Research Square

Preprints are preliminary reports that have not undergone peer review.
They should not be considered conclusive, used to inform clinical practice,
or referenced by the media as validated information.

A Novel EOQ Model for Deteriorating Items with Non-Zero Initial Ramp-Type Demand and Two-Phase Discounting

Rina Khatun
IIEST Shibpur
Shariful Alam
IIEST Shibpur
Khairul Habib Alam
research@alamkh.com

SRM University

Research Article

Keywords: Inventory EOQ management, Advertisement effect, Two-phase discounting scheme, Non-zero initial demand, General Ramp-type demand

Posted Date: September 16th, 2026

DOI: https://doi.org/10.21203/rs.3.rs-10359163/v1

License: This work is licensed under a Creative Commons Attribution 4.0 International License.
Read Full License

Additional Declarations: No competing interests reported.

A Novel EOQ Model for Deteriorating Items with Non-Zero Initial Ramp-Type Demand and Two-Phase Discounting

Rina Khatun[1], Shariful Alam[1], Khairul Habib Alam[2*]

[1]Department of Mathematics, IIEST Shibpur, College Road, Howrah, 711103, WB, India.
[2]Department of Mathematics, SRM University, Delhi-NCR, Sonepat, 131029, Haryana, India.

*Corresponding author(s). E-mail(s): research@alamkh.com; khairulhabib@srmuniversity.ac.in;
Contributing authors: 2024map019.rina@students.iiests.ac.in; salam@math.iiests.ac.in;

Abstract

This study introduces and demonstrates an EOQ-type inventory management model for deteriorating items. It is tailored to take into account more realistic features of demand and inventory behavior. In reality, demand for many products usually starts at zero, it often starts at a positive level because of factors such as advertisement or advance bookings. Building on this idea, the model considers an initially positive, linearly increasing demand that finally changes into a ramp-type demand, which stabilizes after a specific time. The deterioration of items is considered proportional to time, showing a slight decrease in item quality or utility during storage time. In order to deal with this, the model uses different discount policies for the pre-deterioration and post-deterioration stages, which are operational techniques that increase demand and decrease inventory losses. The model is presented for a single-item system with no shortages, constant holding costs, and an infinite planning horizon. Closed-form formulas for the optimal inventory policy are constructed and demonstrated using MATLAB-based numerical illustrations and sensitivity analysis with graphical representations. These formulas also show how key parameters such as discount levels, the deterioration rate, and advertisement-driven initial demand impact system performance. This

work extends existing ramp-type demand models by using a two-phase discounting policy and specifically considering non-zero initial demand, which provides a deeper understanding of optimal inventory management.

Keywords: Inventory EOQ management, Advertisement effect, Two-phase discounting scheme, Non-zero initial demand, General Ramp-type demand

MSC 2020: 90B05 , 90B60 , 91B24

1 Introduction

Managing the inventory of deteriorating products is a very important for industries such as food production, medicine, fashion, and electronics. These products have to deal with two parallel problems, they are likely to deteriorate over time and their demand patterns change periodically. The classical EOQ models, which consider a constant demand rate and no deterioration effect, are not suitable in such cases. Real-world demand is rarely stable, it usually starts at a nonzero level (caused by advertising or pre-orders), increases slowly, and finally stabilizes, similar to a ramp-type demand pattern. At the same time, deterioration is natural, medicines lose their efficiency, food deteriorates, and electronic products get expired.

Over the last few decades, researchers have developed inventory theory by integrating deterioration, shortages, trade credit, and advertising into EOQ models. For example, in 2006, Manna et al. [1] proposed an EOQ framework with a ramp-type demand function and time-dependent deterioration, while Skouri et al. [2] extended it with a general time-dependent demand function. Wu and Zhao [3] demonstrated the role of the stock-dependent demand function and trade credit, showing that advertising and product display have a significant effect on people's purchasing habits. Recently, Rabiu and Musa [4] considered discounting policies to control demand during deterioration. Other studies introduced non-zero initial demand to demonstrate the reality of product launch events and advertising-driven early sales.

Even after all these developments, almost all models handle these characteristics separately. There is still no complete model that considers all the following factors

- non-zero initial demand due to advertising,
- general ramp-type demand function with stabilization,
- time-dependent linear deterioration, and
- dual discounts applied before and after deterioration.

In order to fulfill this requirement, this research presents an infinite-horizon inventory model without shortages that integrates all of these characteristics. This will offer managers a more advanced tool for pricing and replenishment decisions.

