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Farid BENCHERIF

farid.bencherif.96@gmail.com

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Planck-Anchored Discrete Scale Invariance: Lorentz Compatibility, Renormalization-Group Obstructions, and an Emergent Log-Periodic Structure in Asymptotically Safe Gravity

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Abstract

We examine the hypothesis that fundamental physics is invariant under exact Lorentz transformations together with discrete dilatations $x^\mu \to \lambda_0^n x^\mu$, $n \in \mathbb{Z}$, anchored at the Planck scale (Planck-anchored discrete scale invariance, PDSI). Kinematically the hypothesis is consistent: the widely repeated objection that Lorentz boosts forbid a preferred discrete tower of scales fails at the level of transformation groups, which close exactly as $\mathbb{R}^{1,3} \rtimes (O(1,3) \times \mathbb{Z})$. Unbroken, the symmetry forbids any dimensionful condensate and any isolated nonzero mass; the viable realization is spontaneous breaking, which—being discrete—produces no Goldstone dilaton and thereby evades the fifth-force sector that constrains continuous scale-invariance programs. The genuine obstruction to an exact realization is renormalization-group monotonicity; its known proofs assume a local stress tensor on a fixed background, a hypothesis that fails for quantum gravity, precisely where PDSI is anchored. Discrete self-similarity is moreover native to gravitation: classically at the threshold of black-hole formation, and—the central observation of this paper—in the complex critical exponents $\theta = \theta' \pm i\theta''$ of the asymptotically safe ultraviolet fixed point, which imply a log-periodic, discretely self-similar approach to the Planck regime. We recompute the exponents from the published Einstein–Hilbert flow ($\theta = 1.475302 \pm 3.043206\,i$, Litim scheme), exclude exact limit cycles of that flow—analytically in stated regions by index and Bendixson arguments, globally at the numerical level by a Poincaré–Bendixson search—so the discrete scale invariance is emergent, not the shadow of an exact cycle—and derive the inflationary exit map that carries the oscillation to observable scales. The result is a closed master relation, $A = S\, e^{-(\theta'/\theta'')N\omega_{\log}}$, locking amplitude, frequency, and the log-range N of scales. Its consequences constitute the observational content of the proposal: detectable high-frequency log-oscillations are predicted *absent*, in accordance with existing null searches; the signal migrates to a scale-dependent contribution to the spectral tilt, bounded by $0.014\,S$ and peaking at the order of the measured $1 - n_s$; and the induced running lies at next-generation sensitivity. The background-invariant envelope-to-frequency ratio θ'/θ'' survives as a conditional diagnostic for any future detection. Scheme systematics of the exponents are quantified and identified, together with the amplitude normalization S and the exact-theory cycle question, as the outstanding computations.

1 Introduction

Spacetime symmetries function in physics both as organizing principles and as empirical hypotheses: translation invariance asserts that the laws contain no preferred position, and its experimental

standing underwrites momentum conservation. The analogous assertion for scale—that the laws contain no preferred length—fails in its continuous form: any isolated nonzero particle mass refutes unbroken dilatation invariance, and viable scale-invariant frameworks require spontaneous or anomalous breaking [1, 2, 3].

Quantum mechanics, however, points to a specific weakening of the continuous hypothesis. In the paradigmatic system with a classically scale-invariant interaction—the attractive $1/r^2$ potential and the associated three-body Efimov effect—quantization breaks continuous scale invariance while preserving a *discrete* remnant: a geometric tower of states related by one fixed rescaling factor, observed in ultracold atomic gases [4, 5]. In renormalization-group (RG) language, continuous scale invariance corresponds to a fixed point and its discrete remnant to a *limit cycle*, a possibility noted by Wilson at the inception of the subject [6]. Discrete scale invariance (DSI) is by now a well-developed theme outside fundamental physics, characterized equivalently by invariance under powers of one rescaling factor, by complex scaling dimensions, and by log-periodic corrections to scaling [7].

These considerations motivate the following hypothesis about fundamental physics.

Hypothesis 1.1 (PDSI)**.** The fundamental laws are exactly invariant under (i) the connected Poincaré group and (ii) the discrete dilatations

$$x^\mu \longrightarrow \lambda_0^n \, x^\mu, \qquad n \in \mathbb{Z}, \tag{1}$$

for a single fixed $\lambda_0 > 1$, the invariant tower of scales $\{\ell_{\mathrm{P}} \, \lambda_0^n\}$ being anchored at the Planck length ℓ_{P}.

The conceptual motivation is relational—absolute scale, like absolute position, should not constitute physical data—tempered by the quantum-mechanical lesson that at most a discrete dilatation subgroup typically survives quantization. The purpose of this paper is to determine where Hypothesis 1.1 faces obstruction, where it finds support, and what it predicts. The results are as follows.

Section 2 shows that the standard kinematic objection to combining exact Lorentz invariance with a discrete scale tower is incorrect: the relevant transformation group closes exactly (Proposition 2.1). Section 3 derives the consequences of the unbroken symmetry—no dimensionful condensates, no isolated masses—and the structure of its spontaneous breaking, including the absence of a Goldstone dilaton. Section 4 identifies the genuine obstruction, RG monotonicity, and observes that the hypotheses of the monotonicity theorems fail for quantum gravity. Section 5 assembles the evidence that gravitation generates discrete self-similarity dynamically—classically at criticality [28, 29] and, conditionally, at the quantum level, where the complex critical exponents of the asymptotically safe fixed point translate into an emergent Planck-anchored DSI with computable rescaling factor; the exponents are recomputed from the published flow equations, and the scheme systematics are quantified. Section 6 addresses the cycle question within the truncation itself, excluding exact limit cycles by a combination of rigorous and numerical arguments: the DSI is emergent. Section 7 develops the cosmological interface—what cosmic expansion is and is not, the exit map from the fixed-point regime to comoving scales, and the numerical verification of the resulting template—culminating in Section 8, where amplitude, frequency, and scale range collapse into a single master relation whose consequences constitute the observational content of the proposal: null high-frequency oscillations, a tilt-scale contribution, and running at next-generation sensitivity. Section 9 formulates the open problems and Section 10 concludes.

