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A Unified Space–Time Finite Element Solution Framework for Electromagnetic–Mechanical Coupling Problems

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Abstract

Electromagnetic–mechanical coupling problems are widely encountered in engineering applications. Traditional methods mostly adopt a solution paradigm that combines spatial finite element semi-discretization with temporal integration schemes. This paradigm may introduce numerical dissipation, energy drift, or fail to preserve the underlying symplectic structure during long-term transient simulations. To address these issues, this paper introduces a space–time finite element mesh and establishes a unified space–time discrete variational framework for electromagnetic–mechanical coupling problems, thereby avoiding the conventional treatment that separates spatial discretization from temporal integration. Based on the discrete variational formulation, a unified one-step recursive scheme applicable to both nondissipative and dissipative systems is constructed, enabling the displacement field and electromagnetic fields to be solved within a unified discrete framework. Through three benchmark examples—including pure electromagnetic field problems, elastodynamic problems, and sequentially coupled electromagnetic–mechanical problems—the proposed method demonstrates its capability to accurately solve different types of physical problems within a unified framework and preserve the symplectic structure of nondissipative systems. This paper provides a unified computational approach for electromagnetic

fields, structural dynamics, and electromagnetic–mechanical coupling problems, enabling accurate and long-term stable transient simulations.

Keywords: electromagnetic-mechanical coupling, space-time finite element method, discrete variational method, symplecticity-preserving, recursive scheme

1 Introduction

The coupling between electromagnetic fields and mechanical deformation is a common physical phenomenon in power equipment, pulsed electromagnetic devices, and precision electromagnetic structures. When conductors are subjected to large-amplitude alternating currents, time-varying electromagnetic fields induce eddy currents within the conductor, which interact with the magnetic field to generate transient Lorentz forces, thereby exciting elastic vibrations and dynamic deformations of the structure [1]. Although the deformation amplitudes remain small under normal operating conditions, the transient electromagnetic forces may reach tens or even hundreds of megapascals. Repeated electromagnetic loading can induce structural fatigue and cumulative damage, while excitation frequencies close to the natural frequencies of the structure may lead to resonant amplification [2]. Therefore, the accurate prediction of the transient mechanical response of conducting structures under electromagnetic forces is of considerable engineering significance for the safety assessment and fatigue life evaluation of power equipment.

In response to this engineering demand, researchers have carried out extensive numerical modeling and simulation work for various electromagnetic–mechanical coupling problems. In the field of electromagnetic forming, Yu et al. [1] implemented a sequential coupling simulation of electromagnetic tube compression on the ANSYS multiphysics platform, employing spatial finite element discretization combined with Newmark time integration. In the analysis of electromagnetic actuators and electric machines, Abba and Rachek [3] discretized the electromagnetic and displacement fields using two-dimensional triangular elements and treated the time derivatives with a weighted finite difference scheme, thereby establishing a transient coupled dynamic model for electromagnetic actuators. Merzouki et al. [4] constructed a strongly coupled model of the two-dimensional transient magnetic field and multi-turn winding circuits for permanent magnet synchronous motors, with triangular elements for spatial discretization and a Galerkin scheme for time marching, solving the resulting nonlinear algebraic system via Newton–Raphson iteration to achieve high-fidelity transient simulation of the motor start-up process. Rüberg et al. [5] developed a coupled finite element–boundary element simulation framework, discretizing the solid region with finite elements and the surrounding air region with boundary elements, and solved the quasi-static eddy current Maxwell problem using the backward Euler method. Gebhart et al. [6] addressed the difficulty of enforcing the Coulomb gauge constraint in nondissipative magneto-mechanical coupling problems by proposing three energy-based constraint schemes grounded in variational principles, thereby achieving a fully coupled solution of the static magneto-mechanical problem. Lei et al. [7] employed

an upwind finite volume method to solve the eddy current problem and simulated the complete transient electromagnetic–mechanical coupling process of a metallic projectile under pulsed electromagnetic coil excitation, from current-driven acceleration to free flight. Wang et al. [8] investigated the bidirectional magneto-mechanical coupling in magnetic shape memory alloys, spatially discretizing the displacement and magnetic fields using the absolute nodal coordinate formulation and the finite element method, respectively, and integrating the equations of motion in time with the generalized- α method to study the dynamic mechanical response of the material under cyclic compression. It can be seen from the above works that the prevailing numerical approaches for electromagnetic–mechanical coupling overwhelmingly follow the paradigm of spatial finite element semi-discretization paired with a dedicated time integrator. Within this paradigm, spatial discretization and temporal integration are two relatively independent methodological components: spatial accuracy is governed by finite element interpolation theory, whereas temporal accuracy is controlled by the integration scheme and step size [9]. While symplectic time-marching schemes can be combined with spatial FEM discretizations to preserve geometric structure, such constructions require deliberate, model-specific design [10, 11]. Without a tailored structure-preserving integrator, long-duration transient simulations may suffer from numerical dissipation or energy drift.

To fundamentally overcome the limitations inherent in the space–time separation paradigm, space-time finite element methods were introduced [12]. These methods treat time as an independent dimension on an equal footing with space and perform a unified discretization over the space-time domain, with temporal integration accomplished by finite element interpolation and numerical quadrature in the same manner as spatial integration. Consequently, the variational structure of the continuous problem can be naturally extended to the discrete level, providing a methodological foundation for preserving the geometric properties of the discrete system. In the context of electromagnetic problems, space-time finite element methods have achieved a series of significant advances. Gangl et al. [13] discretized the two-dimensional magneto-quasi-static eddy current equation using tensor-product space-time finite elements on a three-dimensional space-time computational domain, proved the unique solvability of both the continuous and discrete space-time variational formulations via the Babuška–Nečas theory, and substantially enhanced the computational efficiency of transient eddy current simulations of rotating electric machines through space-time parallelization. Hauser et al. [14] addressed the vectorial wave equation derived from the time-dependent Maxwell equations by constructing two conforming tensor-product space-time finite element formulations—a Galerkin–Petrov scheme and, via a modified Hilbert transformation, a Galerkin–Bubnov scheme; the former was shown to be conditionally stable, while the latter achieved unconditional stability, both attaining optimal convergence rates in space-time norms. Miyashita et al. [15] reformulated the scalar and vector Poisson equations under the Coulomb gauge as a single Hodge–Laplace problem on a four-dimensional space-time domain within the framework of finite element exterior calculus, and employed cubic space-time elements to solve a periodic steady-state electromagnetic field problem under a unified mathematical framework. The above works have achieved high-accuracy numerical

solutions to single-field electromagnetic problems using different space-time finite element strategies, yet none has incorporated the mechanical field into the discrete framework. In the domain of solid mechanics and multi-physics coupling, space-time finite element methods grounded in variational principles have also progressed rapidly. Wriggers et al. [16] addressed transient wave propagation and impact problems in one-dimensional elastic bars using the space-time virtual element method based on Hamilton’s variational principle, endowing the space-time mesh with the capability of locally varying time-step sizes; this framework was subsequently extended to two-dimensional elastodynamics [17]. Antonietti et al. [18] constructed a space-time discontinuous Galerkin discretization on tensor-product prismatic space-time meshes for the viscoelastic wave equation, and proved the stability and *hp*-convergence of the scheme in the energy norm through an appropriate choice of numerical fluxes. Junker et al. [19] proposed a unified variational material modeling framework on the space-time cylinder by extending the Hamilton principle of stationary action, employing continuous Galerkin discretization in space and discontinuous Galerkin discretization in time, and incorporating thermo-mechanically coupled wave propagation, viscoelasticity, elastoplasticity with hardening, and gradient-enhanced damage into a single space-time variational discrete system, for which well-posedness was established. From a more general mathematical perspective, Celledoni et al. [20] constructed arbitrary-order tensor-product space-time finite element discretizations and rigorously proved local and global energy conservation laws as well as discrete multisymplectic conservation laws, thereby providing a theoretical foundation for the structure-preserving analysis of space-time finite element methods. In addition, at the continuum variational level, Ricoeur and Wingen [21] established a unified weak formulation for coupled electrodynamic–thermomechanical four-field problems by merging Hamilton’s and Jourdain’s variational principles; however, this work remained at the level of continuum variational theory and did not provide a space-time-discrete solution scheme. It can be seen from the foregoing that current space-time finite element methods have achieved substantial progress for single-field electromagnetic problems as well as for mechanical and thermo-mechanical coupling problems. Nevertheless, the extension of space-time finite element methods to electromagnetic–mechanical coupling problems has rarely been reported, and the existing electrodynamic–mechanical coupling theory at the continuum variational level also lacks a corresponding space-time discrete solution scheme. This extension is nontrivial because electromagnetic and mechanical fields usually involve different finite element spaces and governing structures.

Based on the above background, this paper presents a unified space-time finite element framework for electromagnetic–mechanical sequential coupling problems, grounded in Hamilton’s variational principle. At the continuum level, the variational functionals for the full-wave electromagnetic field and for elastodynamics are separately formulated, and the total variational equation of the coupled system is constructed by incorporating the virtual work of the Lorentz force under the assumption of sequential coupling. At the discrete level, hexahedral–prismatic space-time grid elements are introduced: the magnetic vector potential is discretized with edges, while the electric scalar potential and the displacement field are discretized with nodes. A discrete variational principle for the coupled system is established on the space-time

grid, from which the discrete variational equations are derived. By partitioning and reordering the degrees of freedom according to time steps, a single-step recurrence solution scheme is obtained that involves only the states at two adjacent time instants. For the sequentially coupled problem, the electromagnetic and mechanical subsystems can each be advanced independently. Theoretical analysis shows that, in the nondissipative case, the transfer matrix of the recurrence scheme automatically satisfies the symplectic condition, and the discrete evolution preserves the symplectic structure of the continuum problem.

The remainder of this paper is organized as follows. Section 2 presents the governing equations of the electromagnetic–mechanical coupling problem in their continuous form and develops the associated variational principle. Section 3 introduces the space-time grid elements for discretizing the coupled system, establishes the space-time discrete variational principle for the electromagnetic–mechanical coupling problem, constructs a unified single-step recurrence solution scheme for the two fields, and proves the symplecticity of this scheme in the conservative case. Section 4 verifies the correctness of the proposed method through three numerical examples. Section 5 summarizes the work presented in this paper.

2 Governing equations of the electromagnetic–mechanical coupling problem

Consider a three-dimensional solid conductor occupying a domain $\Omega \subset \mathbb{R}^3$ as the control volume. Its electromagnetic field boundaries are divided into Γ_B and Γ_H , and its displacement field boundaries are divided into Γ_u and Γ_σ , with the boundaries corresponding to the same physical field being mutually non-overlapping. This paper solves for the electromagnetic field over the entire control domain as well as the deformation of the conductor itself induced by the Lorentz force and other external forces. The problem is formulated under the assumptions of linear isotropy, small deformations, and the absence of mechanical damping. The influence of displacement field variations on the electromagnetic field is neglected, and the motional electromotive force generated by conductor displacement as well as its effect on the geometric boundaries of the electromagnetic field are not taken into account; hence, this constitutes a sequential coupling problem. This section presents the governing equations of the problem in detail, including the governing equations of the electromagnetic field and those of elastodynamics accounting for electromagnetic forces, and constructs a unified continuous energy functional for the electromagnetic and displacement fields.

2.1 Governing equations of the full-wave electromagnetic field

For any electromagnetic field problem, the fundamental equations are Maxwell’s equations, which describe the relationships among the physical quantities of the

electromagnetic field. Their differential form is given as follows [22]

$$\begin{cases} \nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_s; \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \\ \nabla \cdot \mathbf{B} = 0; \\ \nabla \cdot \mathbf{D} = \rho_s, \end{cases} \quad (1)$$

where $\nabla \times$ denotes the curl operator, $\nabla \cdot$ denotes the divergence operator, $\mathbf{H}$ is the magnetic field intensity (A/m), $\mathbf{E}$ is the electric field intensity (V/m), $\mathbf{B}$ is the magnetic flux density (T), $\mathbf{D}$ is the electric flux density (C/m²), $\mathbf{J}_c$ is the conduction current density, $\mathbf{J}_s$ is the impressed source current density (both in A/m²), and ρ_v is the volumetric free charge density (C/m³). To render the above system uniquely solvable, appropriate boundary conditions must be supplied. On the outer boundary Γ_B , the normal component of the magnetic flux density is prescribed to be zero, and on the outer boundary Γ_H , the tangential component of the magnetic field intensity $\mathbf{H}$ is prescribed to be zero:

$$\begin{aligned} \mathbf{n} \cdot \mathbf{B} &= 0, \text{ on } \Gamma_B, \\ \mathbf{n} \times \mathbf{H} &= \mathbf{0}, \text{ on } \Gamma_H, \end{aligned} \quad (2)$$

where $\mathbf{n}$ denotes the outward unit normal vector on the boundary surface.

