Supplemental Material: Exact Localized Information Capacity of a Local Observable in a Hilbert-Space-Shattered System

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This Supplemental Material provides: (S1) the model, its elementary move, and the height representation; (S2) the derivation of the exact universal capacity as a commutant projection and its status as the tight Mazur bound; (S3) the infinite-time autocorrelation, its degeneracy-resolved form, and the non-universal excess; (S4) the reduction of the Holevo capacity to a classical mutual information and the bit densities; (S5) the root-density SLIOMs and their statistical localization; (S6) the trace-monoid structure and the structural obstruction to strictly-local complete integrals of motion; (S7) the thermodynamic limit — the closed-form sector count, the subadditivity proof that the capacity density exists, its value $\chi_\infty = 1.09(1)$, the exact block factorization of sector orbits, and the t–J_z second model with its fully closed-form capacity $\chi_\infty = 2/3$; and (S8) numerical methods and reproducibility. All numbers quoted in the main text are reproduced by the accompanying `mblc` package (S7).

MODEL, ELEMENTARY MOVE, AND HEIGHT REPRESENTATION

The chain has local states $S_i^z \in \{-1, 0, +1\}$ ($i = 0, \ldots, L-1$), Hilbert-space dimension $D = 3^L$, and conserves charge $N = \sum_i S_i^z$ and dipole $P = \sum_i i\, S_i^z$. We take the maximal range-three charge- and dipole-conserving dynamics: two basis states are connected when they differ on a window of at most three consecutive sites while preserving that window's charge and dipole. On two sites these two constraints fix both spins, so there are no two-site moves. On three sites the constraints $\delta a + \delta b + \delta c = 0$ and $\delta b + 2\delta c = 0$ force $(\delta a, \delta b, \delta c) = k\,(1, -2, 1)$, and with $S^z \in \{-1, 0, 1\}$ only $k = \pm 1$ is allowed, i.e. the single elementary move

$$(a, +1, c) \;\longleftrightarrow\; (a+1, -1, c+1), \tag{S1}$$

which is exactly the three-site term $H_3 = -\sum_j S_{j-1}^+ (S_j^-)^2 S_{j+1}^+ + \text{H.c.}$ of Ref. [1]. The Krylov-sector count grows exponentially (Table S1), the hallmark of strong fragmentation [2, 3].

Passing to the height field $h_j = \sum_{i \le j} S_i^z$, one has $S_j^z = h_j - h_{j-1}$ with $h_{-1} \equiv 0$, and the elementary move becomes a *gated adjacent transposition*:

Lemma (height rule). A move acts on the window $(i, i+1, i+2)$, $i \le L-3$, exactly when $|h_i - h_{i+1}| = 1$; it swaps

$$h_i \leftrightarrow h_{i+1}, \tag{S2}$$

and is permitted iff both outer heights lie in $\{m, m+1\}$ with $m = \min(h_i, h_{i+1})$, i.e. $h_{i-1} \in \{m, m+1\}$ (with $h_{-1} = 0$) and $h_{i+2} \in \{m, m+1\}$.

Proof. Eq. (S1) changes only the on-site values at $i, i+1, i+2$, hence only h_i, h_{i+1} (the partial sums through $i-1$ and through $i+2$ are unchanged). The move requires the central site to be ± 1, i.e. $h_{i+1} = h_i \pm 1$; either sign swaps the two heights. Validity is that the three post-move on-site values $h_i^{\text{new}} - h_{i-1}$, $h_{i+1}^{\text{new}} - h_i^{\text{new}}$, $h_{i+2} - h_{i+1}^{\text{new}}$ lie in $\{-1, 0, 1\}$; combining with the pre-move validity gives exactly $h_{i-1}, h_{i+2} \in \{m, m+1\}$. □ (Verified exhaustively against the enumerated move set for $L \le 7$; `tests/test_height_rule.py`.)

The dynamics therefore only *permutes* the height values, gated by their immediate neighbours; this underlies both the exact capacity and the locality obstruction below.