2 Literature Review

A number of factors are important to consider when studying inventories of deteriorating items. Among these, demand is most important since it manages the entire inventory system, and the rate of deterioration measures the inherent nature of the considered items. In addition, characteristics such as pricing and discount policies, allowances or restrictions on shortages, inflationary effects, and the time value of money also an important impact on inventory management. By considering different combinations of these characteristics, researchers can formulate different kinds of inventory models, each tailored to capture specific real-world situations.

Classical and Deteriorating Inventory Models

For perishable or degradable items, the deterioration rate is another important characteristic in the study of inventory management, which represents a gradual decrease in product quality or usefulness. Depending on the quality of the items and the conditions under which they are stored, researchers will have to consider many important factors when studying deterioration. At the beginning, EOQ models considered constant demand and ignored the consideration of deterioration effects. Following that, the research in this field got started, and Ghare and Schrader [5] introduced deterioration into inventory systems for the first time. Since then, researchers have developed models that demonstrate deterioration effects under different conditions, such as constant deterioration [6, 7], exponential decay [8, 9], or Weibull distributions [10, 11].

Ramp-Type and Time-Dependent Demand Models

In the literature, inventory models generally demonstrated two categories of demand functions, deterministic and stochastic. Deterministic demand can be expressed in different forms, such as constant demand [12, 13], time-dependent demand [14, 15], stock-dependent demand [16, 17], and price-dependent demand [18]. On the other hand, stochastic demand considers uncertainty and is usually expressed by stochastic processes such as Brownian motion or probability distributions such as Poisson or normal distribution. Ramp-type demand is a specific form of deterministic demand in which demand increases continuously to a maximum level and then stabilizes after a specific time. To represent real purchase behavior, Hill [19] introduced ramp-type demand models. Manna and Chaudhuri [1] further extended this model by demonstrating an EOQ model with a ramp-type demand function and time-dependent deterioration. In order to address different production and shortage phases, Skouri et al. [2] proposed a model for partial treatments of stabilization. Recent work of Chauhan et al. [20] has used ramp-type demand to seasonal and fashion goods, considering inflation, backorder discounts, and increasing holding costs.

Demand Influences: Advertising and Pricing

Demand can be significantly affected by advertising, especially for new items, where effective use of advertising can increase sales. To demonstrate this, researchers have integrated advertising into inventory models in different ways. In 2013, Shah et al.

[21] proposed a model with generalized costs, deterioration, and non-linear price-advertising effects. In 2015, Palanivel and Uthayakumar [22] studied inflation-linked pricing with advertisement-sensitive demand. In 2017, Manna et al. [23] demonstrated a model for deteriorating items with exponential advertising. Later on, these works extended using integration of demand to advertising frequency, perishables with trade credit [24], reliability [25], Weibull deterioration with credit [26], and non-instantaneous deterioration under payment delays [27].

In parallel to advertising, pricing techniques with applicable discounts, play an important role in inventory research. Weatherford et al. [28] and Petruzzi et al. [29] demonstrated independent impacts of temporary discounts in inventory models, which motivated further research on dynamic pricing. For instance, the researchers in [30], [31], and [32] investigated pricing inventory interactions under different assumptions. This concept further extended in 2000 by Khouja et al. [33], who considered multiple discounts to sell excess inventory. The research of Rabiu and Musa [4] introduced selling-price discounts in ramp-type deteriorating models, showing their significance in maximizing profit. However, most of these models considers a single type of discount and do not differentiate between pre- and post-deterioration phases.

Recent Reviews and Extensions

Deteriorating inventory research is classified at the standard level by detailed reviews of Raafat [34] and Li et al. [35]. More recently, Karimi [36] provided a post-2018 classification of deteriorating inventory models into 18 classes and 33 subclasses, including demand structures, deterioration functions, shortages, financial conditions, and solution methods. Although there are improvements, these reviews show that there is still scope for models that integrate different real-world challenges, such as advertising-driven demand, time-dependent deterioration, and dual-discount techniques.

Research Gaps

From the above review, several gaps are identified:

1. **Non-Zero Initial Demand Due to Advertising:** Most of the classical inventory models considers ramp-type demand which starts from zero. Only few of them considers non-zero initial demand that comes from advertising or pre-orders, which is important to realistic modeling.
2. **General Ramp-Type Demand Function:** Past research has generally used functional forms (linear, trapezoidal, exponential) for ramp-type demand. There is more to know about the flexibility in accepting any general positive function $f(t)$ that stabilizes at a fixed period r.
3. **Dual Discount Mechanism:** Existing inventory models usually consider a single price or a backorder discount. The role of separate pre-deterioration (d_a) and post-deterioration (d_b) demand discounts is rarely considered.