For problems without eddy current dissipation, the above electromagnetic field quantities satisfy the following constitutive relations

$$\begin{aligned} \mathbf{B} &= \mu \mathbf{H}, \\ \mathbf{D} &= \varepsilon \mathbf{E}, \end{aligned} \quad (3)$$

where μ is the magnetic permeability (H/m), numerically equal to the product of the vacuum permeability μ_0 and the relative permeability μ_r ($\mu = \mu_0 \mu_r$); ε is the permittivity (F/m), numerically equal to the product of the vacuum permittivity ε_0 and the relative permittivity ε_r ($\varepsilon = \varepsilon_0 \varepsilon_r$). For cases involving eddy current dissipation, the constitutive relation between the current density and the electric field intensity must also be introduced, corresponding to Ohm's law at the microscopic level

$$\mathbf{J}_c = \sigma \mathbf{E}, \quad (4)$$

where σ is the electrical conductivity (S/m).

Since a curl field is divergence-free, it follows from Eq. (1) that a vector potential $\mathbf{A}$ can be introduced to represent $\mathbf{B}$ via its curl:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (5)$$

where $\mathbf{A}$ is called the magnetic vector potential (T·m). Substituting Eq. (5) into the second equation of (1) yields

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = \mathbf{0}. \quad (6)$$

Since a scalar gradient field is irrotational, a scalar potential function φ can be introduced to facilitate the representation of $\mathbf{E}$:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi, \quad (7)$$

where ∇ denotes the gradient operator, and φ denotes the electric scalar potential in volts (V). Substituting Eqs. (5) and (7) into Eq. (1) together with the boundary condition (2) in combination with the constitutive equations (3)–(4), we can derive the full-wave Maxwell equations

$$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{A} + \sigma \dot{\mathbf{A}} + \sigma \nabla \varphi + \varepsilon \ddot{\mathbf{A}} + \varepsilon \nabla \dot{\varphi} = \mathbf{J}_s \quad (8)$$

and the boundary conditions

$$\begin{aligned} \mathbf{n} \times \mathbf{A} &= \mathbf{0} \text{ on } \Gamma_B, \\ \frac{1}{\mu} \nabla \times \mathbf{A} \times \mathbf{n} &= \mathbf{0} \text{ on } \Gamma_H. \end{aligned} \quad (9)$$

formulated in the $\mathbf{A}$ - φ gauge. Equation (8) only yields a vector equation for $\mathbf{A}$; an additional scalar constraint equation for φ is required to close the system of equations. Taking the divergence of both sides of Eq. (8), we can obtain the current continuity equation expressed in terms of $\mathbf{A}$ and φ

$$\nabla \cdot \left(\sigma \dot{\mathbf{A}} + \sigma \nabla \varphi + \varepsilon \ddot{\mathbf{A}} + \varepsilon \nabla \dot{\varphi} \right) = \nabla \cdot \mathbf{J}_s. \quad (10)$$

Equations (8)–(10) constitute the governing equations of electromagnetic fields for full-wave scenarios.

2.2 Governing equations of elastodynamics

For elastodynamic problems, three fundamental sets of equations are involved: the momentum balance equations, constitutive equations, and geometric equations, which are introduced separately here.

1. momentum balance equations

$$\rho \ddot{\mathbf{u}} = \mathbf{L}^T \boldsymbol{\sigma} + \mathbf{f}_v, \quad (11)$$

where ρ denotes the material density with units of kg/m³; $\mathbf{u}$ represents the displacement vector in meters (m); $\boldsymbol{\sigma}$ is the Cauchy stress vector with units of pascals (Pa); $\mathbf{f}_v$ stands for the volume force density, whose unit is also Pa; and

$$\mathbf{L}^T = \begin{bmatrix} \frac{\partial}{\partial x} & & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ & \frac{\partial}{\partial y} & & \\ & & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ & & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix},$$

represents a matrix of differential operators.

In sequentially coupled electromagnetic-mechanical problems, Lorentz force $\mathbf{f}_L$ arises when electric current passes through the conductor:

$$\mathbf{f}_L = \mathbf{J}_c \times \mathbf{B} = \sigma \left(-\dot{\mathbf{A}} - \nabla\varphi \right) \times (\nabla \times \mathbf{A}). \quad (12)$$

The Lorentz force also acts as a body force. Accordingly, Eq. (11) can be rewritten when the Lorentz force is taken into account:

$$\rho \ddot{\mathbf{u}} = \mathbf{L}^T \boldsymbol{\sigma} + \mathbf{f}_v + \mathbf{f}_L. \quad (13)$$

2. elastic constitutive equations

For linear isotropic materials, the constitutive equations are given by Hooke's law, which describes the stress-strain relationship:

$$\boldsymbol{\sigma} = \mathbf{D}_E \boldsymbol{\varepsilon}, \quad (14)$$

where $\mathbf{D}_E$ is the elastic matrix with units of Pa; $\boldsymbol{\varepsilon}$ denotes the engineering strain vector, which is dimensionless.

3. geometric equations

The geometric equations describe the geometric relationship between strain and displacement

$$\boldsymbol{\varepsilon} = \mathbf{L} \mathbf{u}, \quad (15)$$

which hold under the small-strain assumption. By substituting Eqs. (14) and (15) into Eq. (13), the displacement-formulated governing equations of elastodynamics accounting for Lorentz force are derived:

$$\rho \ddot{\mathbf{u}} = \mathbf{L}^T \mathbf{D}_E \mathbf{L} \mathbf{u} + \mathbf{f}_v + \mathbf{f}_L. \quad (16)$$

The corresponding boundary conditions read

$$\begin{aligned} \mathbf{u} &= \bar{\mathbf{u}} \text{ on } \Gamma_u, \\ \boldsymbol{\sigma} \cdot \mathbf{n} &= \bar{\mathbf{f}} \text{ on } \Gamma_\sigma. \end{aligned} \quad (17)$$

The first expression in Eq. (17) represents the prescribed boundary displacement $\bar{\mathbf{u}}$, corresponding to the Dirichlet boundary condition, while the second one denotes

the prescribed boundary load $\bar{\mathbf{f}}$, which corresponds to the Neumann boundary condition.

2.3 Variational principle for continuous electromechanical coupled systems

The aforementioned governing equations are expressed in differential form. The theoretical foundation of the space-time finite element method lies in the variational principle, which converts the solution of the original partial differential equations into a stationary-value problem of the action functional; the geometric structure of the system can be naturally preserved after discretization. In this section, the continuous variational formulations for the electromagnetic field and elastodynamics are derived separately to verify their equivalence to the differential governing equations. Finally, a unified variational framework for the coupled system is established under the sequential coupling assumption.

2.3.1 Continuous variational functional for full-wave electromagnetic fields

For full-wave electromagnetic fields formulated with the $\mathbf{A}$ - φ representation, the Lagrangian density of the conservative system equals the difference between the electric energy density and magnetic energy density:

$$L_{EM} = \frac{1}{2} \mathbf{E}^T \varepsilon \mathbf{E} - \frac{1}{2} \mathbf{B}^T \nu \mathbf{B} + \mathbf{A}^T \mathbf{J}_s - \rho_s \varphi, \quad (18)$$

where $\nu = \frac{1}{\mu}$. Substituting the potential relations (5) and (7) yields

$$L_{EM}(\mathbf{A}, \varphi) = \frac{1}{2} \varepsilon \left(\dot{\mathbf{A}} + \nabla \varphi \right)^T \left(\dot{\mathbf{A}} + \nabla \varphi \right) - \frac{1}{2} \nu (\nabla \times \mathbf{A})^T (\nabla \times \mathbf{A}) + \mathbf{A}^T \mathbf{J}_s - \rho_s \varphi. \quad (19)$$

Integrating the Lagrangian density over the space-time cylinder $\Omega \times [t_0, t_1]$ yields the action functional for electromagnetic fields.

$$\begin{aligned} S_{EM}(\mathbf{A}, \varphi) = & \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{1}{2} \varepsilon \left(\dot{\mathbf{A}} + \nabla \varphi \right)^T \left(\dot{\mathbf{A}} + \nabla \varphi \right) - \frac{1}{2} \nu (\nabla \times \mathbf{A})^T (\nabla \times \mathbf{A}) + \mathbf{A}^T \mathbf{J}_s - \rho_s \varphi \right] d\Omega dt \\ & + \mathbf{A}_0^T \mathbf{\Pi}_{A,0} - \mathbf{A}_1^T \mathbf{\Pi}_{A,1} + \varphi_0 \Pi_{\varphi,0} - \varphi_1 \Pi_{\varphi,1}, \end{aligned} \quad (20)$$

where $\mathbf{\Pi}_A$ denotes the conjugate canonical generalized momentum of the magnetic vector potential $\mathbf{A}$.

$$\mathbf{\Pi}_{A,0} = \frac{\partial S_{EM}}{\partial \mathbf{A}_0} = \int_{\Omega} \varepsilon \left(\dot{\mathbf{A}}_0 + \nabla \varphi_0 \right) d\Omega, \quad \mathbf{\Pi}_{A,1} = -\frac{\partial S_{EM}}{\partial \mathbf{A}_1} = \int_{\Omega} \varepsilon \left(\dot{\mathbf{A}}_1 + \nabla \varphi_1 \right) d\Omega, \quad (21)$$

The subscript digits denote the corresponding time instants, where 0 stands for the instant $t = t_0$ and 1 for $t = t_1$. Π_{φ} is the conjugate canonical generalized momentum

associated with the electric scalar potential φ , defined in the same form as in Eq. (21).

$$\Pi_{\varphi,0} = \frac{\partial S_{EM}}{\partial \varphi_0} = 0, \quad \Pi_{\varphi,1} = -\frac{\partial S_{EM}}{\partial \varphi_1} = 0. \quad (22)$$

This generalized momentum is identically zero, since S_{EM} only involves the spatial gradient $\nabla \varphi$ of φ and contains no time derivative $\dot{\varphi}$. The variation corresponding to Eq. (20) reads:

$$\begin{aligned} \delta S_{EM}(\mathbf{A}, \varphi) = & \int_{t_0}^{t_1} \int_{\Omega} \left[\varepsilon \left(\delta \dot{\mathbf{A}} + \nabla \delta \varphi \right)^T \left(\dot{\mathbf{A}} + \nabla \varphi \right) - \nu (\nabla \times \delta \mathbf{A})^T (\nabla \times \mathbf{A}) + \delta \mathbf{A}^T \mathbf{J}_s - \delta \varphi \rho_s \right] d\Omega dt \\ & + \delta \mathbf{A}_0^T \Pi_{A,0} - \delta \mathbf{A}_1^T \Pi_{A,1} + \delta \varphi_0 \Pi_{\varphi,0} - \delta \varphi_1 \Pi_{\varphi,1}. \end{aligned} \quad (23)$$

2.3.2 Continuous variational functional of elastodynamics

For conservative elastodynamic systems formulated in terms of displacement, the true displacement renders the action functional of the system stationary. The corresponding Lagrangian density is given by:

$$L_{mech} = \frac{1}{2} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} \boldsymbol{\varepsilon}^T \mathbf{D}_E \boldsymbol{\varepsilon} + \mathbf{u}^T \mathbf{f}_v. \quad (24)$$

Substituting the strain-displacement relation (15) gives

$$L_{mech} = \frac{1}{2} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} (\mathbf{L} \mathbf{u})^T \mathbf{D}_E (\mathbf{L} \mathbf{u}) + \mathbf{u}^T \mathbf{f}_v. \quad (25)$$

Integrating the Lagrangian density over the space-time cylinder $\Omega \times [t_0, t_1]$, yields the action functional for displacement fields.

$$S_{mech}(\mathbf{u}) = \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{1}{2} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} (\mathbf{L} \mathbf{u})^T \mathbf{D}_E (\mathbf{L} \mathbf{u}) + \mathbf{u}^T \mathbf{f}_v \right] d\Omega dt + \mathbf{u}_0^T \mathbf{p}_0 - \mathbf{u}_1^T \mathbf{p}_1, \quad (26)$$

where $\mathbf{p}$ denotes the conjugate canonical generalized momentum of displacement $\mathbf{u}$

$$\mathbf{p}_0 = \frac{\partial S_{mech}}{\partial \mathbf{u}_0} = \int_{\Omega} \rho \dot{\mathbf{u}}_0 d\Omega, \quad \mathbf{p}_1 = -\frac{\partial S_{mech}}{\partial \mathbf{u}_1} = \int_{\Omega} \rho \dot{\mathbf{u}}_1 d\Omega \quad (27)$$

and the subscripts represent the corresponding time instants. Then, the variation corresponding to Eq. (26) is written as

$$\delta S_{mech}(\mathbf{u}) = \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{1}{2} \rho \delta \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} \delta \mathbf{u}^T \mathbf{L}^T \mathbf{D}_E \mathbf{L} \mathbf{u} + \delta \mathbf{u}^T \mathbf{f}_v \right] d\Omega dt + \delta \mathbf{u}_0^T \mathbf{p}_0 - \delta \mathbf{u}_1^T \mathbf{p}_1. \quad (28)$$

2.3.3 Unified total variational formulation for sequentially coupled system

The total action functional S of the coupled continuous electromagnetic-mechanical field is the sum of the electromagnetic field action S_{EM} , the displacement field action S_{mech} , and the coupling action S_{couple} [23, 24].

$$S = S_{EM}(\mathbf{A}, \varphi) + S_{mech}(\mathbf{u}) + S_{couple}(\mathbf{A}, \varphi, \mathbf{u}), \quad (29)$$

where $S_{couple} = \int_{t_0}^{t_1} \int_{\Omega} \mathbf{u}^T \mathbf{f}_L d\Omega dt = \int_{t_0}^{t_1} \int_{\Omega} \mathbf{u}^T \left[\sigma(-\dot{\mathbf{A}} - \nabla\varphi) \times (\nabla \times \mathbf{A}) \right] d\Omega dt$ denotes the work exerted by the Lorentz force over the displacement field.