UNIVERSAL CAPACITY AS A COMMUTANT PROJECTION

Equip operators with the infinite-temperature inner product $\langle X, Y \rangle = \frac{1}{D} \mathrm{Tr}(X^\dagger Y)$. The commutant $\mathcal{C}$ is the algebra of operators commuting with every term of

TABLE S1. Krylov-sector count K, its logarithm per site, and the exact capacity density χ/L (Sec.). K obeys $K_L = 2K_{L-1} + K_{L-2} + 2$ exactly for all $L \le 13$ (Sec.); the two densities converge to $\log_2(1+\sqrt{2}) = 1.2716$ and $\chi_\infty = 1.09(1)$ respectively.

L	K	$\log_2 K/L$	χ/L
4	57	1.4582	1.4182
5	139	1.4238	1.3675
6	337	1.3994	1.3296
7	815	1.3815	1.3003
8	1969	1.3679	1.2772
9	4755	1.3572	1.2586
10	11481	1.3487	1.2434
11	27719	1.3417	1.2306
12	66921	1.3358	1.2199
13	161563	1.3309	1.2107

the dynamics. For classical (product-basis) fragmentation, a *diagonal* operator $O = \sum_s o_s |s\rangle\langle s|$ commutes with the dynamics iff o_s is constant on each Krylov sector; hence the diagonal part of $\mathcal{C}$ is spanned by the sector projectors $\{P_\alpha\}$, which are mutually orthogonal, $\langle P_\alpha, P_\beta\rangle = \delta_{\alpha\beta}\mathrm{Tr}P_\alpha/D$.

For a diagonal local observable A (e.g. $A = S^z_i$) the orthogonal projection onto this diagonal commutant is the sector-averaged operator

$$\mathcal{P}_{\mathcal{C}}A = \sum_\alpha \frac{\langle P_\alpha, A\rangle}{\langle P_\alpha, P_\alpha\rangle} P_\alpha = \sum_\alpha \frac{\mathrm{Tr}(AP_\alpha)}{\mathrm{Tr}P_\alpha} P_\alpha, \tag{S3}$$

whose squared norm defines the universal capacity

$$C_{\mathrm{univ}}(A) = \|\mathcal{P}_{\mathcal{C}}A\|^2 = \frac{1}{D}\sum_\alpha \frac{[\mathrm{Tr}(AP_\alpha)]^2}{\mathrm{Tr}P_\alpha}. \tag{S4}$$

Three properties follow immediately. (i) *Closed form*: Eq. (S4) is a finite sum over sectors, evaluated analytically from the commutant. (ii) *Hamiltonian independence*: it depends only on $\{P_\alpha\}$, not on any amplitudes, so it is shared by every dynamics with these sectors. (iii) *Tightness as a Mazur bound* [4, 5]: the Mazur bound from any set of conserved charges $\{Q_k\}$ equals $\|\mathcal{P}_{\mathrm{span}\{Q_k\}}A\|^2$; maximizing over all conserved diagonal charges is the projection onto the full diagonal commutant span$\{P_\alpha\}$, returning Eq. (S4). Since A is diagonal, only diagonal conserved charges overlap its infinite-time autocorrelation, so C_{univ} is the *tightest Mazur bound obtainable from the commutant* for A, not merely one choice of charges. (The general program of extracting optimal Mazur bounds from commutant algebras is due to Ref. [6]; the content added here is the identification of this bound as the Hamiltonian-independent retained norm of A, tight for diagonal observables, together with its bits-valued counterpart in Sec. .) Values are quoted unnormalized; the infinite-temperature norm of the observable is $\langle A, A\rangle = \mathrm{Tr}(A^2)/D = 2/3$ for $A = S^z$. For $A = S^z$ at the central site of $L = 6$, Eq. (S4) gives $C_{\mathrm{univ}} = 0.30636$, versus 0.12064 from the two global charges $\{N, P\}$ alone.