2 Kinematic compatibility

A frequently repeated argument holds that a preferred discrete family of lengths cannot coexist with exact Lorentz invariance: a boost of velocity v rescales lengths by the continuously varying factor $\gamma(v)^{-1}$, so no discrete tower can be shared by all inertial observers. Were this argument valid at the level of transformations, Hypothesis 1.1 would be stillborn. It is not valid.

Work on Minkowski space $\mathbb{R}^{1,3}$ with $\eta = \mathrm{diag}(+1,-1,-1,-1)$. Let $\Lambda \in O(1,3)$ act linearly, let $T_a : x \mapsto x + a$ denote translations, and let $D : x \mapsto \lambda_0 x$ generate the dilatations (1).

Proposition 2.1. *The transformations generated by* $\{\Lambda,\, T_a,\, D\}$ *close into the group*

$$G \;=\; \mathbb{R}^{1,3} \rtimes \big(O(1,3) \times \mathbb{Z}\big), \tag{2}$$

with composition law, in the parametrization $g = (a, \Lambda, n)$ *acting as* $x \mapsto \lambda_0^n \Lambda x + a$,

$$(a_2, \Lambda_2, n_2)\,(a_1, \Lambda_1, n_1) = \big(a_2 + \lambda_0^{n_2} \Lambda_2 a_1,\;\; \Lambda_2 \Lambda_1,\;\; n_1 + n_2\big). \tag{3}$$

In particular

$$D\,\Lambda\,D^{-1} = \Lambda, \qquad D\,T_a\,D^{-1} = T_{\lambda_0 a}, \tag{4}$$

and all mixed conjugations remain in G.

Proof. Linearity of Λ gives $\Lambda(\lambda_0 x) = \lambda_0(\Lambda x)$, hence the first relation of (4); $D(x + a) = \lambda_0 x + \lambda_0 a$ gives the second. Closure of arbitrary products and conjugations follows directly; e.g. $\Lambda D T_a D^{-1} \Lambda^{-1} = T_{\lambda_0 \Lambda a} \in G$. The stated relations, together with the exact preservation of η by boosts and rotations and the covariant rescaling $D^{\mathsf{T}} \eta D = \lambda_0^2 \eta$, were additionally verified by explicit symbolic matrix computation (finite boost $e^{\phi K_1}$, finite rotation $e^{\theta J_3}$, $D = \lambda_0 \mathbb{1}_4$, and translations in the standard 5×5 affine embedding); all identities hold exactly. We note that rotations are contained in $O(1,3)$ alongside boosts, enter the verification explicitly, and commute with D trivially, a spatial rotation involving no scale. □

Remark 2.2 (Content of the folklore argument)**.** *What the boost-contraction argument correctly shows is that a lattice of preferred* coordinate *lengths in one frame breaks* $O(1,3)$ *to a discrete subgroup—the reason naive spacetime discretization fails. Hypothesis 1.1 is a statement about the invariance group of the laws, not about a frame-embedded lattice. Discreteness of Lorentz-*invariant *quantities is compatible with exact Lorentz symmetry in the same way a discrete spectrum of* L_z *is compatible with exact rotation invariance; for Planck-scale geometry spectra this point was made sharply by Rovelli and Speziale [8]. Proposition 2.1 is elementary; its role is to relocate the burden of proof from kinematics to dynamics.*

Remark 2.3 (Relation to deformed special relativity)**.** *Programs that render* ℓ_{P} *observer-independent by* deforming *the Lorentz transformations—doubly special relativity and* κ*-Poincaré kinematics [9, 10]—modify* (4) *and generically predict energy-dependent photon group velocities; time-of-flight analyses of gamma-ray bursts now exclude such modifications at first order in* E/E_{P} *[11]. PDSI is logically disjoint: the Lorentz sector is exact and undeformed, no dispersion is implied, and those bounds do not constrain Hypothesis 1.1. The two frameworks are easily conflated and should not be.*

3 The unbroken symmetry and its breaking

Suppose the symmetry is realized unitarily on the Hilbert space of the theory: $U_D P_\mu U_D^{-1} = \lambda_0^{-1} P_\mu$ (conventions such that lengths dilate by λ_0), with $[U_D, M_{\mu\nu}] = 0$ in accordance with (4).