Taking the variation of Eq. (29) yields

$$\begin{aligned} \delta S &= \delta S_{EM}(\mathbf{A}, \varphi) + \delta S_{mech}(\mathbf{u}) + \delta S_{couple}(\mathbf{A}, \varphi, \mathbf{u}) \\ &= \delta \mathbf{A}^T \frac{\partial S_{EM}}{\partial \mathbf{A}} + \delta \varphi \frac{\partial S_{EM}}{\partial \varphi} + \delta \mathbf{u}^T \frac{\partial S_{mech}}{\partial \mathbf{u}} + \delta \mathbf{A}^T \frac{\partial S_{couple}}{\partial \mathbf{A}} + \delta \varphi \frac{\partial S_{couple}}{\partial \varphi} + \delta \mathbf{u}^T \frac{\partial S_{couple}}{\partial \mathbf{u}} \\ &= \left(\delta \mathbf{A}^T \frac{\partial S_{EM}}{\partial \mathbf{A}} + \delta \varphi \frac{\partial S_{EM}}{\partial \varphi} + \delta \mathbf{A}^T \frac{\partial S_{couple}}{\partial \mathbf{A}} + \delta \varphi \frac{\partial S_{couple}}{\partial \varphi} \right) + \delta \mathbf{u}^T \left(\frac{\partial S_{mech}}{\partial \mathbf{u}} + \frac{\partial S_{couple}}{\partial \mathbf{u}} \right). \end{aligned} \quad (30)$$

Since the present work adopts a one-way sequential coupling assumption for the electromagnetic-mechanical problem, the structural displacements have a negligible effect on the electromagnetic field distribution and boundary geometry under the small-deformation condition. Consequently, the electromagnetic subsystem can be solved independently of the displacement field [25]. In this case, the coupling term acts only on the mechanical subsystem, while the electromagnetic field variation is unaffected by the displacement, so that $\partial S_{couple}/\partial \mathbf{A}$ and $\partial S_{couple}/\partial \varphi$ are both zero. The Lorentz force produced by the electromagnetic field acts as a body force load, and the work done by this body force enters the mechanical subsystem solely through the virtual work form $\delta \mathbf{u}^T \partial S_{couple}/\partial \mathbf{u} = \int_{t_0}^{t_1} \int_{\Omega} \delta \mathbf{u}^T \mathbf{f}_L d\Omega dt$. Substituting Eqs. (23) and (28) into Eq. (30) yields

$$\begin{aligned} \delta S &= \int_{t_0}^{t_1} \int_{\Omega} \left[\varepsilon (\delta \dot{\mathbf{A}} + \nabla \delta \varphi)^T (\dot{\mathbf{A}} + \nabla \varphi) - \nu (\nabla \times \delta \mathbf{A})^T (\nabla \times \mathbf{A}) + \delta \mathbf{A}^T \mathbf{J}_s - \delta \varphi \rho_s \right] d\Omega dt \\ &\quad + \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{1}{2} \rho \delta \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} \delta \mathbf{u}^T \mathbf{L}^T \mathbf{D}_E \mathbf{L} \mathbf{u} + \delta \mathbf{u}^T \mathbf{f}_v \right] d\Omega dt + \int_0^{\Delta t} \int_{\Omega} \delta \mathbf{u}^T \mathbf{f}_L d\Omega dt \\ &\quad + \delta \mathbf{A}_0^T \boldsymbol{\Pi}_{A,0} - \delta \mathbf{A}_1^T \boldsymbol{\Pi}_{A,1} + \delta \varphi_0 \Pi_{\varphi,0} - \delta \varphi_1 \Pi_{\varphi,1} + \delta \mathbf{u}_0^T \mathbf{p}_0 - \delta \mathbf{u}_1^T \mathbf{p}_1. \end{aligned} \quad (31)$$

The above constitutes the variational equations of the coupled electromagnetic-mechanical field under sequential coupling.

When eddy current dissipation is taken into account, Joule heating constitutes an irreversible thermodynamic process that cannot be incorporated into the Lagrangian functional of a conservative system. Following the principle of virtual work and Rayleigh dissipation function theory, it is therefore introduced as an additional variational term, ensuring that the sign of the dissipated power is consistent with the

second law of thermodynamics:

$$\delta R = - \int_{t_0}^{t_1} \int_{\Omega} (\delta \mathbf{A} + \nabla \delta \varphi)^T \sigma \left(\dot{\mathbf{A}} + \nabla \varphi \right) d\Omega dt. \quad (32)$$

Then, Eq. (31) becomes

$$\begin{aligned} \delta S = & \int_{t_0}^{t_1} \int_{\Omega} \left[\varepsilon \left(\delta \dot{\mathbf{A}} + \nabla \delta \varphi \right)^T \left(\dot{\mathbf{A}} + \nabla \varphi \right) - \nu (\nabla \times \delta \mathbf{A})^T (\nabla \times \mathbf{A}) + \delta \mathbf{A}^T \mathbf{J}_s - \delta \varphi \rho_s \right] d\Omega dt \\ & - \int_{t_0}^{t_1} \int_{\Omega} (\delta \mathbf{A} + \nabla \delta \varphi)^T \sigma \left(\dot{\mathbf{A}} + \nabla \varphi \right) d\Omega dt \\ & + \int_{t_0}^{t_1} \int_{\Omega} \left[\frac{1}{2} \rho \delta \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \frac{1}{2} \delta \mathbf{u}^T \mathbf{L}^T \mathbf{D}_E \mathbf{L} \mathbf{u} + \delta \mathbf{u}^T \mathbf{f}_v \right] d\Omega dt + \int_0^{\Delta t} \int_{\Omega} \delta \mathbf{u}^T \mathbf{f}_L d\Omega dt \\ & + \delta \mathbf{A}_0^T \boldsymbol{\Pi}_{A,0} - \delta \mathbf{A}_1^T \boldsymbol{\Pi}_{A,1} + \delta \varphi_0 \Pi_{\varphi,0} - \delta \varphi_1 \Pi_{\varphi,1} + \delta \mathbf{u}_0^T \mathbf{p}_0 - \delta \mathbf{u}_1^T \mathbf{p}_1. \end{aligned} \quad (33)$$

Setting the variation of the above expression to zero yields the continuous form of the electromagnetic–mechanical coupling equations. In the next section, the field variables will be interpolated using space-time elements, and the resulting discrete expressions will be substituted into the continuous variational equations to seek the stationary value of the functional over a finite-dimensional approximation subspace, ultimately leading to the discrete recurrence scheme.

3 Space-time discrete variational framework for the electromagnetic–mechanical coupling problem

In solving electromagnetic–mechanical coupling problems, conventional approaches typically follow the paradigm of spatial semi-discretization combined with a time integrator: the spatial derivatives are first discretized by the finite element method to obtain a system of ordinary differential equations in time, which is then advanced step by step using a dedicated time integrator. For conservative systems, the spatial discretization preserves the symplectic geometric structure of the Hamiltonian system in the temporal direction; however, the introduction of a time integrator may destroy this structure. Unless a symplectic integrator is manually constructed for the specific problem at hand, the numerical solution will inevitably introduce spurious numerical damping or energy drift, compromising the accuracy of long-term simulations. In contrast, the space-time finite element method performs a unified discretization in both space and time, with the temporal integration accomplished by finite element interpolation and numerical quadrature, rather than by an external time-stepping scheme. This “time-embedded-in-space” treatment intrinsically preserves the variational structure of the continuous problem: by seeking the stationary value of the continuous action restricted to a finite-dimensional subspace, the discrete evolution mapping derived from the variational formulation naturally satisfies the symplectic condition for conservative systems without introducing an additional time integration scheme. Even for dissipative systems involving Joule losses, the unified variational discretization of

the space-time finite element method retains the positive semi-definiteness of the dissipation term and the correct sign of the power balance, thereby ensuring consistency with the physical dissipation mechanism and improving long-term numerical stability.

Based on the continuous total variational equation established in Section 2, this section employs hexahedral–prismatic space-time grid elements, with edge elements for the magnetic vector potential $\mathbf{A}$ and nodal elements for the electric scalar potential φ and the displacement field $\mathbf{u}$, to construct the space-time discrete variational functional of the electromagnetic–mechanical coupling system. Through variational integration, single-step recurrence solution schemes are derived for the electromagnetic and mechanical subsystems, and the recurrence scheme is shown to be symplectic for conservative systems.

3.1 Hexahedral–prismatic space-time grid element

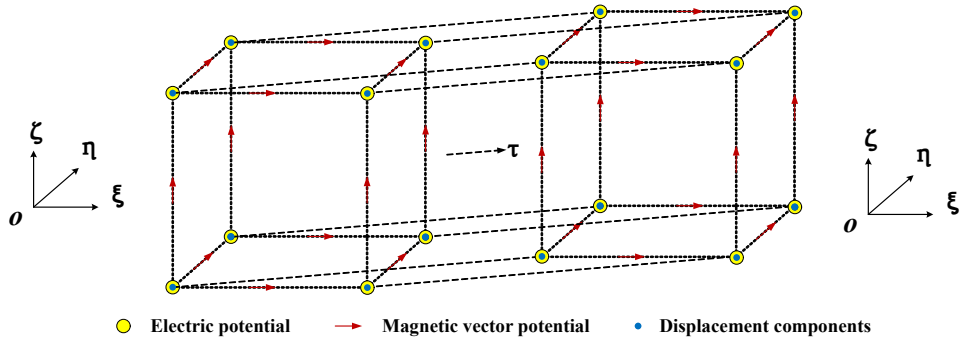


Fig. 1: Schematic of the hexahedral–prismatic space-time grid element

Consider the hexahedral–prismatic space-time grid element illustrated in Fig. 1. The discrete field variables consist of the electric scalar potential φ , the magnetic vector potential $\mathbf{A}$, and the displacement field $\mathbf{u}$. The potentials φ and the displacements $\mathbf{u}$ are discretized using nodal elements, with four degrees of freedom per node — one for the electric scalar potential and three for the displacement components. The magnetic vector potential $\mathbf{A}$ is discretized using edge elements, with one degree of freedom per edge representing the tangential integral of $\mathbf{A}$ along that edge. This space-time grid element is a four-dimensional element encompassing the three spatial dimensions and the temporal dimension, which are mutually orthogonal. The shape functions of the space-time element depend on both the spatial coordinates and the time coordinate, and can therefore be expressed in the following form

$$\begin{aligned} N(\mathbf{x}, t) &= \hat{N}(\mathbf{x}) \Phi(t), \\ \mathbf{W}(\mathbf{x}, t) &= \hat{\mathbf{W}}(\mathbf{x}) \Phi(t), \end{aligned} \tag{34}$$

where $\hat{N}(\mathbf{x})$ denotes the spatial nodal shape function and $\hat{\mathbf{W}}(\mathbf{x})$ the spatial edge shape function, both of which depend only on the spatial coordinates, while $\Phi(t)$ is the temporal shape function, depending solely on time. Accordingly, for the space-time grid element shown in Fig. 1, the electric scalar potential, displacement, and magnetic vector potential at any interior point can be interpolated in the following form

$$\begin{aligned}\varphi(\mathbf{x}, t) &= \mathbf{N}_\varphi \boldsymbol{\varphi}^e, \\ \mathbf{u}(\mathbf{x}, t) &= \mathbf{N}_u \mathbf{u}^e, \\ \mathbf{A}(\mathbf{x}, t) &= \mathbf{W} \mathbf{A}^e,\end{aligned}\tag{35}$$

where

$$\begin{aligned}\mathbf{N}_\varphi &= [N_1 \ N_2 \ \cdots \ N_{16}] = \hat{\mathbf{N}} \otimes [\Phi_0 \ \Phi_1], \\ \mathbf{N}_u &= [N_1 \otimes \mathbf{I}_3 \ N_2 \otimes \mathbf{I}_3 \ \cdots \ N_{16} \otimes \mathbf{I}_3] \\ &= \hat{\mathbf{N}} \otimes \mathbf{I}_3 \otimes [\Phi_0 \ \Phi_1] \\ &= \hat{\mathbf{N}}_u \otimes [\Phi_0 \ \Phi_1], \\ \mathbf{W} &= [\mathbf{W}_1 \ \mathbf{W}_2 \ \cdots \ \mathbf{W}_{24}] = \hat{\mathbf{W}} \otimes [\Phi_0 \ \Phi_1],\end{aligned}\tag{36}$$