4. **Time-Proportional Deterioration:** In ramp-type demand problems, the simpler and more practical linear deterioration of time does not receive much consideration as Weibull or exponential deterioration.
5. **Unified Infinite-Horizon Framework:** Previous inventory models consider advertising, ramp-demand, deterioration, and discounts separately. Very few models integrate all these factors under an infinite horizon with no shortages (and later, with shortages). This can increase both theoretical flexibility and practical use.

This paper bridges the above research gaps by developing an inventory model without shortages. It simultaneously considers advertising-induced non-zero initial demand, a general ramp-type demand function stabilizing after a fixed time. In addition, the model integrates time-linear deterioration and distinct demand discounts before and after deterioration. These characteristics are considered in our order-specific model under infinite-horizon. The study provides a more advanced and practical tool for industries managing deteriorating products under dynamic demand conditions.

3 Assumptions and notations

This section lists the notations used and assumptions considered in developing the model.

Notations

I	Inventory at time $t = 0$.
$I(t)$	Inventory level at time t.
$I_d(t)$	Inventory level after deterioration time.
I_d	Inventory at deterioration time $t = d$.
d	Time when deterioration started.
d_1	Discount on demand before deterioration.
d_2	Discount on demand after deterioration.
t_1	Inventory cycle time.
$D(t)$	Deterioration rate.
$D_1(t)$	Demand rate before deterioration time.
$D_2(t)$	Demand rate after deterioration time.
o	Ordering cost per order.
A	Advertisement cost per cycle.
h	Holding cost per unit per time.
H	Holding cost per cycle.
c_d	Deterioration cost per unit.
DC	Deterioration cost per cycle.
AC	Average total inventory cost per time.
C	Purchasing cost per unit.
P	Selling price per unit.
AP	Average profit per cycle time.

Assumptions

We complete the inventory model considering the following environment:

- The inventory model deals with a single type of item.
- Planning horizon is assumed to be infinite.
- Holding cost remains constant over time.
- Replenishment rate is considered unlimited.
- Lead time for replenishment is assumed to be negligible.
- Deterioration rate varies proportionally with time.
- The model does not allow shortages.
- Demand at the beginning is non-zero because of advertisement.
- Advertisement cost is constant per cycle.

4 Problem Description

We consider a single-item inventory management model for deteriorating items over an infinite planning horizon. The model is motivated by real-world situations where demand is influenced by advertising and evolves dynamically with time, while products gradually decreases their quality due to deterioration. The model is considered to run without shortages, with replenishment cycles repeat in the same way. The following notations and assumptions explain the formulation.

Pre-deterioration demand $D_1(t)$: Before deterioration begins ($0 \leq t \leq d$), demand is modeled as a linear, time-dependent function

$$D_1(t) = \alpha + \beta t, \tag{1}$$

where $\alpha > 0, \beta \geq 0$. The constant α is the positive initial demand because of advertising, promotional campaigns, or advance bookings. The parameter β is the growth in demand over time because of increasing customer awareness or expanding market acceptance.

Post-deterioration demand $D_2(t)$: After deterioration begins ($t \geq d$), demand follows a ramp-type function

$$D_2(t) = \begin{cases} f(t), & t < r \\ f(r), & t \geq r \end{cases}, \tag{2}$$

where $f(t)$ is a general positive function. This formulation represents a gradual increase in demand up to a stabilization point $t = r$, after that demand remains constant at $f(r)$. A structure like this is consistent with product life-cycle behavior, where demand increases rapidly in the initial stages and then reaches its peak level as the market develops.

Discount-adjusted demand: To balance customers demand, discounts are incorporated into both demand phases. Before deterioration, the effective demand is scaled by

$$d_a = (1 - d_1)^{-p}, \qquad p \in \mathbb{R}, \tag{3}$$

while after deterioration, demand is adjusted by

$$d_b = (1 - d_2)^{-q}, \qquad q \in \mathbb{R}. \tag{4}$$

Here, d_1 and d_2 are the discounts, and p, q are parameters controlling their intensity. This structure reflects how discounts stimulate sales differently in the pre- and post-deterioration periods.

Deterioration function: The deterioration of items is a time-dependent function and proportional to the storage duration:

$$D(t) = \xi t, \tag{5}$$

for some constant $\xi > 0$. This linear form represents situations where items gradually lose quality at a rate increasing with time, such as food spoilage, chemical degradation, or technology obsolescence. The parameter ξ is the sensitivity of the item to deterioration with time.