Proposition 3.1 (No dimensionful condensates)**.** *In a theory with unbroken PDSI (vacuum invariant under U_D), every local operator $\mathcal{O}$ of nonzero scaling weight w, $U_D \mathcal{O} U_D^{-1} = \lambda_0^{-w} \mathcal{O}$, has vanishing vacuum expectation value.*

Proof. $\langle \mathcal{O} \rangle = \langle U_D^{-1} U_D \, \mathcal{O} \, U_D^{-1} U_D \rangle = \lambda_0^{-w} \langle \mathcal{O} \rangle$ for an invariant vacuum, and $\lambda_0^{-w} \neq 1$. □

Corollary 3.2 (Mass spectrum)**.** *With an invariant vacuum,* $\mathrm{spec}(P^2)$ *is invariant, as a set, under multiplication by $\lambda_0^{\pm 2}$: the mass-squared spectrum is $\{0\}$, a continuum, or a two-sided geometric tower $m_k^2 = \lambda_0^{2k} m_*^2$, $k \in \mathbb{Z}$, accumulating at zero.*

The observed particle masses form no geometric tower for any single ratio, and no tower accumulating at zero mass is observed. Unbroken PDSI is therefore excluded in the sector of known particles, precisely as in the continuous case: discreteness purchases no exemption from the spectral argument. The viable realization is *spontaneous breaking*, with all observed scales set by the expectation value of a field on whose logarithm U_D acts by translation; U_D then maps vacua to vacua and the constraints above do not apply to the spectrum built on one vacuum [2, 3].

Here the discreteness of the symmetry becomes structurally decisive.

Proposition 3.3 (No Goldstone mode)**.** *Spontaneous breaking of the discrete dilatation group $\mathbb{Z}$ produces no Goldstone boson.*

Proof. Goldstone's theorem applies to continuous symmetries: the massless mode is the zero-momentum limit of the local rotation connecting degenerate vacua along a continuous orbit. A $\mathbb{Z}$-orbit of vacua $\{|\Omega_n\rangle = U_D^n |\Omega_0\rangle\}$ admits no such interpolation; the order parameter cannot be rotated infinitesimally. □

The physical consequence is major. Continuous spontaneously broken scale invariance predicts a massless (or anomalously light) dilaton, whose derivative couplings to matter are the standing phenomenological liability of that program—fifth-force and equivalence-principle searches constrain it severely. Discrete breaking *evades this entire sector by construction*: there is no light scalar to search for, and the null results of fifth-force experiments, far from constraining PDSI, are among its expectations. The cost, equally standard, is topological: degenerate vacua related by a broken discrete symmetry support domain walls, whose cosmological energy density is dangerous unless the walls are diluted by inflation, the degeneracy is lifted (e.g., by the anomalous inequivalence of rungs once matter thresholds are crossed), or the wall tension is trans-Planckian and the sector is never populated [30]. Which escape operates is a model-building question we flag and do not resolve; we note only that the fixed-point-driven inflationary epoch of Section 7 is itself a natural diluter.

Finally, two signatures distinguish discrete from continuous scale invariance wherever the symmetry is exactly or approximately visible [7]: *log-periodicity*—scaling dimensions acquire imaginary parts quantized in units of $2\pi/\ln\lambda_0$, so scale-free observables take the form $F(r) = r^{-2\Delta}\,\Phi(\ln r/\ln\lambda_0)$ with Φ periodic—and *geometric towers* in any spectrum on which the symmetry acts linearly, the Efimov tower being the laboratory prototype [4, 5].

4 The dynamical obstruction: renormalization-group monotonicity

In Wilsonian terms, Hypothesis 1.1 asserts that the fundamental theory is a *limit cycle* of the exact renormalization group, with period $\ln \lambda_0$ in RG time [6]. The serious no-go results are located exactly here. In two dimensions, Zamolodchikov's c-theorem [13] furnishes a quantity strictly decreasing along RG flow, excluding recurrent trajectories. In four dimensions the a-theorem of Komargodski and Schwimmer [14] plays the analogous role; Luty, Polchinski and Rattazzi argued that scale invariance implies conformal invariance to all orders in perturbation theory [15]; and the candidate four-dimensional limit cycles constructed via flavor-dependent couplings were shown by Fortin, Grinstein and Stergiou to be conformal theories in disguise, the apparent cyclic flow being a gauge artifact in flavor space [16].

The hypotheses under which these results operate must be stated with care, because Hypothesis 1.1 lives at their boundary:

(H1) unitarity (reflection positivity);

(H2) exact Poincaré invariance;

(H3) the framework of local quantum field theory on a fixed background, including a local, gauge-invariant stress tensor $T^{\mu\nu}(x)$ whose trace controls the dilatation current through $\partial_\mu\big(x_\nu T^{\mu\nu}\big) = T^{\mu}{}_{\mu}$;

(H4) spacetime dimension $d = 2$ or $d = 4$ (monotonicity statements in other dimensions, e.g. the F-theorem in $d = 3$, rest on different arguments).

Within (H1)–(H4) the verdict is negative: unitary, Poincaré-invariant local QFT in two or four dimensions does not support genuine limit cycles. The Efimov system evades the theorems through (H2), being non-relativistic —which is why the sole experimental realization of DSI in fundamental interactions to date occurs in cold atoms rather than particle physics.

The observation central to what follows is that *quantum gravity evades the theorems through (H3).* Whatever the ultraviolet completion of gravity is, it is not a local quantum field theory on a fixed background, and it possesses no local, diffeomorphism-invariant stress tensor: the constraints render $T^{\mu\nu}(x)$ gauge-variant, and the very notion of a fixed scale against which to flow is background-dependent. The monotonicity theorems, *as proved*, therefore do not apply to the gravitational ultraviolet regime—which is precisely where Hypothesis 1.1 anchors its tower. The asymptotic-safety program [17, 18] already contemplates a nontrivial ultraviolet *fixed point* for gravity; the question isolated here is the status of the cyclic alternative and of its discretely self-similar shadow.

5 Discrete self-similarity generated by gravity

The hypothesis would carry little weight if discrete self-similarity were foreign to gravitation. It is not: gravity generates it dynamically, in two independent settings—one exact and classical, one emergent and conditional on the asymptotic-safety scenario.