$\boldsymbol{\varphi}^e$ denotes the element nodal electric potential vector, $\mathbf{u}^e$ the element nodal displacement vector, and $\mathbf{A}^e$ the element edge tangential integral vector of the magnetic vector potential, with $\otimes$ representing the tensor product. Since the space-time grid element shown in Fig. 1 contains only two temporal nodes, each of the vectors $\boldsymbol{\varphi}^e$, $\mathbf{u}^e$ and $\mathbf{A}^e$ can be decomposed into components at $t = t_0$ and components at $t = t_1$:

$$\boldsymbol{\varphi}^e = [\boldsymbol{\varphi}_0^{eT} \ \boldsymbol{\varphi}_1^{eT}]^T, \mathbf{u}^e = [\mathbf{u}_0^{eT} \ \mathbf{u}_1^{eT}]^T, \mathbf{A}^e = [\mathbf{A}_0^{eT} \ \mathbf{A}_1^{eT}]^T.\tag{37}$$

In Eq. (36), $\hat{\mathbf{N}}_u = \hat{\mathbf{N}} \otimes \mathbf{I}_3$, where $\hat{\mathbf{N}} = [\hat{N}_1 \ \hat{N}_2 \ \cdots \ \hat{N}_8]$ is a row vector composed of the nodal shape functions of the standard eight-node first-order hexahedral Lagrange element, and $\hat{\mathbf{W}} = [\hat{\mathbf{W}}_1 \ \hat{\mathbf{W}}_2 \ \cdots \ \hat{\mathbf{W}}_{12}]$ is a matrix formed by the edge shape functions of the conventional first-order hexahedral Nédélec element. The explicit forms of these functions are not the focus of this paper; the reader is referred to [26].

3.2 Space-time discrete energy functional

When the electromagnetic field is discretized using the space-time grid elements introduced in Section 3.1, the electric field intensity and magnetic flux density within an element can be expressed in the following matrix-vector form

$$\begin{aligned}\mathbf{E} &= -\dot{\mathbf{W}} \mathbf{A}^e - \nabla \mathbf{N} \boldsymbol{\varphi}^e = -\hat{\mathbf{W}} \dot{\boldsymbol{\Phi}}_A \mathbf{A}^e - \nabla \hat{\mathbf{N}} \boldsymbol{\Phi}_\varphi \boldsymbol{\varphi}^e, \\ \mathbf{B} &= \nabla \times \mathbf{W} \mathbf{A}^e = \nabla \times \hat{\mathbf{W}} \boldsymbol{\Phi}_A \mathbf{A}^e,\end{aligned}\tag{38}$$

where $\boldsymbol{\Phi}_A = [\Phi_0 \ \Phi_1] \otimes \mathbf{I}_{12}$ and $\boldsymbol{\Phi}_\varphi = [\Phi_0 \ \Phi_1] \otimes \mathbf{I}_8$ are the temporal shape function matrices, whose dimensions are determined by the interpolation targets. Substituting the above expressions into the continuous action variation equation of the electromagnetic field, Eq. (23), yields the electromagnetic field action variation at the element

level

$$\delta S_{EM} = \delta \mathbf{X}^T \mathbf{K}_1 \mathbf{X} + \delta \mathbf{X}^T \mathbf{F}_{EM} + \delta \mathbf{X}_0^T \mathbf{\Pi}_0 - \delta \mathbf{X}_1^T \mathbf{\Pi}_1 - \delta \mathbf{X}^T \mathbf{K}_2 \mathbf{X}, \quad (39)$$

where $\mathbf{X} = [\mathbf{A}^{eT} \ \boldsymbol{\varphi}^{eT}]$, $\mathbf{X}_0 = [\mathbf{A}_0^{eT} \ \boldsymbol{\varphi}_0^{eT}]^T$, $\mathbf{X}_1 = [\mathbf{A}_1^{eT} \ \boldsymbol{\varphi}_1^{eT}]^T$; $\mathbf{\Pi}_0$ and $\mathbf{\Pi}_1$ correspond to the generalized momenta of the electromagnetic field at $t = 0$ and $t = \Delta t$, respectively, and are automatically defined by the following relations

$$\mathbf{\Pi}_0 = \frac{\partial S_{EM}}{\partial \mathbf{X}_0}, \mathbf{\Pi}_1 = -\frac{\partial S_{EM}}{\partial \mathbf{X}_1}, \quad (40)$$

which correspond respectively to the continuous forms in Eqs. (21) and (22);

$$\mathbf{K}_1 = \int_{t_0}^{t_1} \int_{\Omega} \begin{bmatrix} \dot{\Phi}_A^T \hat{\mathbf{W}}^T \varepsilon \hat{\mathbf{W}} \dot{\Phi}_{\varphi} - \Phi_A^T (\nabla \times \mathbf{W})^T \boldsymbol{\nu} (\nabla \times \mathbf{W}) \Phi_A & \dot{\Phi}_A^T \mathbf{W}^T \varepsilon \nabla \mathbf{N} \Phi_{\varphi} \\ \Phi_{\varphi}^T (\nabla \mathbf{N})^T \varepsilon \mathbf{W} \dot{\Phi}_A & \Phi_{\varphi}^T (\nabla \mathbf{N})^T \varepsilon \nabla \mathbf{N} \Phi_{\varphi} \end{bmatrix} d\Omega dt, \quad (41)$$

is the stiffness matrix contributed by the conservative part of the electromagnetic field;

$$\mathbf{K}_2 = \int_{t_0}^{t_1} \int_{\Omega} \begin{bmatrix} \Phi_A^T \hat{\mathbf{W}}^T \sigma \hat{\mathbf{W}} \dot{\Phi}_A & \Phi_A^T \hat{\mathbf{W}}^T \sigma \nabla \hat{\mathbf{N}} \Phi_{\varphi} \\ \Phi_{\varphi}^T (\nabla \hat{\mathbf{N}})^T \sigma \hat{\mathbf{W}} \dot{\Phi}_A & \Phi_{\varphi}^T (\nabla \hat{\mathbf{N}})^T \sigma \nabla \hat{\mathbf{N}} \Phi_{\varphi} \end{bmatrix} d\Omega dt, \quad (42)$$

is the stiffness matrix contributed by the dissipative part of the electromagnetic field;

$$\mathbf{F}_{EM} = \begin{bmatrix} \int_{t_0}^{t_1} \int_{\Omega} \mathbf{W}^T \mathbf{J}_s d\Omega dt \\ \int_{t_0}^{t_1} \int_{\Omega} -\mathbf{N}^T \rho_s d\Omega dt \end{bmatrix}, \quad (43)$$

is the space-time load vector of the externally applied excitation of the electromagnetic field.

Then, the displacement field is discretized using the space-time grid elements, and the strain and stress within an element can be expressed as

$$\begin{aligned} \varepsilon &= \hat{\mathbf{B}} \Phi_u \mathbf{u}^e, \\ \sigma &= \mathbf{D}_E \varepsilon = \mathbf{D}_E \hat{\mathbf{B}} \Phi_u \mathbf{u}^e, \end{aligned} \quad (44)$$

where $\hat{\mathbf{B}}$ is the strain-displacement matrix (constructed from the spatial gradients of the nodal shape functions), $\mathbf{D}_E$ is the elasticity matrix, and $\Phi_u = [\Phi_0 \ \Phi_1] \otimes \mathbf{I}_{24}$ is the temporal shape function matrix for the displacement field. Based on Eq. (44), the tensor-product form of the space-time shape functions allows the temporal interpolation and the spatial interpolation within a space-time grid element to be carried out separately. Substituting the above expressions into the continuous action variation equation of the displacement field, Eq. (28), yields the displacement field action variation at the element level

$$\delta S_{mech} = \delta \mathbf{u}^{eT} \mathbf{K}_u \mathbf{u}^e + \delta \mathbf{u}^{eT} \mathbf{F}_u + \delta \mathbf{u}_0^{eT} \mathbf{p}_0 - \delta \mathbf{u}_1^{eT} \mathbf{p}_1, \quad (45)$$

where $\mathbf{p}_0$ corresponds to the generalized momentum at $t = t_0$ and $\mathbf{p}_1$ corresponds to that at $t = t_1$, satisfying the following relations

$$\mathbf{p}_0 = \frac{\partial S_{mech}}{\partial \mathbf{u}_0^e}, \mathbf{p}_1 = -\frac{\partial S_{mech}}{\partial \mathbf{u}_1^e}, \quad (46)$$

corresponding to the Eq. (27); and

$$\begin{aligned} \mathbf{K}_u &= \int_{t_0}^{t_1} \int_{\Omega} \left(\dot{\Phi}_u^T \hat{N}_u^T \rho \hat{N}_u \Phi_u - \Phi_u^T \hat{B}^T D_E \hat{B} \Phi_u \right) d\Omega dt, \\ \mathbf{F}_u &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_u^T \hat{N}_u^T \mathbf{f}_v d\Omega dt, \end{aligned} \quad (47)$$

are the stiffness matrix of the discretized displacement field and the load vector excluding the Lorentz force, respectively.

Adding Eq. (39) and Eq. (45) and introducing the work done by the Lorentz force via the principle of virtual work yields the variation of the total action functional S of the electromagnetic-mechanical coupled system at the element level

$$\begin{aligned} \delta S &= \delta \mathbf{X}^T \mathbf{K}_1 \mathbf{X} + \delta \mathbf{X}^T \mathbf{F}_{EM} + \delta \mathbf{X}_0^T \mathbf{\Pi}_0 - \delta \mathbf{X}_1^T \mathbf{\Pi}_1 - \delta \mathbf{X}^T \mathbf{K}_2 \mathbf{X} \\ &+ \delta \mathbf{u}^{eT} \mathbf{K}_u \mathbf{u}^e + \delta \mathbf{u}^{eT} \mathbf{F}_u + \delta \mathbf{u}_0^{eT} \mathbf{p}_0 - \delta \mathbf{u}_1^{eT} \mathbf{p}_1 + \delta \mathbf{u}^{eT} \mathbf{F}_L, \end{aligned} \quad (48)$$

where $\mathbf{F}_L = \int_{t_0}^{t_1} \int_{\Omega} \Phi_u^T \hat{N}_u^T \mathbf{f}_L d\Omega dt$. Setting the variation in Eq. (48) to zero yields

$$\begin{cases} (\mathbf{K}_1 - \mathbf{K}_2) \mathbf{X} + \mathbf{F}_{EM} + \mathbf{Q}_0^T \mathbf{\Pi}_0 - \mathbf{Q}_1^T \mathbf{\Pi}_1 = 0 \\ \mathbf{K}_u \mathbf{u}^e + \mathbf{F}_u + \mathbf{F}_L + \hat{\mathbf{Q}}_0^T \mathbf{p}_0 - \hat{\mathbf{Q}}_1^T \mathbf{p}_1 = 0 \end{cases}. \quad (49)$$

where $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\hat{\mathbf{Q}}_0$, and $\hat{\mathbf{Q}}_1$ are Boolean extraction matrices satisfying the following conditions

$$\mathbf{X}_0 = \mathbf{Q}_0 \mathbf{X}, \mathbf{X}_1 = \mathbf{Q}_1 \mathbf{X}, \mathbf{u}_0^e = \hat{\mathbf{Q}}_0 \mathbf{u}^e, \mathbf{u}_1^e = \hat{\mathbf{Q}}_1 \mathbf{u}^e. \quad (50)$$

When $\mathbf{X}_0$, $\mathbf{\Pi}_0$, $\mathbf{u}_0^e$, and $\mathbf{p}_0$ are known, Eq. (49) can be solved directly as a linear system; however, this is not the most efficient computational approach. In the next section, the matrices involved in the variational process are analyzed in detail, and a simpler recurrence solution scheme is derived.

3.3 Recurrence solution scheme based on the space-time grid element

To improve computational efficiency, the global discrete variational equations obtained in Section 3.2 are reordered by partitioning the degrees of freedom according to their temporal nodes. The global variables are split into the state at $t = t_0$, i.e. $\mathbf{X}_0$, $\mathbf{\Pi}_0$, $\mathbf{u}_0$, $\mathbf{p}_0$, and the state at $t = t_1$, i.e. $\mathbf{X}_1$, $\mathbf{\Pi}_1$, $\mathbf{u}_1$, $\mathbf{p}_1$. Eliminating the intermediate variables through block-matrix simplification yields an implicit recurrence scheme that relates the states at two adjacent time instants.