INFINITE-TIME AUTOCORRELATION AND THE NON-UNIVERSAL EXCESS

The infinite-temperature, infinite-time-averaged autocorrelation is

$$\begin{aligned} C_\infty(A) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{D}\mathrm{Tr}[A(t)A(0)]\,dt \\ &= \frac{1}{D}\sum_E \mathrm{Tr}\big(\Pi_E A\,\Pi_E A\big), \end{aligned} \tag{S5}$$

where Π_E projects onto the eigenspace of energy E; only equal-energy pairs survive the time average (the diagonal ensemble [7]). For a non-degenerate spectrum this reduces to $\frac{1}{D}\sum_n \langle n|A|n\rangle^2$, but the degeneracy-resolved form (S5) is required here: H_3 has exact degeneracies (frozen singleton sectors give zero modes, and the real symmetric hopping matrix has a chiral $E \leftrightarrow -E$ structure), and the naive formula spuriously undercounts, violating $C_\infty \geq C_{\mathrm{univ}}$. Equation (S5) itself is evaluated by exact diagonalization to machine precision; as an independent consistency check, a direct Monte Carlo average of $\langle A(t)A(0)\rangle$ over random times agrees within its sampling error ($< 5 \times 10^{-3}$) for all tested seeds and sites.

TABLE S2. Universal capacity C_{univ} (sector-projector Mazur bound; Hamiltonian independent) versus the infinite-time autocorrelation C_∞ of $A = S^z$ at the central site, without and with dipole-conserving disorder ($V \sim \mathcal{N}(0, 0.5)$). C_∞ entries are mean ± std over ten realizations of the random hopping amplitudes (and, in the last column, of the disorder). The excess $C_\infty - C_{\mathrm{univ}}$ is non-universal: it grows with L and grows further under disorder.

L	C_{univ}	C_∞	C_∞ (disorder)	excess (clean / dis.)
4	0.3457	0.407(15)	0.490(28)	0.061(15) / 0.144(28)
5	0.3320	0.405(11)	0.459(32)	0.073(11) / 0.127(32)
6	0.3064	0.399(5)	0.461(24)	0.092(5) / 0.154(24)
7	0.2805	0.387(5)	0.451(24)	0.107(5) / 0.171(24)

By Mazur's inequality [4, 5] $C_\infty(A) \geq C_{\mathrm{univ}}(A)$, with equality iff each sector is internally thermal [8–10] (eigenstate expectations equal the sector average). Generically the inequality is strict, and the excess $C_\infty - C_{\mathrm{univ}}$ is *non-universal*: it depends on the Hamiltonian. To confirm it is dynamical rather than an artifact, we add a dipole-conserving density–density term $\sum_{i<j} V_{ij} S^z_i S^z_j$ with random V_{ij}. This term is diagonal, leaves the Krylov sectors—and hence C_{univ}—unchanged, but lifts intra-sector degeneracies. Averaged over ten realizations, the excess grows monotonically with L and grows further under disorder (Table S2), confirming it is localization-like dynamical memory sitting on top of the exact universal floor C_{univ}.

FROM HOLEVO CAPACITY TO A CLASSICAL MUTUAL INFORMATION

Encode the initial sector α with the infinite-temperature weight $p_\alpha = \mathrm{Tr}P_\alpha/D$; the infinite-time reduced state on a region R is $\rho^\alpha_R = \mathrm{Tr}_{\bar{R}}(P_\alpha)/\mathrm{Tr}P_\alpha$. The Holevo quantity [11] of the retained ensemble is $\chi_R = S(\sum_\alpha p_\alpha \rho^\alpha_R) - \sum_\alpha p_\alpha S(\rho^\alpha_R)$. For classical fragmentation every ρ^α_R is diagonal, so the von Neumann entropies are Shannon entropies of the region-configuration distribution $q^\alpha_R(c)$, and χ_R collapses to a mutual information

$$\chi_R = I(\alpha : c|_R) = \sum_{\alpha,c} P(\alpha, c)\,\log_2 \frac{P(\alpha, c)}{P(\alpha)P(c)}, \tag{S6}$$