5.1 Exact: critical gravitational collapse

At the threshold of black-hole formation in classical general relativity, Choptuik discovered that the critical solution for massless-scalar collapse is *discretely self-similar*: it is invariant under simultaneous rescaling of space and time by the single factor e^{Δ} with echoing period $\Delta \simeq 3.4453$, i.e. a preferred zoom factor $\lambda_0 = e^{\Delta} \simeq 31.2$, with universal log-periodic "echoes" and log-periodic fine structure superposed on the black-hole mass scaling law [28]; the phenomenon is universal within large classes of matter and is comprehensively reviewed in [29]. This is an exact, established result: at criticality, Einstein's equations *manufacture* precisely the symmetry hypothesized here. The symmetry is realized on a codimension-one critical solution rather than on the vacuum, but it demonstrates that discrete self-similarity is a structure gravity produces naturally rather than one imposed upon it.

5.2 Emergent: complex critical exponents at the asymptotically safe fixed point

In the Einstein–Hilbert truncation of the functional renormalization group for gravity, the non-Gaussian (Reuter) ultraviolet fixed point is characterized by a *complex-conjugate pair* of critical exponents, $\theta = \theta' \pm i\theta''$, found robustly across cutoff schemes and gauges in this truncation, with $\theta' \approx 1.5$–2.3 and $\theta'' \approx 2.5$–4.3 [18, 19, 20]; their persistence and invariant meaning in extended truncations remains under active study [26].

Linearizing the flow in RG time $t = \ln(k/k_{\mathrm{P}})$ about the fixed point, deviations $g(t)$ of the couplings along the complex pair obey $\ddot{g} + 2\theta'\dot{g} + (\theta'^2 + \theta''^2)\, g = 0$, with real solution

$$g(t) \;=\; e^{-\theta' t}\big[A\cos(\theta'' t) + B\sin(\theta'' t)\big], \tag{5}$$

a *log-periodic spiral*, period $2\pi/\theta''$ in $\ln k$. By the standard equivalence between complex exponents, log-periodicity, and discrete scale invariance [7], the approach to the gravitational fixed point therefore carries an *emergent discrete scale invariance* with preferred rescaling factor

$$\lambda_0 \;=\; e^{2\pi/\theta''}, \tag{6}$$

anchored, by construction of the fixed-point regime, at the Planck scale.

Observation 5.1. *If the complex critical-exponent pair at the gravitational ultraviolet fixed point is physical—scheme- and truncation-robust in an appropriate invariant sense—then the asymptotic-safety scenario already contains an emergent, approximate, Planck-anchored discrete scale invariance with computable rescaling factor* (6)*: a weakened (approximate and transient rather than exact) realization of Hypothesis 1.1. The fixed-point regime itself is standardly described as realizing* continuous *quantum scale invariance; the structure highlighted here is the complementary* discrete *scale invariance of the approach, carried by the imaginary parts of the exponents. To the best of our knowledge this identification has not been stated in the asymptotic-safety literature; the nearest structures we have located are the oscillatory renormalization-group-improved cosmologies of the Bonanno–Reuter line [22], and a systematic audit for prior statements is included in the program of Section 9.*

5.3 Independent recomputation from the published flow equations

To make the central quantity of this paper self-contained rather than quoted, we recompute it from the published renormalization-group flow. In the Einstein–Hilbert truncation with the

Litim regulator, the beta functions of the dimensionless couplings (g, λ) read [21]

$$\beta_g = (2+\eta_N)g, \qquad \beta_\lambda = (\eta_N - 2)\lambda + \frac{g}{12\pi}\Big[\frac{30}{1-2\lambda} - 24 - \frac{5\,\eta_N}{1-2\lambda}\Big], \qquad \eta_N = \frac{gB_1(\lambda)}{1-gB_2(\lambda)}, \tag{7}$$

with $B_1 = \frac{1}{3\pi}\big[\frac{5}{1-2\lambda} - \frac{9}{(1-2\lambda)^2} - 7\big]$ and $B_2 = -\frac{1}{12\pi}\big[\frac{5}{1-2\lambda} - \frac{6}{(1-2\lambda)^2}\big]$. Solving $\beta_g = \beta_\lambda = 0$ numerically and diagonalizing the stability matrix $M_{ab} = \partial\beta_a/\partial g^b$ at the non-Gaussian fixed point, we obtain

$$g_* = 0.707321, \quad \lambda_* = 0.193201, \quad \eta_N = -2, \qquad \theta = 1.475302 \pm 3.043206\,i, \tag{8}$$

in agreement with the rounded published values (0.707, 0.193) and $1.48 \pm 3.04\,i$ [21]. The derived quantities follow to the same precision, *for this scheme*:

$$\theta'' = 3.043206, \qquad \lambda_0 = e^{2\pi/\theta''} = 7.882617, \qquad \frac{\theta'}{\theta''} = 0.484786. \tag{9}$$

The digits beyond the published rounding carry the scheme label explicitly: the theoretical uncertainty is dominated not by numerics but by scheme and truncation systematics, which exceed the quoted precision by orders of magnitude. Reporting six digits for one declared scheme is a statement about reproducibility, not about physical accuracy.