Model Formulation

In consideration of the above assumptions and the additional condition that shortages are not allowed, our aim in this paper is to develop and investigate an order-level inventory model for deteriorating items under a finite replenishment rate. At the starting time $t = 0$, the inventory level is I, and at the deterioration time $t = d$, the inventory level is I_d. Before deterioration starting time $t = d$, the inventory level $I(t)$ at time t, is affected by the linear demand $D_1(t)$. After the deterioration time $t = d$, the inventory level $I_d(t)$ at time t is affected by the ramp-type demand $D_2(t)$, which has a fixed stabilization time $t = r$. Finally, when $t = t_1$, the inventory falls to zero, and the process itself starts repeating. This can be seen in Fig. 1.

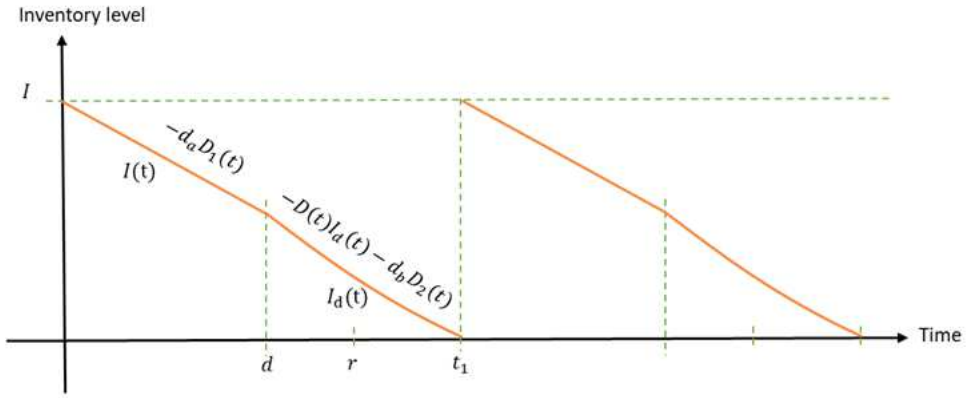


Fig. 1 Diagram representing the model

Thus, we have the following differential equations describing the changes in the inventory level in respect of the time within the cycle time $[0, t_1]$.

$$\frac{dI(t)}{dt} = -d_a D_1(t), \qquad 0 \le t \le d, \tag{6}$$

with boundary conditions $I(0) = I$ and $I_d(d) = I_d$;

$$\frac{dI_d(t)}{dt} = -D(t)I_d(t) - d_b D_2(t), \qquad d \le t \le r, \tag{7}$$

with initial condition $I_d(d) = I_d$;

$$\frac{dI_d(t)}{dt} = -D(t)I_d(t) - d_b D_2(t), \qquad r \le t \le t_1, \tag{8}$$

with terminal condition $I_d(t_1) = 0$.

Utilizing the functions $D_1(t), D_2(t)$ and $D(t)$ defined in the equations (1), (2) and (5) respectively, the differential equations (6), (7) and (8) becomes

$$\frac{dI(t)}{dt} = -d_a \beta t - d_a \alpha, \qquad 0 \le t \le d, \tag{9}$$

with boundary conditions $I(0) = I$ and $I_d(d) = I_d$;

$$\frac{dI_d(t)}{dt} + \xi t I_d(t) = -d_b f(t), \qquad d \le t \le r, \tag{10}$$

with initial condition $I_d(d) = I_d$;

$$\frac{dI_d(t)}{dt} + \xi t I_d(t) = -d_b f(r), \qquad r \le t \le t_1, \tag{11}$$

with terminal condition $I_d(t_1) = 0$.

Now, we solve the three ordinary differential equations on the intervals $0 \le t \le d$, $d \le t \le r$ and $r \le t \le t_1$ respectively, with the given boundary/initial/terminal conditions.

Integrating both sides of equation (9) with respect to t, we get

$$I(t) = \int \big(- d_a \beta\, t - d_a \alpha \big)\, dt = -d_a \beta \frac{t^2}{2} - d_a \alpha t + Constant.$$

Using $I(0) = I$, we have $Constant = I$. Hence,

$$I(t) = I - d_a \left(\frac{\beta}{2} t^2 + \alpha t \right), \qquad 0 \le t \le d. \tag{12}$$

In particular, the inventory level at $t = d$ (which is the initial condition for the next interval) is

$$I_d = I(d) = I - d_a \left(\frac{\beta}{2} d^2 + \alpha d \right). \tag{13}$$

Utilizing the integrating factor $e^{\int \xi t dt} = e^{\frac{\xi t^2}{2}}$ and integrating both sides of equation (10) from $t = d$ to an arbitrary $t \in [d, r]$, we get

$$I_d(t) = e^{-\frac{\xi t^2}{2}} \left(I_d(d)\, e^{\frac{\xi d^2}{2}} - d_b \int_d^t f(s)\, e^{\frac{\xi s^2}{2}}\, ds \right).$$