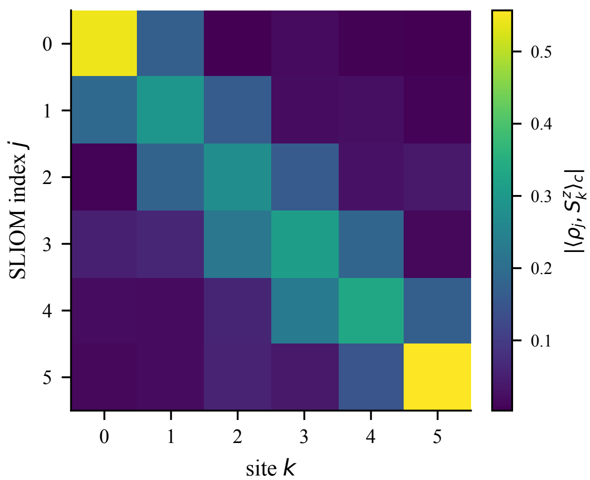

FIG. S1. Statistical localization of the root-density SLIOMs ($L = 6$): the connected overlap $|\langle \rho_j S_k^z \rangle_c|$ is diagonal-dominant, peaking at $k = j$ with $\mathcal{O}(1)$ width.

with c the configuration on R. Taking R to be the whole chain, each configuration is unique, so $\chi = H(\{p_\alpha\}) = -\sum_\alpha p_\alpha \log_2 p_\alpha$: the number of bits retained about which sector the system started in, and the exact per-site density in Table S1. It obeys $\chi \leq \log_2 K$, with equality only for equal-size sectors. χ_R is nondecreasing in R and saturates (main-text Fig. 4b), confirming the retained bits are locally accessible.

ROOT-DENSITY SLIOMS AND STATISTICAL LOCALIZATION

Each Krylov sector possesses a canonical root, its lexicographically minimal configuration (the maximally reduced "defect" state). Define the root-density operators $\rho_j(\text{state}) = [\text{root of its sector}]_j \in \{-1, 0, +1\}$. They are conserved (constant on each sector by construction) and jointly separating—the tuple $(\rho_0, \ldots, \rho_{L-1})$ equals the root, hence the exact sector label—so the algebra they generate is the complete set of sector projectors. They are statistically localized in the sense of Ref. [12]: the connected overlap $|\langle \rho_j S_k^z \rangle_c|$ peaks at $k = j$ with $\mathcal{O}(1)$ bulk variance ($1.17, 1.27, 1.84$ for $L = 5, 6, 7$; Fig. S1).

Their Mazur bounds form the hierarchy (for $A = S^z$, $L = 6$)

$$\underbrace{0.1206}_{\{N,P\}} < \underbrace{0.2337}_{\text{span}\{\rho_j\}} < \underbrace{0.3064}_{\langle \rho_j \rangle_{\text{alg}} = \{P_\alpha\} = C_{\text{univ}}} \leq \underbrace{0.3981}_{C_\infty}. \tag{S7}$$

The linear span of the L generators ρ_j already beats the two-symmetry bound; the *algebra* they generate (their products) spans the K-dimensional space of sector projectors and recovers the exact universal capacity. The gap between the linear and algebra bounds is expected: L linear operators cannot span a K-dimensional projector space, so the full capacity is reached only by including their products.

TRACE-MONOID STRUCTURE AND THE STRICTLY-LOCAL CEILING

By Eq. (S2) the dynamics permutes the height values by adjacent transpositions of values differing by one. This is a partially commutative (trace) monoid on the alphabet of integer heights [13]: two values *commute* (their transposition is a move) iff they differ by exactly one, and are *dependent* iff they are equal or differ by ≥ 2. By the Cartier–Foata projection lemma, the complete invariant of the *free* trace is the height multiset together with the projection of the height sequence onto every dependent value-pair (those differing by ≥ 2), all read off by a single strictly-local left-to-right sweep. This invariant is conserved and, at $L = 6$, separates 239 of the 337 sectors—strictly more than the multipole/multiset invariant $M_k = \sum_j h_j^k$, which separates 171 (Table S3).

The free-trace invariant is nonetheless *incomplete*, and this is structural. The physical swap (S2) is context-gated—by the height rule it requires $h_{i-1}, h_{i+2} \in \{m, m+1\}$ ($m = \min(h_i, h_{i+1})$) and $i \leq L-3$—so the true Krylov sectors *refine* the free-trace classes: two states may be free-trace equivalent (connected by unconstrained swaps) yet lie in distinct sectors because the connecting swaps are blocked. The residual distinction is a frozen ordering of *commuting* (differ-by-one) pairs. For instance, the $L = 6$ states with heights $(-1, -2, -3, -4, -5, -4)$ and $(-1, -2, -3, -4, -4, -5)$ share their multiset and their free-trace invariant, yet occupy different sectors, differing only in whether the height -5 sits between the two -4's at the right edge, where the boundary blocks the swap. A frozen, context-dependent ordering is a *statistically* localized datum, not a strictly local one.