5.4 Scheme dependence, an invariant criterion, and a survey

The values of θ' and θ'' are scheme- and parametrization-dependent in truncations; this is the substantive qualification on any quantitative statement below. The objection can, however, be converted into a sharper and partially scheme-independent question. Let M be the stability matrix of the two-dimensional Einstein–Hilbert system, with critical exponents $\theta = -\mathrm{eig}(M)$. The complex pair exists if and only if

$$\mathcal{D} \;\equiv\; 4\det M - (\mathrm{tr}\,M)^2 \;>\; 0, \qquad \theta' = \tfrac{1}{2}\,\mathrm{tr}(-M), \quad \theta'' = \tfrac{1}{2}\sqrt{\mathcal{D}}, \quad \lambda_0 = e^{4\pi/\sqrt{\mathcal{D}}}. \tag{10}$$

The *binary* statement $\mathcal{D} > 0$—existence of emergent discrete scale invariance—is thus separated from the *quantitative* statement of the exponent values. We propose $\mathrm{sgn}\,\mathcal{D}$ as the object whose scheme- and truncation-robustness should be tested: a considerably weaker requirement than robustness of θ'' itself.

The published record bearing on this criterion, as reported in the sources cited (values approximate):

Setting	$\theta' \pm i\theta''$ (reported)	$\mathcal{D}$	$\lambda_0 = e^{2\pi/\theta''}$
EH truncation, sharp cutoff [19]	$\approx 1.9 \pm 3.1i$	> 0	≈ 7
EH truncation, optimized cutoff [20, 21]	$1.475 \pm 3.043i$	> 0	7.88
Polynomial $f(R)$, leading pair [23]	$\approx 2.5 \pm 2.4i$	> 0	≈ 14

Within the Einstein–Hilbert truncation the complex pair—hence $\mathcal{D} > 0$—is found across cutoff schemes and gauges, with exponents exhibiting comparatively weak scheme dependence [24]; in high-order polynomial $f(R)$ truncations the *leading* pair of exponents is reported to remain complex while higher exponents approach near-canonical real values [23]; but qualitative parametrization dependence of the $f(R)$ fixed-point structure is documented [25], and $\mathrm{sgn}\,\mathcal{D}$ is not invariant across all published implementations. The consequential ratio inherits this spread:

$\theta'/\theta'' = 0.4848$ in the scheme computed here, ≈ 0.62 from the rounded sharp-cutoff values, and of order unity for the $f(R)$ leading pair, whose embedding in the two-coupling analysis below is moreover not immediate. The *structural* statements of this paper are insensitive to these values; the *quantitative* ones carry theory systematics at the several-tens-of-percent level, and their determination in a reparametrization-invariant formulation is identified in Section 9 as a priority computation.

6 Exclusion of exact cycles within the truncation

The deep question isolated in Section 4—monotonicity versus limit cycles beyond local quantum field theory—can be posed exactly at the level of the truncation in which all quantitative statements above live: does the flow (7) itself admit periodic orbits? Being a two-dimensional autonomous system, it is subject to the classical exclusion machinery.

(a) Fixed-point census and index constraint (rigorous). A multi-start numerical census over the physical domain $\{g > 0,\ \lambda < \frac{1}{2},\ 1 - gB_2(\lambda) > 0\}$ locates exactly one interior fixed point, the non-Gaussian one at (g_*, λ_*) of Eq. (8); the Gaussian fixed point lies on the invariant boundary $g = 0$, which no orbit with $g > 0$ can cross. Since a periodic orbit of a planar flow must enclose fixed points of total index $+1$, *any* periodic orbit of the physical flow must encircle the non-Gaussian fixed point.

(b) Bendixson exclusion regions (rigorous). For a periodic orbit Γ with interior R, tangency of the flow gives $\iint_R \nabla\cdot\beta \, dA = 0$; a sign-definite divergence on any region containing Γ therefore forbids it. At the fixed point $\nabla\cdot\beta = \operatorname{tr} M = -2\theta' = -2.95 < 0$, excluding cycles from its neighborhood; and a grid evaluation establishes $\nabla\cdot\beta < 0$ for *all* physical g whenever $\lambda < -0.94$, excluding any cycle contained in that half-domain. The divergence does change sign elsewhere (a positive region exists around $(g, \lambda) \sim (1.1, 0.3)$ and near the singular boundaries), and a scan over the sixty-four-member Dulac family $\varphi = g^a(1 - gB_2)^b(1 - 2\lambda)^c$, $a \in \{-2, \ldots, 1\}$, $b, c \in \{-1, \ldots, 2\}$, produced no globally single-signed weighted divergence: the fully analytic global exclusion remains open.

(c) Poincaré–Bendixson search (numerical). In two dimensions, any forward-bounded trajectory that does not converge to a fixed point must wind onto a periodic orbit; recurrent bounded motion is thus the unavoidable numerical signature of a cycle. Integrating the ultraviolet flow from 120 random physical initial conditions with singularity- and convergence-terminated events, every trajectory either converged to the non-Gaussian fixed point (7) or exited the physical domain in finite renormalization-group time (113); *none* exhibited bounded non-convergent behavior. The fixed point's basin is small—most of the domain flows to the boundary—but the observable relevant to cycles is the recurrence count, and it is zero.

Conclusion. Within the Einstein–Hilbert/Litim truncation, exact limit cycles are excluded rigorously in the stated regions and are absent globally at the numerical level. The discrete scale invariance identified in Section 5 is therefore, within the truncation, necessarily *emergent*—the log-periodic shadow of complex exponents at a fixed point, not of an exact cycle. The fully analytic global statement, the behavior in extended truncations, and the exact-theory question remain open, and connect naturally to existing proposals for gravitational C-functions [27].