For the electromagnetic field, corresponding to the first equation in Eq. (49), the block-reordered equation is

$$\begin{bmatrix} \mathbf{K}_{X,00} & \mathbf{K}_{X,01} \\ \mathbf{K}_{X,10} & \mathbf{K}_{X,11} \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{X,00} & \tilde{\mathbf{K}}_{X,01} \\ \tilde{\mathbf{K}}_{X,10} & \tilde{\mathbf{K}}_{X,11} \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{X,0} \\ \mathbf{F}_{X,1} \end{bmatrix} + \begin{bmatrix} \mathbf{\Pi}_0 \\ -\mathbf{\Pi}_1 \end{bmatrix} = \mathbf{0}, \quad (51)$$

where

$$\begin{aligned} \mathbf{K}_{X,00} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt - K_v \int_{t_0}^{t_1} \Phi_0^2 dt & C_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_0 dt \\ C_\varepsilon^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & K_\varepsilon \int_{t_0}^{t_1} \Phi_0^2 dt \end{bmatrix}, \\ \mathbf{K}_{X,01} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt - K_v \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt & C_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_1 dt \\ C_\varepsilon^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & K_\varepsilon \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \end{bmatrix}, \\ \mathbf{K}_{X,10} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt - K_v \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & C_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_0 dt \\ C_\varepsilon^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & K_\varepsilon \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \end{bmatrix}, \\ \mathbf{K}_{X,11} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt - K_v \int_{t_0}^{t_1} \Phi_1^2 dt & C_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_1 dt \\ C_\varepsilon^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & K_\varepsilon \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \\ \tilde{\mathbf{K}}_{X,00} &= \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & C_\sigma \int_{t_0}^{t_1} \Phi_0^2 dt \\ C_\sigma^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & K_\sigma \int_{t_0}^{t_1} \Phi_0^2 dt \end{bmatrix}, \tilde{\mathbf{K}}_{X,01} = \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & C_\sigma \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ C_\sigma^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & K_\sigma \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \end{bmatrix}, \\ \tilde{\mathbf{K}}_{X,10} &= \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & C_\sigma \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \\ C_\sigma^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & K_\sigma \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \end{bmatrix}, \tilde{\mathbf{K}}_{X,11} = \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & C_\sigma \int_{t_0}^{t_1} \Phi_1^2 dt \\ C_\sigma^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & K_\sigma \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \\ \mathbf{F}_{X,0} &= \begin{bmatrix} \int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{W}}^\top \mathbf{J}_s d\Omega dt \\ -\int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{N}}^\top \rho_s d\Omega dt \end{bmatrix}, \mathbf{F}_{X,1} = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{W}}^\top \mathbf{J}_s d\Omega dt \\ -\int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{N}}^\top \rho_s d\Omega dt \end{bmatrix}. \end{aligned}$$

The detailed derivation and the definitions of the sub-block matrices are provided in Appendix A.

For the displacement field, corresponding to the second equation in Eq. (49), the block-reordered equation is

$$\begin{bmatrix} \mathbf{K}_{u,00} & \mathbf{K}_{u,10} \\ \mathbf{K}_{u,01} & \mathbf{K}_{u,11} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0^e \\ \mathbf{u}_1^e \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{u,0} + \mathbf{F}_{L,0} \\ \mathbf{F}_{u,1} + \mathbf{F}_{L,1} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_0 \\ -\mathbf{p}_1 \end{bmatrix} = \mathbf{0}, \quad (52)$$

where

$$\begin{aligned} \mathbf{K}_{u,00} &= M_\rho \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt - K_E \int_{t_0}^{t_1} \Phi_0^2 dt, \\ \mathbf{K}_{u,01} &= M_\rho \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt - K_E \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt, \\ \mathbf{K}_{u,10} &= M_\rho \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt - K_E \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt, \\ \mathbf{K}_{u,11} &= M_\rho \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt - K_E \int_{t_0}^{t_1} \Phi_1^2 dt, \end{aligned}$$

$$\begin{aligned}\mathbf{F}_{u,0} &= \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{N}}_u^T \mathbf{f}_v d\Omega dt, \quad \mathbf{F}_{u,1} = \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{N}}_u^T \mathbf{f}_v d\Omega dt, \\ \mathbf{F}_{L,0} &= \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt, \quad \mathbf{F}_{L,1} = \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt.\end{aligned}$$

The detailed derivation and the definitions of the sub-block matrices are provided in Appendix B.

Comparing Eqs. (51) and (52) shows that both can be written in the following two-variable recurrence form

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{b}_1 \end{bmatrix} = \mathbf{P} - \mathbf{T} \begin{bmatrix} \mathbf{a}_0 \\ \mathbf{b}_0 \end{bmatrix}. \quad (53)$$

Depending on the integration strategy adopted for the temporal shape functions in the sub-block matrices—such as exact integration or reduced integration—various solution schemes can be constructed, all of which, however, retain the same unified recurrence structure.

3.4 Symplecticity of the recurrence scheme

When eddy current dissipation is absent, the conductivity $\sigma = 0$, the matrix $\tilde{\mathbf{K}}_X$ vanishes automatically, and the electromagnetic and displacement fields become mutually independent conservative systems. In this case, since linear shape functions are employed in the present space-time grid element, it can be readily verified that the following relation holds for any integration strategy of the temporal shape functions:

$$\mathbf{K}_{X,01} = \mathbf{K}_{X,10}^T, \mathbf{K}_{X,00} = \mathbf{K}_{X,11}, \mathbf{K}_{u,10} = \mathbf{K}_{u,01}^T, \mathbf{K}_{u,00} = \mathbf{K}_{u,11}. \quad (54)$$

In this case, the transfer matrix $\mathbf{T}$ in Eqs. (51) and (52) can be uniformly written in the following simplified form

$$\mathbf{T} = \begin{bmatrix} \mathbf{M}_0 & \\ \mathbf{M}_1 & -\mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{M}_1 & \mathbf{I} \\ \mathbf{M}_0^T & \end{bmatrix} = \begin{bmatrix} \mathbf{M}_0^{-1} \mathbf{M}_1 & \mathbf{M}_0^{-1} \\ \mathbf{M}_1 \mathbf{M}_0^{-1} \mathbf{M}_1 - \mathbf{M}_0^T & \mathbf{M}_1 \mathbf{M}_0^{-1} \end{bmatrix}, \quad (55)$$

where $\mathbf{M}_1$ is a symmetric matrix. Since

$$\begin{aligned}& \mathbf{T}^T \begin{bmatrix} \mathbf{I} \\ -\mathbf{I} \end{bmatrix} \mathbf{T} \\ &= \begin{bmatrix} \mathbf{M}_1 \mathbf{M}_0^{-T} & \mathbf{M}_1 \mathbf{M}_0^{-T} \mathbf{M}_1 - \mathbf{M}_0 \\ \mathbf{M}_0^{-T} & \mathbf{M}_0^{-T} \mathbf{M}_1 \end{bmatrix} \begin{bmatrix} -\mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{M}_0^{-1} \mathbf{M}_1 & \mathbf{M}_0^{-1} \\ \mathbf{M}_1 \mathbf{M}_0^{-1} \mathbf{M}_1 - \mathbf{M}_0^T & \mathbf{M}_1 \mathbf{M}_0^{-1} \end{bmatrix} \\ &= \begin{bmatrix} -\mathbf{M}_1 \mathbf{M}_0^{-T} \mathbf{M}_1 + \mathbf{M}_0 & \mathbf{M}_1 \mathbf{M}_0^{-T} \\ \mathbf{M}_0^{-T} \mathbf{M}_1 & \mathbf{M}_0^{-T} \end{bmatrix} \begin{bmatrix} \mathbf{M}_0^{-1} \mathbf{M}_1 & \mathbf{M}_0^{-1} \\ \mathbf{M}_1 \mathbf{M}_0^{-1} \mathbf{M}_1 - \mathbf{M}_0^T & \mathbf{M}_1 \mathbf{M}_0^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} \\ -\mathbf{I} \end{bmatrix},\end{aligned} \quad (56)$$

the transfer matrix $\mathbf{T}$ is symplectic [27]. Therefore, in the absence of dissipation, the recurrence scheme derived from Eqs. (51) and (52) is automatically symplectic, regardless of the integration strategy adopted for the temporal shape functions.

4 Numerical examples

To systematically validate the correctness of the proposed method, three numerical examples are presented in this section, following the progressive logic of “single-field first, then coupled”. These examples verify the performance of the method for nondissipative electromagnetic field problems, elastodynamic problems, and electromagnetic–mechanical coupling scenarios, respectively.

4.1 Cubic resonant cavity

This example considers a standard cubic cavity benchmark [15] to verify the correctness of the electromagnetic space-time recurrence scheme. A unit cube $\Omega = [0, 1]^3$ is taken as the computational domain with perfect electric conductor (PEC) boundary conditions. The material parameters are normalized to unity, with $\varepsilon = 1$ and $\mu = 1$, both expressed in dimensionless form.

To construct a test case that admits an analytical solution, a source term satisfying the following equation is introduced:

$$\begin{aligned} F_{EM}(t, x, y, z) &= \rho dt + J_x dx + J_y dy + J_z dz \\ &= -3\pi^2 \sin(\pi x) \sin(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dt \\ &\quad + 3\pi^2 \cos(\pi x) \sin(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dx \\ &\quad + 3\pi^2 \sin(\pi x) \cos(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dy \\ &\quad - 6\pi^2 \sin(\pi x) \sin(\pi y) \cos(\pi z) [1 - \cos(2\pi t)] dz. \end{aligned} \quad (57)$$

The corresponding analytical solution is

$$\begin{aligned} u_{EM}(t, x, y, z) &= \varphi dt + A_x dx + A_y dy + A_z dz \\ &= \sin(\pi x) \sin(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dt \\ &\quad + \cos(\pi x) \sin(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dx \\ &\quad + \sin(\pi x) \cos(\pi y) \sin(\pi z) [1 - \cos(2\pi t)] dy \\ &\quad - 2 \sin(\pi x) \sin(\pi y) \cos(\pi z) [1 - \cos(2\pi t)] dz, \end{aligned} \quad (58)$$

where $\varphi = 0$ on $\partial\Omega$, $\nabla \cdot \mathbf{A} = 0$, and $\mathbf{A} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, with $\mathbf{n}$ denoting the outward unit normal to the boundary.

The computation is carried out using the space-time finite element recurrence scheme proposed in this paper, with reduced integration adopted for the temporal shape functions. The spatial domain is discretized with a $10 \times 10 \times 10$ hexahedral mesh, and the time interval $[0, 2]$ is uniformly partitioned into 400 subintervals with a step size of $\Delta t = 0.005$, yielding a total simulation duration of 2. The spatial and temporal discretizations are combined into a global space-time mesh via tensor-product

construction. To eliminate the non-uniqueness of the solution arising from the gauge invariance of the magnetic vector potential $\mathbf{A}$, the tree-cotree gauge condition [28] is employed in the numerical computation of $\mathbf{A}$. The implementation proceeds as follows: a connectivity graph is formed by traversing all edges of the mesh; a minimum spanning tree that covers all nodes is designated as the set of tree edges, with the remaining edges forming the cotree; the tangential degrees of freedom of $\mathbf{A}$ on the tree edges are removed from the algebraic system; only the degrees of freedom on the cotree edges participate in the assembly and solution of the system; after solving, the degrees of freedom on the tree edges are restored to zero.