Substituting $I_d(d) = I_d$ obtained in (13), we get

$$I_d(t) = e^{-\frac{\xi t^2}{2}} \left[\left(I - d_a \left(\frac{\beta}{2} d^2 + \alpha d \right) \right) e^{\frac{\xi d^2}{2}} - d_b \int_d^t f(s)\, e^{\frac{\xi s^2}{2}}\, ds \right], \qquad d \le t \le r. \tag{14}$$

Note that $I_d(r)$ is the value at $t = r$ can be obtained from equation (14), that is

$$I_d(r) = e^{-\frac{\xi r^2}{2}} \left[\left(I - d_a \left(\frac{\beta}{2} d^2 + \alpha d \right) \right) e^{\frac{\xi d^2}{2}} - d_b \int_d^r f(s)\, e^{\frac{\xi s^2}{2}}\, ds \right]. \tag{15}$$

Similarly multiplying by integrating factor and integrating the equation (10) between an arbitrary $t \in [r, t_1]$ and the point r, we get

$$I_d(t) = e^{-\frac{\xi t^2}{2}} \left[I_d(r)\, e^{\frac{\xi r^2}{2}} - d_b f(r) \int_r^t e^{\frac{\xi s^2}{2}}\, ds \right].$$

Now substituting $I_d(r)$, we get

$$\begin{aligned} I_d(t) = e^{-\frac{\xi t^2}{2}} \Bigg[& \left(I - d_a \left(\frac{\beta}{2} d^2 + \alpha d \right) \right) e^{\frac{\xi d^2}{2}} - d_b \int_d^r f(s)\, e^{\frac{\xi s^2}{2}}\, ds \\ & - d_b f(r) \int_r^t e^{\frac{\xi s^2}{2}}\, ds \Bigg], \qquad r \le t \le t_1. \end{aligned} \tag{16}$$

Applying the terminal condition $I_d(t_1) = 0$, we obtain the following required inventory function $I_r(t_1)$ depending on t_1 to reach $I(t_1) = 0$ by the terminal time t_1.

$$I_r(t_1) = d_a \left(\frac{\beta}{2} d^2 + \alpha d \right) + d_b e^{-\frac{\xi d^2}{2}} \left[\int_d^r f(s)\, e^{\frac{\xi s^2}{2}}\, ds + f(r) \int_r^{t_1} e^{\frac{\xi s^2}{2}}\, ds \right]. \tag{17}$$

In this case, we use the required inventory function I_r depending on t_1 to calculate the following cost functions and finally we will optimize the average profit.

From the piece wise solutions in equations (12), (14) and (16), we clearly see $I(t)$ depends on I and I depends on t_1 only via the linear factor $I_r(t_1)$.

Consequently, the holding cost can be written as

$$\begin{aligned}
H(t_1) &= h\int_0^{t_1} I(t;t_1)dt \\
&= h\int_0^{d}\left[I_r(t_1) - d_a\left(\frac{\beta}{2}t^2+\alpha t\right)\right]dt \\
&\quad +h\int_d^{r} e^{-\frac{\xi t^2}{2}}\left[\left(I_r(t_1) - d_a\left(\frac{\beta}{2}d^2+\alpha d\right)\right)e^{\frac{\xi d^2}{2}} - d_b\int_d^t f(s)\,e^{\frac{\xi s^2}{2}}\,ds\right]dt \\
&\quad +h\int_r^{t_1} e^{-\frac{\xi t^2}{2}}\left[\left(I_r(t_1) - d_a\left(\frac{\beta}{2}d^2+\alpha d\right)\right)e^{\frac{\xi d^2}{2}} - d_b\int_d^r f(s)\,e^{\frac{\xi s^2}{2}}\,ds\right. \\
&\quad \left. -d_b f(r)\int_r^t e^{\frac{\xi s^2}{2}}\,ds\right]dt. \qquad (18)
\end{aligned}$$

The total units demanded is

$$\begin{aligned}
Q(t_1) &= d_a\int_0^d D_1(t)dt + d_b\int_d^{t_1} D_2(t)dt \\
&= d_a\left(\alpha d+\frac{\beta}{2}d^2\right) + d_b\int_d^r f(t)dt + d_b f(r)(t_1-r). \qquad (19)
\end{aligned}$$

Total purchasing cost is

$$PC(t_1) = CI_r(t_1). \qquad (20)$$

The deterioration cost is

$$\begin{aligned}
DC(t_1) &= c_d\left(I_r(t_1) - Q(t_1)\right) \\
&= c_d\left[I_r(t_1) - d_a\left(\alpha d+\frac{\beta}{2}d^2\right) - d_b\int_d^r f(t)dt - d_b f(r)(t_1-r)\right]. \qquad (21)
\end{aligned}$$