Consequently, every invariant in this strictly-local hierarchy—by which we mean conserved labels computable in a single left-to-right sweep reading a finite window (string-order-like functionals; genuinely finite-support operators are weaker still)—tops out below com-

TABLE S3. Sectors separated by strictly-local conserved data: the multipole/multiset invariant and the free-trace (Cartier–Foata) invariant both fall short of the complete count K. Completeness requires the statistically localized ρ_j.

L	multiset	free-trace	complete K
5	79	99	139
6	171	239	337
7	365	577	815

pleteness: $\{N, P\} \subset$ multiset (171) $\subset$ free-trace invariant (239) $\subset$ complete (337) at $L = 6$ (Table S3). We do not claim a general no-go theorem for all conceivable local constructions in all fragmented models; what is shown is that for this model the natural complete invariant of the underlying trace monoid fails for a structural reason—the gated swaps freeze orderings of commuting pairs—and that the residual labels are, by construction, exactly the statistically localized content carried by the root densities ρ_j. This gives a first-principles account of why the complete integrals of motion of this shattered Hilbert space are "statistically" rather than sharply localized [12]. We also note that $M_1 = \sum_j h_j$ lies in the linear span of the global charges ($M_1 = LN - P$ for sites labelled $0, \ldots, L{-}1$), so only the $k \geq 2$ moments are genuinely new conserved quantities.

THERMODYNAMIC LIMIT: EXACT SECTOR COUNT AND CAPACITY DENSITY

Closed-form commutant dimension

The Krylov-sector count of the spin-1 dipole chain satisfies the inhomogeneous linear recurrence

$$K_L \;=\; 2K_{L-1} + K_{L-2} + 2, \tag{S8}$$

with $K_1 = 3$, $K_2 = 9$ (no three-site window exists), and $K_3 = 27 - 4 = 23$ (exactly four elementary moves, each merging two product states). We have verified Eq. (S8) with zero residual for every $L \leq 13$ (eleven consecutive values, $K_{13} = 161\,563$). Solving it,

$$K_L = \tfrac{1}{2}\left[(2{+}\sqrt{2})(1{+}\sqrt{2})^L + (2{-}\sqrt{2})(1{-}\sqrt{2})^L\right] - 1, \tag{S9}$$

so the count density converges to the silver ratio, $\log_2 K/L \;\to\; \log_2(1 + \sqrt{2}) \;=\; 1.27155$. Equivalently, $K_L = 2P_{L+1} - 1$ with P_n the Pell numbers ($P_1 = 1$, $P_2 = 2$, $P_{n+1} = 2P_n + P_{n-1}$). For comparison, the t–J_z chain has the exact commutant dimension $2^{L+1} - 1$ [6]; for the dipole model only numerical estimates (and lower bounds on frozen-state counts) were previously available [1].

Derivation from a boundary transfer system. In the $(L{+}1)$-site chain every elementary move either lies within the first L sites—preserving the class (σ, s) of (parent sector, appended letter)—or is the last-window move $x(a, {+}1, c) \;\leftrightarrow\; x(a{+}1, {-}1, c{+}1)$ with $a, c \in \{-1, 0\}$. Hence the $(L{+}1)$-sectors are precisely the classes of pairs (σ, s) under a boundary merge relation, and each merge joins exactly two pairs, (σ, c) and $(\sigma', c{+}1)$ with $c \in \{-1, 0\}$ (no triple merges occur; verified). Assign to each sector a *boundary type*: the set of achievable last-two-letter suffixes in its orbit, together with its one-step merge/child profile. Exactly 23 types occur, and the typing closes on itself (one refinement step reaches a fixed point). The **Boundary-Markov Lemma**—the child's type and the merge role are determined by (parent type, appended letter) through a fixed finite rule table—is verified exhaustively at all accessible sizes (rule identity across $L = 5$–8; consequences against direct enumeration for all $L \leq 13$); its proof for arbitrary L is the one remaining step. Granting it, type counts evolve linearly, $n(L{+}1) = M\, n(L)$, with an explicit 23×23 integer matrix satisfying $\mathrm{char}(M) = x^{17}(x{-}1)^2(x^2{-}2x{-}1)(x^2{-}x{+}1)$. By Cayley–Hamilton the sequence $K_L = u^T M^{L-5} n(5)$ obeys the order-23 recurrence with these roots; the closed form Eq. (S9) obeys the same recurrence, since $(x{-}1)(x^2{-}2x{-}1)$ divides $\mathrm{char}(M)$; and the two sequences agree exactly, in integer/radical arithmetic, on 26 consecutive values. They therefore coincide for all $L \geq 5$; $L \leq 4$ is checked directly (`scripts/transfer_system.py`). We note that the transfer system acts on sector boundary *types*, not on configurations: the root language itself is not regular at small window widths (widths 3–5 over-count, consistent with Sec.), which is why a state-level local grammar fails while the sector-level automaton closes.