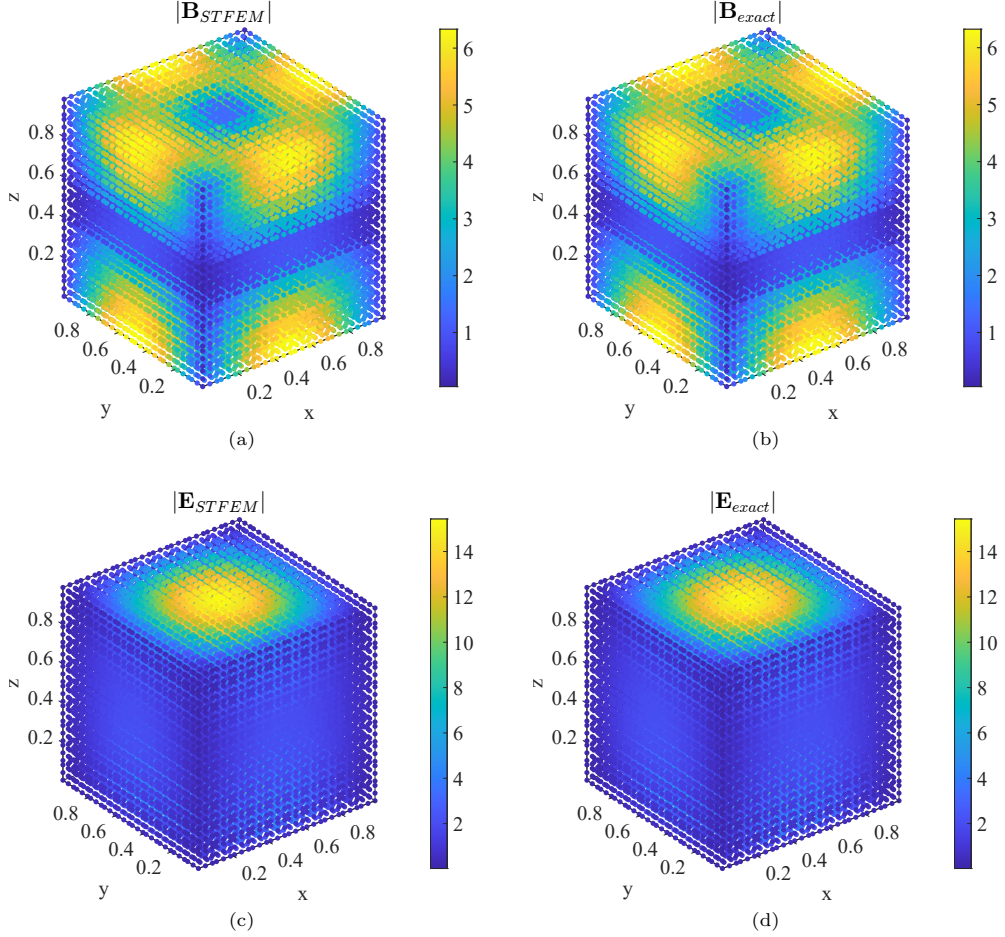


Fig. 2: Contours of the magnetic flux density and the electric field intensity at $t = 0.8$

Figure 2 shows the contours of the magnetic flux density and the electric field intensity at $t = 0.8$. The results obtained by the proposed method are denoted by the

subscript "STFEM", while the exact analytical solution is denoted by the subscript "exact". As shown in Fig. 2, the STFEM solutions agree well with the analytical solutions. To quantify the error more precisely, the following formulas are used to compute the discrete-time L^2 relative errors of the magnetic flux density and the electric field intensity:

$$\begin{aligned} er_{\mathbf{B}} &= \frac{\left(\sum_{k=0}^N \|\mathbf{B}_{STFEM}(t_k) - \mathbf{B}_{exact}(t_k)\|_2^2 \right)^{1/2}}{\left(\sum_{k=0}^N \|\mathbf{B}_{exact}(t_k)\|_2^2 \right)^{1/2}}, \\ er_{\mathbf{E}} &= \frac{\left(\sum_{k=0}^N \|\mathbf{E}_{STFEM}(t_k) - \mathbf{E}_{exact}(t_k)\|_2^2 \right)^{1/2}}{\left(\sum_{k=0}^N \|\mathbf{E}_{exact}(t_k)\|_2^2 \right)^{1/2}}, \end{aligned} \quad (59)$$

where $N = 400$ is the number of time steps. Evaluating Eq. (59) yields a relative error of $er_{\mathbf{B}} = 5.3359 \times 10^{-4}$ for the magnetic flux density $\mathbf{B}$ and $er_{\mathbf{E}} = 6.0226 \times 10^{-4}$ for the electric field intensity $\mathbf{E}$, both remaining at the order of 10^{-4} , which further confirms the accuracy of the proposed scheme.

According to the analysis in Section 3.4, the proposed scheme is automatically symplectic for nondissipative electromagnetic field problems. A simple numerical test is designed here to verify this property. The source term is again constructed according to Eq. (57), and the system's free evolution is studied after the external source is removed. To avoid a sudden energy jump caused by an abrupt removal of the loading, the source term is smoothly attenuated. Specifically, the original loading $F_{EM}(t)$ obtained from the analytical solution is multiplied by a temporal modulation function $g(t)$:

$$\tilde{F}_{EM}(t) = g(t) F_{EM}(t), \quad (60)$$

where

$$g(t) = \begin{cases} 1, & t \leq 0.1; \\ \frac{1}{2} \left[1 + \cos \left(\pi \frac{t - 0.1}{0.05} \right) \right], & 0.1 \leq t \leq 0.15; \\ 0, & t \geq 0.15. \end{cases} \quad (61)$$

Figure 3 presents the temporal evolution of the system energy computed by the STFEM method. As shown in Fig. 3, after the external loading is gradually removed, both the electric and magnetic field energies oscillate over time and exhibit a clear characteristic of mutual conversion. Meanwhile, their sum, i.e., the total electromagnetic energy, remains essentially constant once the external excitation ceases, exhibiting only minor periodic oscillations attributable to numerical errors. This corroborates the symplecticity of the proposed method.

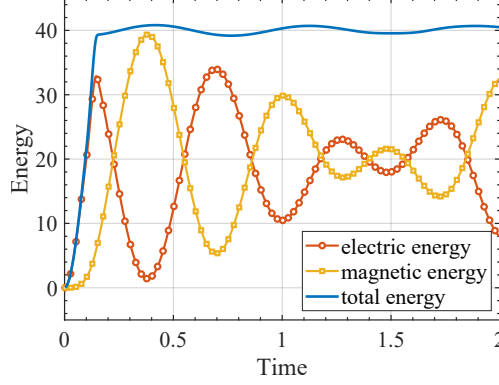


Fig. 3: System energy evolution computed by the STFEM method

4.2 Free vibration of a simply supported thick plate

This example is intended to verify the solution accuracy and symplecticity of the proposed space-time finite element recurrence scheme for purely elastodynamic problems. A simply supported rectangular thick plate is considered, with side lengths $a = b = 1$, thickness $h = 0.1$, elastic modulus $E = 100$, Poisson's ratio $\lambda = 0.3$, and density $\rho = 1$, all expressed in dimensionless form. The spatial domain is discretized with a $10 \times 10 \times 4$ hexahedral mesh, and the time interval is uniformly partitioned into 5000 subintervals with a step size of $\Delta t = 0.001$, yielding a total simulation duration of 5. The spatial and temporal discretizations are combined into a global space-time mesh via tensor-product construction. The initial condition is generated from a static analysis: a uniform pressure $q = 2$ is applied to the top surface of the plate, and the initial displacement field is obtained by solving the static equilibrium equation; the initial velocity is set to zero. At $t = 0$ the external load is removed, and the plate undergoes free vibration.

To assess the solution accuracy, the following three approaches are employed for cross-validation: (1) the space-time finite element recurrence scheme proposed in this work, with two variants distinguished by the integration strategy adopted for the temporal shape functions—STFEM-RI (reduced integration) and STFEM-AI (accurate integration); (2) the built-in ODE45 solver in MATLAB [29, 30], which is based on an embedded Runge–Kutta method and serves as a high-accuracy symplectic numerical reference solution; (3) the backward Euler method [31], which is employed as a non-symplectic reference for comparison.

Figures 4 and 5 show, respectively, the full time-history comparison and the absolute error of the plate center deflection. The space-time finite element solutions obtained with both STFEM-AI and STFEM-RI exhibit stable, constant-amplitude oscillations over the entire time span. The absolute errors of both integration strategies relative to the ode45 reference solution remain below the 10^{-3} level, and no appreciable difference in accuracy is observed between the two strategies. By contrast, the backward Euler solution exhibits monotonic amplitude decay—a characteristic manifestation of the numerical dissipation inherent in non-symplectic schemes—and

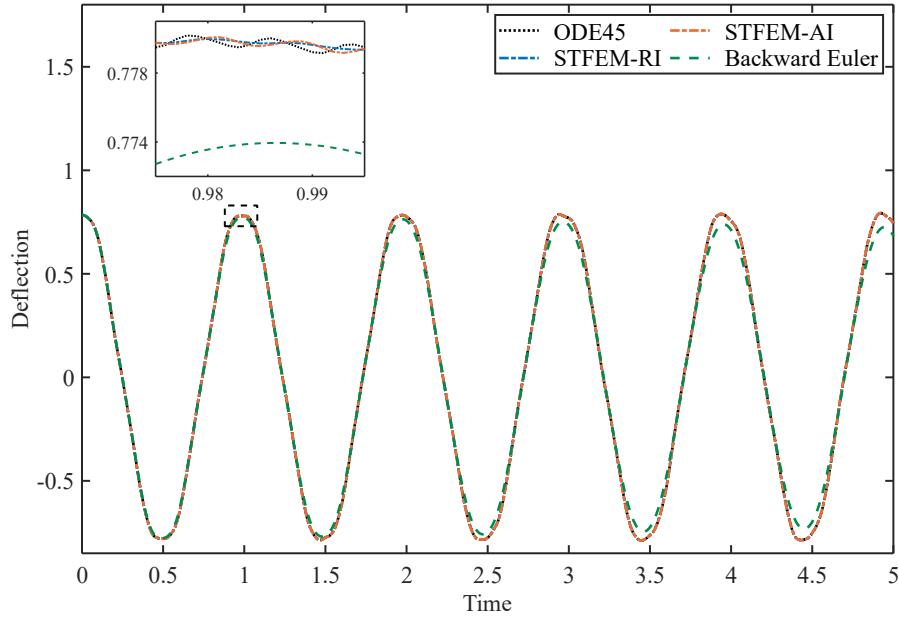


Fig. 4: Full time-history comparison of the plate center deflection computed by different methods

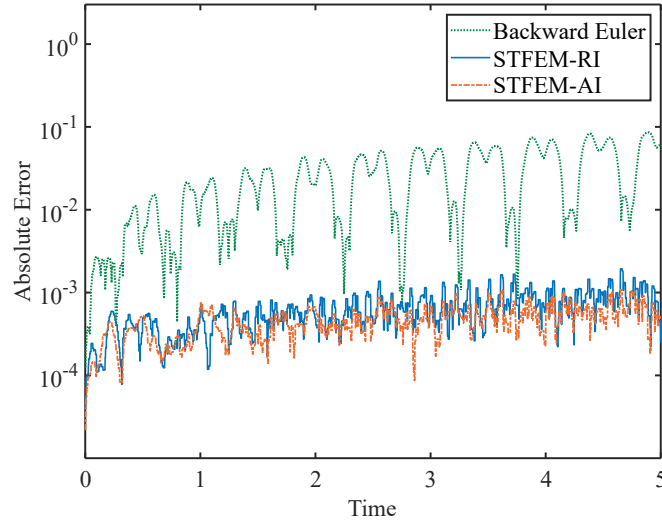


Fig. 5: Absolute error comparison of the plate center deflection computed by different methods

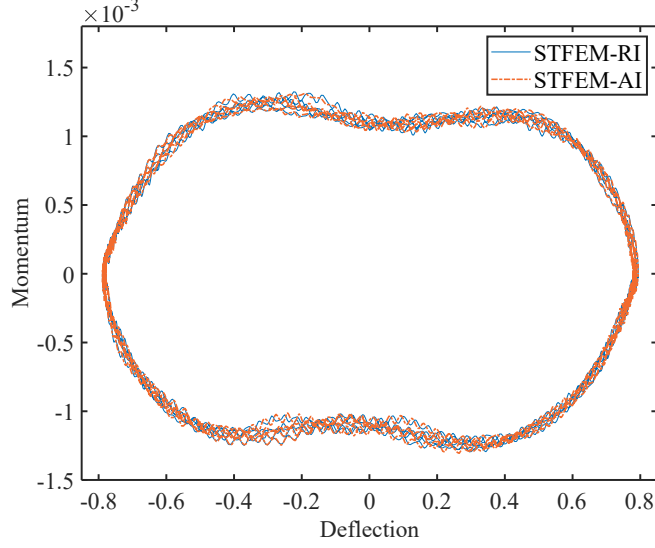


Fig. 6: Phase portrait of the deflection versus generalized momentum at the plate center computed by STFEM

its absolute error accumulates continuously with simulation time. Figure 6 presents the phase portrait of the deflection versus generalized momentum at the plate center. The phase trajectories of both the STFEM-AI and the STFEM-RI form reasonably regular closed ellipses, indicating that the discrete system retains the intrinsic symplectic geometric structure of the continuous problem under both quadrature strategies. In summary, this example demonstrates that the proposed space-time finite element recurrence scheme achieves satisfactory solution accuracy for elastodynamic problems; both quadrature strategies yield results in close agreement with the high-accuracy reference solution, and stand in clear contrast to the numerical dissipation exhibited by the backward Euler method.