Total selling revenue is

$$\begin{aligned} SR(t_1) &= PQ(t_1) \\ &= P\left[d_a\left(\alpha d + \frac{\beta}{2}d^2\right) + d_b\int_d^r f(t)dt + d_b f(r)(t_1 - r)\right]. \end{aligned} \tag{22}$$

Average profit is

$$AP(t_1) = \frac{SR(t_1) - (PC(t_1) + H(t_1) + DC(t_1) + o + A)}{t_1}. \tag{23}$$

To optimize the average profit, we now solve the differential equation

$$\frac{dAP(t_1)}{dt_1} = \frac{1}{t_1}\left(\frac{dSR(t_1)}{dt_1} - \frac{dPC(t_1)}{dt_1} - \frac{dH(t_1)}{dt_1} - \frac{dDC(t_1)}{dt_1}\right) - \frac{1}{t_1}AP(t_1) = 0. \tag{24}$$

Now

$$\frac{dSR(t_1)}{dt_1} = P\frac{dQ(t_1)}{dt_1} = Pd_b f(r),$$
$$\frac{dPC(t_1)}{dt_1} = C\frac{dI_r(t_1)}{d(t_1)} = Cd_b f(r)e^{\frac{\xi}{2}(t_1^2 - d^2)},$$
$$\frac{dDC(t_1)}{dt_1} = c_d\left(\frac{dI_r(t_1)}{dt_1} - \frac{dQ(t_1)}{dt_1}\right) = c_d d_b f(r)\left(e^{\frac{\xi}{2}(t_1^2 - d^2)} - 1\right),$$

and utilizing Leibniz rule with the terminal condition $I(t_1; t_1) = 0$, we get

$$\frac{dH(t_1)}{dt_1} = h\left[I(t_1; t_1) + \int_0^{t_1}\frac{\partial I(t; t_1)}{\partial t_1}dt\right] = h\int_0^{t_1}\frac{\partial I(t; t_1)}{\partial t_1}dt.$$

Rewriting in terms of $I(t; t_1)$ and differentiating the equations (12), (14) and (16) partially with respect to t_1, we get

$$\int_0^{t_1}\frac{\partial I(t; t_1)}{\partial t_1}dt = \int_0^{d}\frac{\partial I(t; t_1)}{\partial t_1}dt + \int_d^{r}\frac{\partial I(t; t_1)}{\partial t_1}dt + \int_r^{t_1}\frac{\partial I(t; t_1)}{\partial t_1}dt.$$

Hence

$$\frac{dH(t_1)}{dt_1} = hd_b f(r)\left[de^{\frac{\xi}{2}(t_1^2 - d^2)} + e^{\frac{\xi t_1^2}{2}}\int_d^r e^{-\frac{\xi t^2}{2}}dt + e^{\frac{\xi t_1^2}{2}}\int_r^{t_1} e^{-\frac{\xi t^2}{2}}dt\right].$$

Consequently, form (24), the optimal condition is

$$d_b f(r)\left(P + c_d - (C + c_d))e^{\frac{\xi}{2}(t_1^2 - d^2)}\right) - \frac{dH(t_1)}{dt_1} - AP(t_1) = 0. \tag{25}$$

5 Numerical Computations and Sensitivity Analysis

Setting the parameter values as $\alpha = 53, \beta = 3, d = 2.23, r = 2.83, \xi = 0.073, d_1 = 0.153, d_2 = 0.253, p = 1.23, q = 0.93, P = 63, C = 23, h = 1.53, c_d = 0.83, o = 123, A = 203$ and considering $f(t) = kt$, where $k = 673$, the optimal (calculated using MATLAB code provided at the end of this section) cycle time is obtained $t_1^* = 4.398487$. Also, the average profit is $AP^* = 33240.415246$ for the inventory is $I^* = 6913.181974$.

To investigate the stability and reliability of the proposed inventory model, a detailed sensitivity analysis (calculated using MATLAB code provided at the end of this section) has been conducted. This is done by varying a key parameter at a time while keeping the remaining parameters fixed. The results are presented in the corresponding tables and illustrated by the three-dimensional plots in the corresponding figures. The tables and figures demonstrate the variations in the optimal cycle time t_1^*, the inventory level I^*, and the average profit AP^*.

Table 1 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of α and β.