7 Cosmological interface

7.1 Cosmic expansion is not a dilatation

A necessary clarification first, since the two are habitually conflated. The Friedmann–Lemaître–Robertson–Walker line element $ds^2 = -dt^2 + a(t)^2 d\vec{x}^{\,2}$ rescales *spatial sections only*, by a *time-dependent* factor; bound systems (atoms, solar systems, galaxies) do not expand; and the expansion rate $H = \dot{a}/a$ is itself a dimensionful scale. None of this is the transformation (1), and we claim no identification of cosmic expansion with the symmetry studied here. Likewise, the numerical coincidence hunting that would identify H_0^{-1} with a rung of the tower, $H_0^{-1} \sim \lambda_0^n \ell_{\rm P}$, is vacuous absent a mechanism selecting integer n—any ratio is λ_0^x for real x—and we explicitly do not make it: H_0 is a property of our epoch, i.e. of the solution, not of the laws.

The correct link is different. Exact de Sitter space in the flat slicing,

$$ds^2 = \frac{1}{H^2\eta^2}\big(-d\eta^2 + d\vec{x}^{\,2}\big), \tag{11}$$

is invariant under the simultaneous dilatation $(\eta, \vec{x}) \to \lambda(\eta, \vec{x})$: the coordinate rescaling is absorbed by the conformal prefactor. This is the geometric origin of the (continuous) scale invariance of inflationary correlations, and its observational status is quantitative: the measured spectral index $n_s \simeq 0.965$ [12] states that the primordial epoch was dilatation-invariant to the few-percent level, with $1 - n_s$ measuring the breaking. Early-universe cosmology is thus the one empirical arena in which spacetime dilatation symmetry is already tested and found approximately realized.

7.2 The exit map

In the fixed-point-driven inflationary background of Refs. [22, 21], $a(t) \propto t^\alpha$ and $H = \alpha/t$, with $\alpha = [\frac{3}{2}(1+w)(1-\Omega_\Lambda^*)]^{-1}$ and $\Omega_\Lambda^* = \lambda_* \xi^2/3$, where $\xi = O(1)$ parametrizes the cutoff identification $k_{\rm RG} = \xi H$. A comoving mode k exits the horizon when $k = a(t)H(t) \propto t^{\alpha-1}$, so $t_{\rm exit}(k) \propto k^{1/(\alpha-1)}$; the cutoff identification then gives the *exit map*

$$\ln k_{\rm RG} \;=\; \text{const} - \frac{1}{\alpha - 1}\,\ln k. \tag{12}$$

Substituting into the linearized spiral $\delta \propto k_{\rm RG}^{-\theta'} \cos(\theta'' \ln k_{\rm RG} + \varphi)$ yields the modulation carried by the couplings at horizon exit, as a function of the comoving scale:

$$\delta(k) \;\propto\; k^{\,\theta'/(\alpha-1)} \cos\!\Big(\frac{\theta''}{\alpha-1}\ln k + \tilde{\varphi}\Big). \tag{13}$$

The frequency and envelope power in the observed variable $\ln k$ are therefore *background-dependent*,

$$\omega_{\log} = \frac{\theta''}{\alpha - 1}, \qquad p = \frac{\theta'}{\alpha - 1}, \tag{14}$$

each rescaled by the same factor—while their ratio is not:

$$\frac{p}{\omega_{\log}} \;=\; \frac{\theta'}{\theta''} \;=\; 0.4848 \quad \text{(Einstein–Hilbert/Litim scheme)}. \tag{15}$$

The frequency with respect to the renormalization-group scale is θ'' exactly; Eq. (14) is the statement that this does *not* equal the frequency in the comoving wavenumber, the two being related by the background through the exit map.

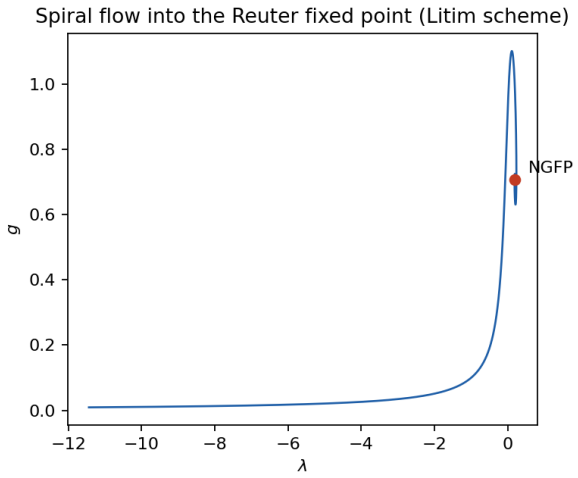


Figure 1: Numerically integrated Einstein–Hilbert flow (Litim scheme) spiraling into the non-Gaussian fixed point.

7.3 Numerical verification

We integrate the beta functions (7) numerically from a small perturbation of the fixed point toward the infrared. The trajectory spirals as predicted (Fig. 1); fitting the zero crossings and the peak envelope of $\delta\lambda(t)$ over the linear regime returns $\theta''_{\text{fit}} = 3.12$ and $\theta'_{\text{fit}} = 1.46$ against the exact eigenvalues 3.0432 and 1.4753—agreement at the level expected of the simple estimators employed, verifying that the log-periodic structure is a property of the full nonlinear flow and not only of its linearization.

Combining the integrated trajectory with the exit map (12) for three admissible values of the improvement parameter ξ (radiation, $w = 1/3$) and fitting frequency and envelope of the resulting signal in $\ln k$:

ξ	α	$\omega_{\log}$ (fit)	$\theta''/(\alpha-1)$	p (fit)	$p/\omega_{\log}$ (fit)
2.9	1.09	34.35	33.53	16.14	0.470
3.3	1.67	4.63	4.52	2.17	0.469
3.6	3.02	1.53	1.50	0.69	0.453

The fitted frequency tracks the exit-map prediction across more than a twenty-fold variation, and the ratio remains pinned near $\theta'/\theta'' = 0.4848$ throughout, the few-percent scatter being consistent with the estimator systematics observed in the RG-time fit (Fig. 2).