Existence of the capacity density: subadditivity

Cut the chain into contiguous pieces of lengths $L_1 + L_2 = L$. Every elementary move of either piece is a move of the whole chain, while the whole chain possesses additional moves crossing the cut. Hence each full-chain sector is a *union* of products of half-chain sectors: the pair label $(\alpha_{L_1}, \alpha_{L_2})$ refines the full-chain sector label α, i.e. α is a function of the pair. Moreover α_{L_1} is a function of the left configuration alone, whose marginal at infinite temperature is exactly the uniform ensemble of the L_1-site problem, so $H(\alpha_{L_1}) = \chi(L_1)$ (and likewise for the right piece). Entropy is monotone under refinement and subadditive,

$$\begin{aligned}\chi(L) = H(\alpha) \;&\leq\; H(\alpha_{L_1}, \alpha_{L_2}) \\ &\leq\; H(\alpha_{L_1}) + H(\alpha_{L_2}) = \chi(L_1) + \chi(L_2),\end{aligned} \tag{S10}$$

so by Fekete's lemma $\chi(L)/L$ converges to $\chi_\infty = \inf_L \chi(L)/L$, and every finite size furnishes a rigorous upper bound (e.g. $\chi_\infty \leq 1.2107$ at $L = 13$; Table S1).

Value of χ_∞

The increments $\Delta(L) = \chi(L) - \chi(L{-}1)$ decrease monotonically ($1.1650, 1.1398, \ldots, 1.1003$ for $L = 4, \ldots, 13$) and converge to χ_∞ (Cesàro); if their monotonicity persists, $\chi_\infty \;\leq\; \Delta(13) \;=\; 1.1003$. Aitken acceleration of the increments gives 1.09737(26), where the quoted spread is the acceleration stability, and the arithmetically independent series $A(L) = \sum_\alpha p_\alpha \log_2 N_\alpha$ (mean log sector size) accelerates to $a = 0.48760(26)$, hence $\chi_\infty = \log_2 3 - a = 1.09736$, consistent at the 10^{-5} level. However, this agreement checks arithmetic, not the extrapolation *model*: the t–J_z chain (Sec.) has an exact $\frac{1}{2}\log_2 L$ correction to $\chi(L)$, and an analogous term here would bias plain geometric acceleration. Indeed the three-parameter fit $\chi(L) = aL + b\log_2 L + c$ on $L = 8$–13 is essentially exact (residual 3×10^{-7}) with $a = 1.080$, $b = 0.17$. We therefore quote, conservatively,

$$\chi_\infty = 1.09(1) \text{ bits per site,} \tag{S11}$$

the spread covering both extrapolation models; the rigorous statements remain the Fekete bound $\chi_\infty \leq 1.2107$ and, under increment monotonicity, ≤ 1.1003. Pinning the value (and ideally a closed form) awaits the renewal analysis below.

Exact block factorization of sector orbits

For each sector define its *active windows* as the three-site window positions at which some state of the sector admits an elementary move, and its *frozen (wall) bonds* as the bonds interior to no active window. The walls partition the chain into blocks, and we find that the sector orbit factorizes *exactly*: the sector size equals the product of the orbit sizes of its blocks evolved independently with open boundaries, for 100% of sectors (unweighted and probability-weighted) at $L = 8, 9, 10$ — an exhaustive check over 18 205 sectors. This renewal structure is what makes the capacity density intensive; turning it into a closed form for χ_∞ (via a generating function over irreducible blocks with boundary-compatibility labels) is left for future work.