α	β	t_1^*	I^*	AP^*
53.000	2.500	4.398775	6912.874483	33227.624146
53.000	2.667	4.398679	6912.976971	33231.887745
53.000	2.833	4.398583	6913.079440	33236.151449
53.000	3.000	4.398487	6913.181892	33240.415246
53.000	3.167	4.398390	6913.284327	33244.679133
53.000	3.333	4.398294	6913.386746	33248.943111
53.000	3.500	4.398198	6913.489147	33253.207183
50.000	3.000	4.400062	6911.626791	33170.533602
51.000	3.000	4.399537	6912.145668	33193.824668
52.000	3.000	4.399012	6912.664031	33217.118586
54.000	3.000	4.397961	6913.699244	33263.714697
55.000	3.000	4.397435	6914.216090	33287.016934
56.000	3.000	4.396910	6914.732429	33310.321938

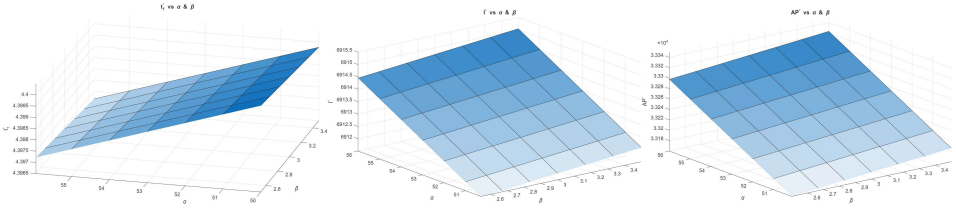


Fig. 2 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of α and β.

The first set of experiments examines the influence of the demand-related parameters α and β (see Table 1 and Fig. 2). It is observed that the cycle time remains

nearly unchanged. There are only minor fluctuations, varying from 4.39 to 4.40 units. This indicates that replenishment intervals are largely unaffected by changes in these parameters. The inventory level demonstrates a slow but steady increase as both α and β increase, showing a moderate sensitivity. In contrast, the average profit is more responsive, showing clear improvement as α increases and, to a lesser extent, with changes in β. This suggests that profitability is more strongly influenced by demand scaling than by the shape of the demand function.

Table 2 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of ξ and d.

ξ	d	t_1^*	I^*	AP^*
0.0750	2.000	4.227623	6897.380717	34919.812652
0.0750	2.067	4.266987	6879.578982	34356.803147
0.0750	2.133	4.307135	6859.576101	33782.468992
0.0750	2.200	4.348040	6837.316490	33197.149650
0.0750	2.267	4.389674	6812.751872	32601.155940
0.0750	2.333	4.432013	6785.841079	31994.771763
0.0750	2.400	4.475032	6756.549968	31378.254257
0.0600	2.200	4.618188	7562.791930	35731.359491
0.0650	2.200	4.519294	7299.536526	34831.892321
0.0700	2.200	4.429705	7058.767290	33989.243549
0.0800	2.200	4.273192	6632.626480	32450.349677
0.0850	2.200	4.204262	6442.603796	31744.377654
0.0900	2.200	4.140511	6265.514260	31075.406540

The second set of experiments considers the impact of the deterioration parameter (ξ) and the demand parameter (d) (see Table 2 and Fig. 3). Here, the effects are more pronounced. As the deterioration rate ξ increases, the cycle time becomes shorter, reflecting the need for quicker replenishment when items are prone to faster decay. Inventory levels decline sharply with higher values of ξ, while average profit also falls significantly, dropping from 35,731 to 31,075 units as ξ increases from 0.06 to 0.09. A similar negative effect is observed with increasing d, where both inventory and profit tend to decline. These findings highlight the critical importance of managing product deterioration, as even small increases in ξ can substantially erode both operational and financial performance.

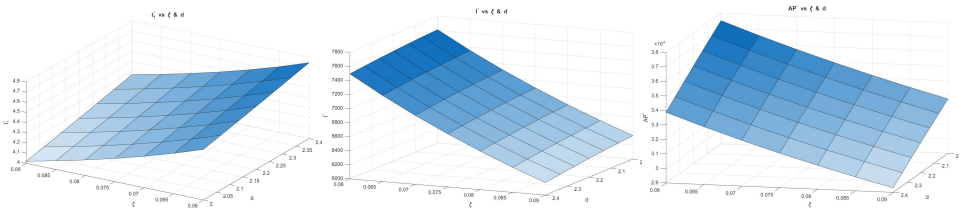


Fig. 3 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of ξ and d.