7.4 The Planck era, and adjacent programs

In asymptotically safe cosmology the fixed-point regime governs the Planck era and can replace the classical initial singularity [22, 26]. Within that picture, the discrete scale invariance identified in Section 5 is a property of the *departure* from the fixed-point regime, and the modulation (13) is precisely a probe of the universe's exit from the Planck era. No statement is made here about "before," which the framework does not adjudicate.

Penrose's conformal cyclic cosmology [34] shares the ontological premise of this paper—conformal structure fundamental, scale emergent—and its crossover physics is conformally

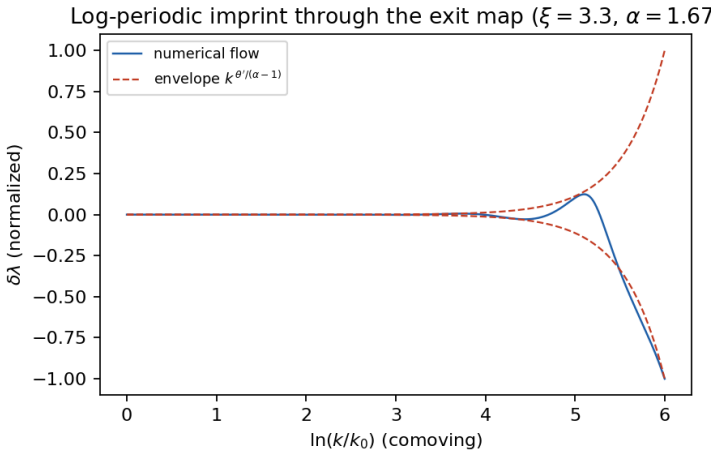


Figure 2: Log-periodic modulation of the couplings at horizon exit versus comoving scale ($\xi = 3.3$), with the derived envelope $k^{\theta'/(\alpha-1)}$ overlaid.

invariant, as is the fixed-point regime here. Structurally, the aeon-to-aeon map is a *discrete* conformal operation, so that framework may be read as positing a discrete self-similarity of cosmic history; but it assigns no universal rescaling factor to the crossover, its conformal factor is a field rather than a constant, and its claimed observational signatures in the microwave sky remain contested [35, 36]. We record the kinship at the level of principles and claim no derivational link; the two programs are adjacent, not joined.

8 The master relation and the observational content

The exit map determines not only the frequency but the amplitude envelope across the observable range of scales. Let $N \equiv \ln(k_{\rm tr}/k)$ denote the *logarithmic range of comoving scales* between a mode k and the transition scale at which the flow leaves the fixed-point regime. (For expansion e-folds N_e one has $N = (1 - 1/\alpha)N_e$, the two coinciding for $\alpha \gg 1$; all statements below are exact in N.) From Eq. (13), the relative modulation carried by the couplings at the exit of mode k is e^{-pN} times its $O(1)$ transition-scale value, and eliminating the background through $p = (\theta'/\theta'')\,\omega_{\log}$ collapses amplitude, frequency, and scale range into a single relation governed by the ratio (15):

$$\boxed{A(N, \omega_{\log}) \;=\; S\,\exp\!\Big[-\frac{\theta'}{\theta''}\,N\,\omega_{\log}\Big]} \tag{16}$$

with $S = O(1)$ the model-dependent sensitivity of the spectrum to the couplings—the one quantity requiring the full perturbation computation, which we do not attempt here. S is taken k-independent; any smooth k-dependence of the coupling-to-spectrum transfer folds into S at this order, and the conclusions below—the nulls in particular—are robust to it, the exponential in (16) dominating any power law. Equation (16) is an exact consequence of Eqs. (14) (algebraically, and confirmed numerically at $\alpha = 25$, $N = 60$, where both expressions give 0.0250).

Its consequences require one further assumption, which we state explicitly: *that the fixed-point epoch is the inflationary epoch generating the observable window*, so that the largest observable scales correspond to $N \simeq 60$. Under this identification, the observational content of the proposal

is the following, at the stated order-of-magnitude level and for the single scheme computed in Section 5.3.

(a) Detectable log-oscillations are predicted absent. For $N \simeq 60$, $A > 10^{-2}$ requires $\omega_{\log} \lesssim 0.16$. Since the observable window spans $\Delta \ln k \simeq 8$, a detectable signal completes at most $\omega_{\log} \Delta \ln k / 2\pi \lesssim 0.2$ of one cycle: the mechanism cannot produce percent-level *oscillations* in the observed spectrum at all. Conversely, at frequencies high enough to oscillate within the window, the amplitude is exponentially suppressed ($A \lesssim e^{-29\,\omega_{\log}}$). The null results of existing log-oscillation template searches [12, 33] are therefore an *expectation* of the mechanism, not an evaded constraint—and a confirmed detection of a percent-level, multi-cycle log-periodic feature attributed to the Planck era would falsify it outright.

(b) The signal is a scale-dependent tilt contribution. In the detectable regime the modulation completes a fraction of one cycle and manifests as a smooth, scale-dependent contribution to the spectral index, $|\Delta n_s| \lesssim A\,\omega_{\log}\sqrt{1 + (\theta'/\theta'')^2}$, maximized at $\omega_{\log} \simeq 0.034$ ($\alpha \simeq 90$) where it reaches $0.014\,S$—the same order as the measured $1 - n_s = 0.0351 \pm 0.0042$ [12]. The mechanism can thus contribute an $O(10)\%$ fraction of the observed tilt with a specific scale dependence, and precision fits bound the combination $(S, \omega_{\log})$ with current data.