4.3 Three-dimensional rectangular direct current busbar

This example is intended to verify the correctness of the proposed space-time finite element recurrence scheme for electromagnetic-mechanical coupling problems. Consider a long straight busbar conductor as shown in Fig. 7, with an axial length $L = 100$ mm and a rectangular cross-section with width $w = 4$ mm and thickness $d = 2$ mm. The conductor is grounded and clamped at the $z = 0$ end, satisfying $\varphi = 0$, while the $z = L$ end is current-fed and mechanically free. A perfect magnetic conductor (PMC) boundary condition, $\mathbf{n} \times \mathbf{A} = \mathbf{0}$, is prescribed on the face $y = 0$. At $t = 0$ s, an external source current is applied to the end face $z = L$, with the total current gradually

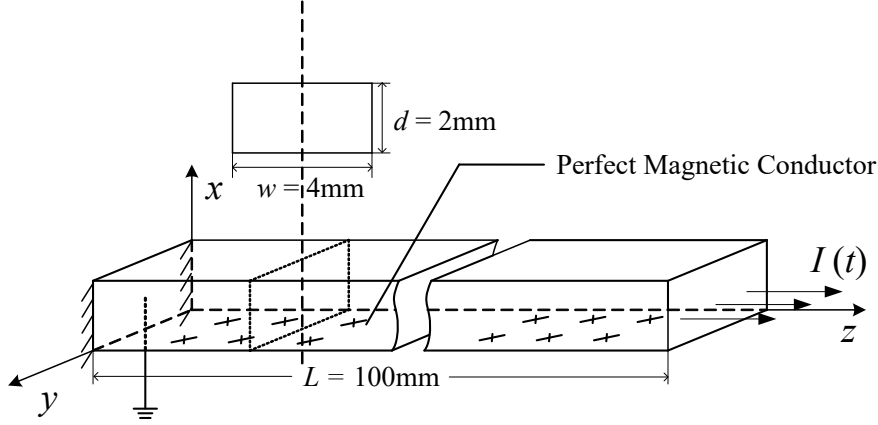


Fig. 7: Schematic diagram of the busbar example

ramping up from 0 A to $I_{\max} = 50$ A according to the following time variation:

$$I = \begin{cases} 50 \left(\frac{t}{0.05} \right) \text{ A}, & 0 \text{ s} \leq t \leq 0.05 \text{ s}; \\ 50 \text{ A}, & 0.05 \text{ s} \leq t \leq 0.08 \text{ s}. \end{cases} \quad (62)$$

The current load is uniformly applied to the cross-sectional nodes as a source term in the φ equation and is subsequently transferred to the magnetic field via electromagnetic coupling.

The conductor is modeled as an isotropic material with electrical conductivity $\sigma = 10^6$ S/m, magnetic permeability $\mu = 4\pi \times 10^{-7}$ H/m, permittivity $\varepsilon = 8.854 \times 10^{-12}$ F/m, elastic modulus $E = 1.2 \times 10^{11}$ Pa, Poisson's ratio $\lambda = 0.3$, and density $\rho = 8960$ kg/m³. The current flowing through the conductor alters the internal electric and magnetic field distributions, thereby generating a Lorentz force that acts on the conductor and induces structural displacements.

The electromagnetic field distribution is first computed using the full-wave recurrence scheme for eddy-current problems proposed in Section 3, with the gauge condition enforced via the tree-cotree gauge [28]. The Lorentz force distribution is then evaluated from the computed electromagnetic fields, and subsequently supplied as a body force load to the mechanical recurrence scheme of Section 3 for solving the conductor displacements. Reduced integration is adopted for the temporal shape functions in both schemes. The spatial domain is discretized with a $4 \times 4 \times 100$ hexahedral mesh, and the time interval of total duration 0.08 s is uniformly partitioned into 800 subintervals with a step size $\Delta t = 1 \times 10^{-4}$ s. The spatial and temporal discretizations are combined into a global space-time mesh via tensor-product construction.

With the PMC boundary condition applied on one face and assuming a uniform current density $J_z = I(t)/(dw)$, the closed-form solution of the one-dimensional

Poisson equation can serve as the reference solution for the magnetic field:

$$B_x(y, t) = \frac{\partial A_z}{\partial y} = \mu_0 J_z (w - y), \quad B_z = B_y = 0. \quad (63)$$

The corresponding Lorentz force density can be evaluated analytically:

$$f_y(y, t) = J_z B_x = \mu J_z^2 (w - y), \quad f_x = f_z = 0. \quad (64)$$

This force is directed along the positive y -axis and decays linearly from its maximum value at $y = 0$ to zero at $y = w$. The cross-sectional static force density is then given by

$$q(t) = \int_0^d \int_0^w f_y(y, t) dy dx = \frac{1}{2} \mu J_z^2 d w^2. \quad (65)$$

Based on the Euler–Bernoulli beam theory, when a cantilever beam is subjected to a time-varying uniform line load q , the dynamic deflection at the free end can be expressed in the following analytical form via the modal superposition method [32]:

$$w(x, t) = \sum_{n=1}^{\infty} \varphi_n(x) \frac{1}{\omega_n} \int_0^t \frac{q(\tau) \int_0^L \varphi_n(s) ds}{\rho A \int_0^L \varphi_n^2(s) ds} \sin[\omega_n(t - \tau)] d\tau, \quad (66)$$

where $\varphi_n(x) = \cosh(\beta_n x) - \cos(\beta_n x) - \alpha_n [\sinh(\beta_n x) - \sin(\beta_n x)]$, with β_n satisfying the characteristic equation $\cosh(\beta_n L) \cos(\beta_n L) + 1 = 0$, $\alpha_n = [\cosh(\beta_n L) + \cos(\beta_n L)] / [\sinh(\beta_n L) + \sin(\beta_n L)]$, and $\omega_n = \beta_n^2 \sqrt{EI/(\rho A)}$, in which A is the cross-sectional area and I the area moment of inertia.

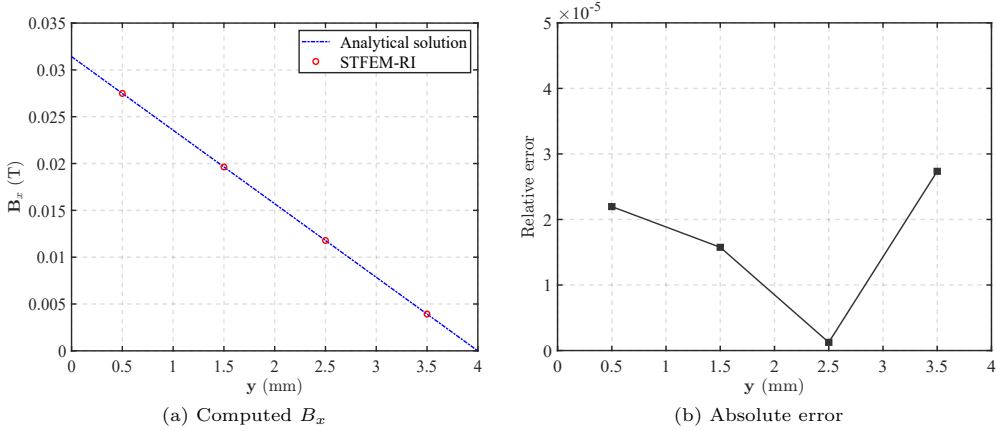


Fig. 8: Computed B_x and its absolute error at $x = 0.75$ mm, $z = 49.5$ mm inside the conductor at $t = 0.08$ s

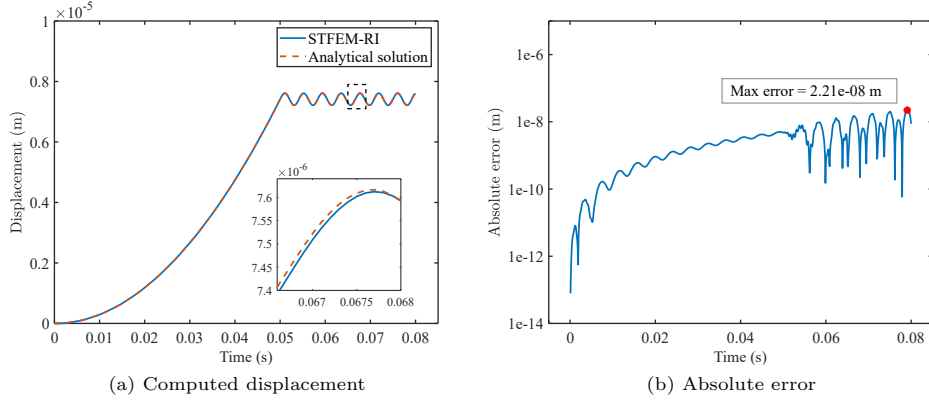


Fig. 9: Computed displacement at the midpoint of the free end face of the conductor and its absolute error

Figures 8(a) and 8(b) respectively present the comparison of the computed B_x at $x = 0.75$ mm, $z = 49.5$ mm inside the conductor at $t = 0.08$ s and the corresponding absolute error. Figures 9(a) and 9(b) respectively show the comparison between the transient displacement response at the midpoint of the free end face of the conductor at $t = 0.08$ s and the reference solution based on Eq. (66), together with the corresponding absolute error. As shown in Fig. 8(b), the absolute errors in the magnetic field results remain on the order of 10^{-5} , confirming the correctness of the electromagnetic coupling computation. It can be observed from Figs. 9(a) and 9(b) that the displacement results are consistent with the trend of the reference solution: in the phase $t < 0.05$ s, the displacement grows as $I^2(t) \propto t^2$, and stabilizes thereafter for $t > 0.05$ s, while the absolute error remains steadily at the order of 10^{-8} . In summary, this example establishes a full-chain electromagnetic-to-mechanical coupling verification for the rectangular cross-section busbar, from the electromagnetic field to the displacement field. The comparison with the analytical reference solutions confirms the accuracy of the proposed space-time finite element-based coupling framework.

5 Conclusion

This paper has presented a unified space-time finite element framework for electromagnetic-mechanical sequential coupling problems. Grounded in Hamilton's variational principle, the framework introduces hexahedral-prismatic space-time grid elements and establishes a discrete variational principle together with a single-step recurrence solution scheme for the coupled system within a unified variational discrete formulation. The scheme has been shown to automatically preserve the symplectic structure in the nondissipative case.

The correctness of the proposed method has been validated through three numerical examples. In single-field verification of electromagnetic problems, the space-time

finite-element solutions agree well with analytical or high-accuracy reference solutions. In the long-time simulation of the free vibration of a simply supported thick plate, both the displacement amplitude and the total energy obtained from the symplectic recurrence scheme remain stable, standing in clear contrast to the numerical dissipation exhibited by the backward Euler method. In the coupled example of a direct-current busbar under short-circuit conditions, the transient results of the electromagnetic-mechanical sequential coupling are consistent with the reference solutions, demonstrating that the present framework accurately captures the structural dynamic response driven by the Lorentz force.

The above results indicate that the proposed space-time finite element framework can simultaneously ensure the accuracy and long-time numerical stability of electromagnetic field solutions, elastodynamic solutions, and their sequential coupling, without the need for additional structure-preserving integrators. The present work is based on the small-deformation assumption and does not account for the feedback of the displacement field on the electromagnetic field distribution. Future work may consider extending the framework to two-way coupling or nonlinear problems and incorporating space-time adaptive discretization strategies to further enhance computational efficiency.

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Data availability. The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

Conflict of interest. The authors declare that they have no conflict of interest.

Appendix A

This appendix presents a detailed analysis of the matrices appearing in the first equation of Eq. (49) and derives the recurrence solution scheme for the electromagnetic field. We begin with a detailed examination of the sub-block matrices within the block matrix $\mathbf{K}_1$. Let

$$\mathbf{K}_1 = \begin{bmatrix} \mathbf{K}_{1,00} & \mathbf{K}_{1,01} \\ \mathbf{K}_{1,10} & \mathbf{K}_{1,11} \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{K}_{1,00} &= \int_{t_0}^{t_1} \int_{\Omega} \dot{\Phi}_A^T \hat{\mathbf{W}}^T \varepsilon \hat{\mathbf{W}} \dot{\Phi}_A - \Phi_A^T (\nabla \times \mathbf{W})^T \nu (\nabla \times \mathbf{W}) \Phi_A d\Omega dt \\
&= \int_{t_0}^{t_1} \dot{\Phi}_A^T M_{\varepsilon} \dot{\Phi}_A dt - \int_{t_0}^{t_1} \Phi_A^T K_v \Phi_A dt \\
&= \begin{bmatrix} M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt & M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt \\ M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt & M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt \end{bmatrix} - \begin{bmatrix} K_v \int_{t_0}^{t_1} \Phi_0^2 dt & K_v \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ K_v \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & K_v \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \tag{A1}
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{1,01} &= \int_{t_0}^{t_1} \int_{\Omega} \dot{\Phi}_A^T \mathbf{W}^T \varepsilon \nabla N \Phi_{\varphi} d\Omega dt = \int_{t_0}^{t_1} \dot{\Phi}_A^T C_{\varepsilon} \Phi_{\varphi} dt \\
&= \begin{bmatrix} C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_0 dt & C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_1 dt \\ C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_0 dt & C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_1 dt \end{bmatrix}, \tag{A2}
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{1,10} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_{\varphi}^T (\nabla N)^T \varepsilon \mathbf{W} \dot{\Phi}_A d\Omega dt = \int_{t_0}^{t_1} \Phi_{\varphi}^T C_{\varepsilon}^T \dot{\Phi}_A dt \\
&= \begin{bmatrix} C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt \\ C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt \end{bmatrix}, \tag{A3}
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{1,11} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_{\varphi}^T (\nabla N)^T \varepsilon \nabla N \Phi_{\varphi} d\Omega dt = \int_{t_0}^{t_1} \Phi_{\varphi}^T K_{\varepsilon} \Phi_{\varphi} dt \\
&= \begin{bmatrix} K_{\varepsilon} \int_{t_0}^{t_1} \Phi_0^2 dt & K_{\varepsilon} \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ K_{\varepsilon} \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & K_{\varepsilon} \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix} \tag{A4}
\end{aligned}$$

and

$$M_{\varepsilon} = \int_{\Omega} \hat{\mathbf{W}}^T \varepsilon \hat{\mathbf{W}} d\Omega, \tag{A5}$$

$$K_v = \int_{\Omega} (\nabla \times \hat{\mathbf{W}})^T \nu (\nabla \times \hat{\mathbf{W}}) d\Omega, \tag{A6}$$

$$C_{\varepsilon} = \int_{\Omega} \hat{\mathbf{W}}^T \varepsilon \nabla \hat{N} d\Omega, \tag{A7}$$

$$K_{\varepsilon} = \int_{\Omega} (\nabla \hat{N})^T \varepsilon \nabla \hat{N} d\Omega. \tag{A8}$$