Second model: t–J_z, a fully closed-form capacity

To demonstrate that the construction applies wholesale to classically fragmented systems, we treat the t–J_z chain [6, 12] in the no-double-occupancy basis: site values $\{\downarrow, 0, \uparrow\}$, kinetic moves $(\sigma, 0) \leftrightarrow (0, \sigma)$ on adjacent pairs, no spin exchange (the Ising J_z term is diagonal and does not affect the sectors). Particles cannot pass one another, so the conserved sector label is the left-to-right *spin word* w; all particle-position configurations with the same word are connected by hops. Hence $K = \sum_{n=0}^{L} 2^n = 2^{L+1} - 1$, reproducing the commutant dimension of Ref. [6], and the sector of a word with n particles has size $\binom{L}{n}$, i.e. $p_w = \binom{L}{n}/3^L$.

The capacity follows from the chain rule. Given n, all 2^n words are equiprobable, and at infinite temperature $n \sim \mathrm{Bin}(L, 2/3)$, so

$$\begin{aligned}\chi(L) = H(w) = H(n) + \mathbb{E}[n] &= \tfrac{2}{3}L + H\big[\mathrm{Bin}(L, \tfrac{2}{3})\big] \\ &= \tfrac{2}{3}L + \tfrac{1}{2}\log_2 L + O(1),\end{aligned} \tag{S12}$$

and the capacity density converges—with the same $O(\log L/L)$ correction as the dipole chain—to

$$\chi_\infty = \tfrac{2}{3} \text{ bit per site, exactly,} \tag{S13}$$

the particle density times one bit per retained spin: positions thermalize, the spin sequence is eternal. The count density is $\log_2 K/L \to 1$, again strictly above χ_∞. All statements were verified against brute-force sector enumeration for $L \leq 8$ ($K = 2^{L+1} - 1$ exactly; χ matches the closed form to 10^{-9}), and the correctness invariant holds on this model too ($C_{\mathrm{univ}} = 0.2807 \leq C_\infty = 0.4327$ at $L = 6$, central site, seed 0).

NUMERICAL METHODS AND REPRODUCIBILITY

All results are produced by the `mblc` Python package (`numpy`/`scipy`). Basis states are enumerated as tuples in $\{-1, 0, 1\}^L$; Krylov sectors are the connected components of the move graph (`scipy.sparse.csgraph.connected_components`). Exact diagonalization (`numpy.linalg.eigh`) is guarded by an explicit `N_max`; the autocorrelation study uses $L \leq 7$ ($D \leq 2187$), while the sector combinatorics (capacity, SLIOMs, trace invariant) run to $L = 10$ ($D = 59049$).

The key routines are `commutant.krylov_sectors` and `commutant_dimension` (K); `capacity.sector_entropy` and `information_capacity` (Eq. (S6)); `autocorr.sector_projector_mazur_bound` (Eq. (S4)), `build_hamiltonian`, and `infinite_time_autocorrelation` (Eq. (S5)); and `sliom.build_sliom`, `sliom_mazur_hierarchy`, `build_multipole_slioms`, and `build_trace_slioms`. The tables and figures are regenerated by `scripts/scaling.py`, `scripts/gap_analysis.py`, `scripts/thermo_limit.py` (the $L \leq 13$ sector census, recurrence check, Aitken acceleration, and exhaustive factorization test of Sec. ; runtime a few seconds), and `scripts/make_figures.py`; the t–J_z model of Sec. is `models.tjz_model` with its closed forms enforced in `tests/test_tjz.py`; the correctness invariant $C_\infty \geq C_{\mathrm{univ}}$ (with both positive) is enforced in `tests/test_capacity.py`, and the SLIOM properties in `tests/test_sliom.py`. Hamiltonian amplitudes and disorder use fixed seeds; the random-time cross-check of Eq. (S5) agrees with the closed form to $< 5 \times 10^{-3}$.

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