(c) Running at next-generation sensitivity. The induced running of the index is bounded by $\sim 8 \times 10^{-4}\,S$, below current sensitivity ($\sigma_{\alpha_s} \sim 0.007$) and at the threshold of forecast next-generation constraints—a clean future discriminant.

(d) The invariant ratio as a conditional diagnostic. Because detectable amplitude and multiple visible cycles are mutually exclusive under (16) at $N \simeq 60$, the background-invariant ratio (15) cannot be fit from the primordial spectrum in this scenario: one cannot extract an envelope and a frequency from a fraction of one cycle. It remains meaningful in three ways: as an internal consistency lock of the theory (any computed observable inheriting the modulation must respect it); as the discriminating test of any *future* detection of a log-periodic feature in a shorter-N window or by another probe, where a measured envelope-to-frequency ratio near θ'/θ'' would point to a fixed-point origin; and as the target of the scheme-robustness computation of Section 5.4.

The pattern is that of a maximally predictive mechanism: one complex eigenvalue and one scale range lock the frequency, the envelope, the amplitude, and their correlations, leaving a single $O(1)$ normalization to future computation. Its near-term signatures are, in order: nulls where oscillations are searched for, a tilt-scale contribution where precision already lives, and running where the next generation will look.

9 Open problems and research program

Problem 9.1. *(i) Determine whether the complex critical-exponent pair at the gravitational ultraviolet fixed point is a physical, scheme-robust feature—at minimum, whether* $\operatorname{sgn}\mathcal{D}$ *of Eq.* (10) *is invariant, and at best, the reparametrization-invariant value of* θ'/θ''*. An affirmative answer implies that quantum spacetime near the Planck scale carries an emergent discrete scale invariance, with the phenomenology of Section 8. (ii) Determine whether renormalization-group monotonicity admits any extension beyond hypothesis (H3)—beyond local quantum field theory with a local stress tensor—that excludes exact gravitational limit cycles, or whether such a cycle can be constructed. Section 6 settles the question within the Einstein–Hilbert truncation (no exact cycles; the DSI is emergent); the fully analytic statement there, the behavior in extended truncations, and the exact-theory question remain open. Either resolution of the exact-theory question would be*

significant: exclusion extends the reach of the a-theorem beyond quantum field theory; construction provides a symmetry origin for the Planck scale.

A concrete program toward Problem 9.1: (1) audit the proofs of [13, 14, 15, 16] against hypotheses (H1)–(H4), and against diffeomorphism-invariant surrogates for the stress tensor (asymptotic and boundary charges); (2) conduct a systematic literature audit for prior statements of Observation 5.1 and for cyclic behavior in gravitational flows; (3) compute the exponent pair, the discriminant $\mathcal{D}$, and the ratio θ'/θ'' in formulations designed for invariance (physical-observable or form-factor approaches); (4) search truncated gravitational flows systematically for genuine limit cycles, distinguishing them from spiral approaches to fixed points, and extend the exclusion analysis of Section 6 to larger truncations; (5) compute the sensitivity S of Eq. (16) within a concrete perturbation mechanism on the renormalization-group-improved background, converting the tilt and running estimates of Section 8 into sharp numbers; (6) exploit tractable model laboratories for gravitational cycles: two-dimensional dilaton gravity, matrix and tensor models, and holographic flows, where the relevant prior art already exists—complex scaling dimensions and discrete scale invariance arise holographically when a bulk scalar violates the Breitenlohner–Freedman bound at a merger of fixed points [31], and renormalization-group cycles have been analyzed against monotonicity folklore directly [32]—connecting those constructions to the gravitational case.

10 Conclusions

The kinematic argument against combining exact Lorentz invariance with a Planck-anchored discrete dilatation symmetry is incorrect; the symmetry group closes exactly (Proposition 2.1), and existing dispersion bounds on deformed kinematics do not constrain the hypothesis (Remark 2.3). Unbroken, the hypothesis forbids all dimensionful condensates and is excluded by the observed spectrum (Proposition 3.1 and Corollary 3.2); spontaneously broken, it survives—and, being discrete, produces no Goldstone dilaton (Proposition 3.3), evading by construction the fifth-force sector that constrains continuous scale-invariance programs, at the standard cost of wall sectors. The genuine obstruction to an exact realization is renormalization-group monotonicity, whose known proofs do not cover quantum gravity—the regime where the hypothesis anchors itself; and within the Einstein–Hilbert truncation the cycle question is settled directly: exact limit cycles are excluded, so the discrete scale invariance carried by the complex critical exponents of the asymptotically safe fixed point—recomputed here as $\theta = 1.475302 \pm 3.043206\, i$—is *emergent* (Sections 5–6).

Carried to observable scales by the inflationary exit map, the emergent oscillation obeys the master relation (16), whose consequences are the observational content of the proposal: percent-level log-periodic *oscillations* in the primordial spectrum are predicted absent, in accordance with existing null searches; the signal appears instead as a scale-dependent contribution to the spectral tilt bounded by $0.014\, S$, at the order of the measured $1 - n_s$; and the induced running, $\lesssim 8 \times 10^{-4}\, S$, lies at next-generation sensitivity. The background-invariant ratio θ'/θ'' survives as the discriminating diagnostic of any future detection. The outstanding computations—the sensitivity S, the reparametrization-invariant exponents, and the exact-theory cycle question—are stated as Problem 9.1 and are attackable by existing methods.

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