Next, the sub-block matrices within $\mathbf{K}_2$ are analyzed. Let

$$\mathbf{K}_2 = \begin{bmatrix} K_{2,00} & K_{2,01} \\ K_{2,10} & K_{2,11} \end{bmatrix},$$

where

$$\begin{aligned} K_{2,00} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_A^T \hat{\mathbf{W}}^T \sigma \hat{\mathbf{W}} \dot{\Phi}_A d\Omega dt = \int_{t_0}^{t_1} \Phi_A^T M_{\sigma} \dot{\Phi}_A dt \\ &= \begin{bmatrix} M_{\sigma} \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & M_{\sigma} \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt \\ M_{\sigma} \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & M_{\sigma} \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt \end{bmatrix}, \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} K_{2,01} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_A^T \hat{\mathbf{W}}^T \sigma \nabla N \Phi_{\varphi} d\Omega dt = \int_{t_0}^{t_1} \Phi_A^T C_{\sigma} \Phi_{\varphi} dt \\ &= \begin{bmatrix} C_{\sigma} \int_{t_0}^{t_1} \Phi_0^2 dt & C_{\sigma} \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ C_{\sigma} \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & C_{\sigma} \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} K_{2,10} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_{\varphi}^T (\nabla N)^T \sigma \hat{\mathbf{W}} \dot{\Phi}_A d\Omega dt = \int_{t_0}^{t_1} \Phi_{\varphi}^T C_{\sigma} \dot{\Phi}_A dt \\ &= \begin{bmatrix} C_{\sigma}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & C_{\sigma}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt \\ C_{\sigma}^T \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & C_{\sigma}^T \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt \end{bmatrix}, \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} K_{2,11} &= \int_{t_0}^{t_1} \int_{\Omega} \Phi_{\varphi}^T (\nabla \hat{N})^T \sigma \nabla \hat{N} \Phi_{\varphi} d\Omega dt = \int_{t_0}^{t_1} \Phi_{\varphi}^T K_{\sigma} \Phi_{\varphi} dt \\ &= \begin{bmatrix} K_{\sigma} \int_{t_0}^{t_1} \Phi_0^2 dt & K_{\sigma} \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ K_{\sigma} \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & K_{\sigma} \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix} \end{aligned} \quad (\text{A12})$$

and

$$M_{\sigma} = \int_{\Omega} \hat{\mathbf{W}}^T \sigma \hat{\mathbf{W}} d\Omega, \quad K_{\sigma} = \int_{\Omega} (\nabla \hat{N})^T \sigma \nabla \hat{N} d\Omega, \quad C_{\sigma} = \int_{\Omega} \hat{\mathbf{W}}^T \sigma \nabla \hat{N} d\Omega.$$

Then, analyzing the load term yields

$$\mathbf{F}_{EM} = \begin{bmatrix} \int_{t_0}^{t_1} \int_{\Omega} \mathbf{W}^T \mathbf{J}_s d\Omega dt \\ \int_{t_0}^{t_1} \int_{\Omega} -\mathbf{N}^T \rho_s d\Omega dt \end{bmatrix} = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_A^T \int_{\Omega} \hat{\mathbf{W}}^T \mathbf{J}_s d\Omega dt \\ - \int_{t_0}^{t_1} \Phi_{\varphi}^T \int_{\Omega} \hat{\mathbf{N}}^T \rho_s d\Omega dt \end{bmatrix} = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{W}}^T \mathbf{J}_s d\Omega dt \\ \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{W}}^T \mathbf{J}_s d\Omega dt \\ - \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{N}}^T \rho_s d\Omega dt \\ - \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{N}}^T \rho_s d\Omega dt \end{bmatrix}. \quad (\text{A13})$$

Substituting Eqs. (A1)–(A11) into the first equation of Eq. (49) and reordering the degrees of freedom according to the temporal nodes yields

$$\begin{bmatrix} \mathbf{K}_{X,00} & \mathbf{K}_{X,01} \\ \mathbf{K}_{X,10} & \mathbf{K}_{X,11} \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{X,00} & \tilde{\mathbf{K}}_{X,01} \\ \tilde{\mathbf{K}}_{X,10} & \tilde{\mathbf{K}}_{X,11} \end{bmatrix} \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{X,0} \\ \mathbf{F}_{X,1} \end{bmatrix} + \begin{bmatrix} \mathbf{\Pi}_0 \\ -\mathbf{\Pi}_1 \end{bmatrix} = \mathbf{0}, \quad (\text{A14})$$

where

$$\begin{aligned} \mathbf{K}_{X,00} &= \begin{bmatrix} M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt - K_v \int_{t_0}^{t_1} \Phi_0^2 dt & C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_0 dt \\ C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & K_{\varepsilon} \int_{t_0}^{t_1} \Phi_0^2 dt \end{bmatrix}, \\ \mathbf{K}_{X,01} &= \begin{bmatrix} M_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt - K_v \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt & C_{\varepsilon} \int_{t_0}^{t_1} \dot{\Phi}_0 \Phi_1 dt \\ C_{\varepsilon}^T \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & K_{\varepsilon} \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{X,10} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt - \mathbf{K}_v \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & \mathbf{C}_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_0 dt \\ \mathbf{C}_\varepsilon^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & \mathbf{K}_\varepsilon \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \end{bmatrix}, \\
\mathbf{K}_{X,11} &= \begin{bmatrix} M_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt - \mathbf{K}_v \int_{t_0}^{t_1} \Phi_1^2 dt & \mathbf{C}_\varepsilon \int_{t_0}^{t_1} \dot{\Phi}_1 \Phi_1 dt \\ \mathbf{C}_\varepsilon^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & \mathbf{K}_\varepsilon \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \\
\tilde{\mathbf{K}}_{X,00} &= \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & \mathbf{C}_\sigma \int_{t_0}^{t_1} \Phi_0^2 dt \\ \mathbf{C}_\sigma^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_0 dt & \mathbf{K}_\sigma \int_{t_0}^{t_1} \Phi_0^2 dt \end{bmatrix}, \tilde{\mathbf{K}}_{X,01} = \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & \mathbf{C}_\sigma \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ \mathbf{C}_\sigma^\top \int_{t_0}^{t_1} \Phi_0 \dot{\Phi}_1 dt & \mathbf{K}_\sigma \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \end{bmatrix}, \\
\tilde{\mathbf{K}}_{X,10} &= \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & \mathbf{C}_\sigma \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \\ \mathbf{C}_\sigma^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_0 dt & \mathbf{K}_\sigma \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt \end{bmatrix}, \tilde{\mathbf{K}}_{X,11} = \begin{bmatrix} M_\sigma \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & \mathbf{C}_\sigma \int_{t_0}^{t_1} \Phi_1^2 dt \\ \mathbf{C}_\sigma^\top \int_{t_0}^{t_1} \Phi_1 \dot{\Phi}_1 dt & \mathbf{K}_\sigma \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix}, \\
\mathbf{F}_{X,0} &= \begin{bmatrix} \int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{W}}^\top \mathbf{J}_s d\Omega dt \\ - \int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{N}}^\top \rho_s d\Omega dt \end{bmatrix}, \mathbf{F}_{X,1} = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{W}}^\top \mathbf{J}_s d\Omega dt \\ - \int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{N}}^\top \rho_s d\Omega dt \end{bmatrix}.
\end{aligned}$$

Appendix B

This appendix presents a detailed analysis of the matrices appearing in the second equation of Eq. (49) and derives the recurrence solution scheme for the displacement field. We begin with an analysis of the block matrix $\mathbf{K}_u$:

$$\begin{aligned}
\mathbf{K}_u &= \int_{t_0}^{t_1} \int_\Omega \left(\dot{\Phi}_u^\top \hat{\mathbf{N}}_u^\top \rho \hat{\mathbf{N}}_u \dot{\Phi}_u - \Phi_u^\top \hat{\mathbf{B}}^\top \mathbf{D}_E \hat{\mathbf{B}} \Phi_u \right) d\Omega dt \\
&= \int_{t_0}^{t_1} \dot{\Phi}_u^\top \mathbf{M}_\rho \dot{\Phi}_u dt - \int_{t_0}^{t_1} \Phi_u^\top \mathbf{K}_E \Phi_u dt \\
&= \begin{bmatrix} \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt & \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt \\ \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt & \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt \end{bmatrix} - \begin{bmatrix} \mathbf{K}_E \int_{t_0}^{t_1} \Phi_0^2 dt & \mathbf{K}_E \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt \\ \mathbf{K}_E \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt & \mathbf{K}_E \int_{t_0}^{t_1} \Phi_1^2 dt \end{bmatrix},
\end{aligned} \tag{B15}$$

where

$$\begin{aligned}
\mathbf{M}_\rho &= \int_\Omega \hat{\mathbf{N}}_u^\top \rho \hat{\mathbf{N}}_u d\Omega, \\
\mathbf{K}_E &= \int_\Omega \hat{\mathbf{B}}^\top \mathbf{D}_E \hat{\mathbf{B}} d\Omega.
\end{aligned}$$

Then, analyzing the load vector yields

$$\begin{aligned}
\mathbf{F}_u &= \int_{t_0}^{t_1} \int_\Omega \Phi_u^\top \hat{\mathbf{N}}_u^\top \mathbf{f}_v d\Omega dt = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{N}}_u^\top \mathbf{f}_v d\Omega dt \\ \int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{N}}_u^\top \mathbf{f}_v d\Omega dt \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{u,0} \\ \mathbf{F}_{u,1} \end{bmatrix}, \\
\mathbf{F}_L &= \int_{t_0}^{t_1} \int_\Omega \Phi_u^\top \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt = \begin{bmatrix} \int_{t_0}^{t_1} \Phi_0 \int_\Omega \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt \\ \int_{t_0}^{t_1} \Phi_1 \int_\Omega \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{L,0} \\ \mathbf{F}_{L,1} \end{bmatrix}.
\end{aligned} \tag{B16}$$

Substituting Eqs. (B15)–(B16) into the second equation of Eq. (49) and reordering the degrees of freedom according to the temporal nodes yields

$$\begin{bmatrix} \mathbf{K}_{u,00} & \mathbf{K}_{u,10} \\ \mathbf{K}_{u,01} & \mathbf{K}_{u,11} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0^e \\ \mathbf{u}_1^e \end{bmatrix} + \begin{bmatrix} \mathbf{F}_{u,0} + \mathbf{F}_{L,0} \\ \mathbf{F}_{u,1} + \mathbf{F}_{L,1} \end{bmatrix} + \begin{bmatrix} \mathbf{p}_0 \\ -\mathbf{p}_1 \end{bmatrix} = \mathbf{0}, \tag{B17}$$

where

$$\begin{aligned}
\mathbf{K}_{u,00} &= \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_0^2 dt - \mathbf{K}_E \int_{t_0}^{t_1} \Phi_0^2 dt, \\
\mathbf{K}_{u,01} &= \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_0 \dot{\Phi}_1 dt - \mathbf{K}_E \int_{t_0}^{t_1} \Phi_0 \Phi_1 dt, \\
\mathbf{K}_{u,10} &= \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_1 \dot{\Phi}_0 dt - \mathbf{K}_E \int_{t_0}^{t_1} \Phi_1 \Phi_0 dt, \\
\mathbf{K}_{u,11} &= \mathbf{M}_\rho \int_{t_0}^{t_1} \dot{\Phi}_1^2 dt - \mathbf{K}_E \int_{t_0}^{t_1} \Phi_1^2 dt, \\
\mathbf{F}_{u,0} &= \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{N}}_u^T \mathbf{f}_v d\Omega dt, \quad \mathbf{F}_{u,1} = \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{N}}_u^T \mathbf{f}_v d\Omega dt, \\
\mathbf{F}_{L,0} &= \int_{t_0}^{t_1} \Phi_0 \int_{\Omega} \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt, \quad \mathbf{F}_{L,1} = \int_{t_0}^{t_1} \Phi_1 \int_{\Omega} \hat{\mathbf{N}}_u \mathbf{f}_L d\Omega dt.
\end{aligned}$$

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