehensive models and theories capable of describing complex material behavior, such as those found in multi-dimensional and multi-phase systems. These models will require experimental verification to ensure accuracy and to enable the tailoring of material properties for specific applications. In conclusion, this study's findings suggest numerous avenues for both the practical use and continued investigation of fractal materials and their thermoelastic behaviour. Incorporating the concepts of memory and nonlocality is a particularly exciting direction for future research in materials science and engineering, potentially leading to significant advancements across diverse industries.

8. Declarations

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8.2 Conflicts of Interest

The author has no conflict of interest to declare that are relevant to the content of this article.

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