Table 3 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of d and r.

d	r	t_1^*	I^*	AP^*
2.200	2.600	4.359455	6389.408148	31605.952442
2.200	2.700	4.367736	6623.928771	32476.176634
2.200	2.800	4.376922	6855.110462	33279.865567
2.200	2.900	4.386861	7082.549559	34018.901098
2.200	3.000	4.397420	7305.824036	34695.241049
2.200	3.100	4.408481	7524.493102	35310.924461
2.000	2.800	4.255542	6913.969686	35023.274686
2.100	2.800	4.315369	6887.054204	34164.264459
2.300	2.800	4.440107	6817.966356	32371.181144
2.400	2.800	4.504838	6775.485992	31439.176865
2.500	2.800	4.571033	6727.568635	30484.686107

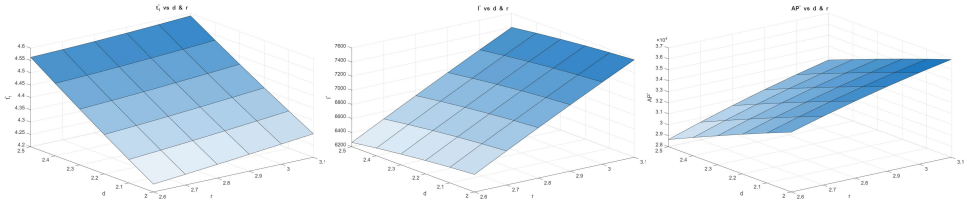


Fig. 4 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of d and r.

Now the joint effect of the starting point of deterioration (d) and the replenishment rate (r) is analyzed in Table 3 and Fig. 4. Both parameters influence the cycle time, which lengthens steadily with higher values of either d or r. In this case, the inventory level increases considerably with larger r and decreases as d becomes larger. This shows a mixed response, demonstrating a trade-off between demand intensity and replenishment capacity. The effect on profitability is particularly noteworthy. As r increases from 2.6 to 3.1, average profit increases consistently, reaching above 35,000 units. Again, increasing d leads to a decline in profit, with values dropping below 31,000 units at $d = 2.5$. This demonstrates that while a higher replenishment rate supports profitability, excessive demand growth puts pressure on costs and reduces overall profit.

The results in Table 4 and Fig. 5 show that variations in d_1 have only a marginal impact. For example, when d_1 increases from 0.12 to 0.17 (with ξ fixed at 0.075), the cycle time changes only from 4.3680 to 4.3659, inventory increases slightly from 6825.47 to 6827.38, and profit rises from 32,868.87 to 32,962.67. In contrast, ξ demonstrates a much stronger effect. Increasing xi from 0.06 to 0.09 (with $d_1 = 0.15$) shortens the cycle time from 4.6371 to 4.1593, reduces inventory from 7553.77 to 6253.49, and decreases profit significantly from 35,452.53 to 30,810.36. This highlights ξ as a critical factor, with higher values directly declining system performance.

Table 4 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of ξ and d_1.

ξ	d_1	t_1^*	I^*	AP^*
0.0750	0.1200	4.368026	6825.151835	32869.594987
0.0750	0.1300	4.367633	6825.563480	32887.433487
0.0750	0.1400	4.367228	6825.985500	32905.736982
0.0750	0.1500	4.366813	6826.418285	32924.523139
0.0750	0.1600	4.366387	6826.862251	32943.810458
0.0750	0.1700	4.365949	6827.317806	32963.618696
0.0600	0.1500	4.637125	7553.765541	35452.529824
0.0650	0.1500	4.538152	7289.780080	34554.916260
0.0700	0.1500	4.448510	7048.392633	33714.349947
0.0800	0.1500	4.291952	6621.285607	32180.139221
0.0850	0.1500	4.223023	6430.890019	31476.702113
0.0900	0.1500	4.159285	6253.488086	30810.361136

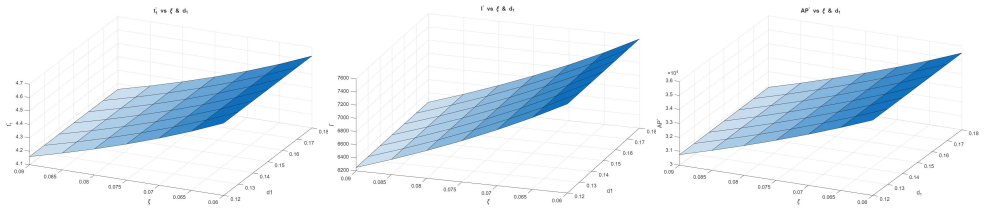


Fig. 5 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of ξ and d_1.

Table 5 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of d_2 and p.

d_2	p	t_1^*	I^*	AP^*
0.25	1.15	4.398774	6887.209992	33104.040019
0.25	1.18	4.398628	6887.354333	33110.503211
0.25	1.21	4.398481	6887.499353	33116.998909
0.25	1.24	4.398333	6887.645059	33123.527260
0.25	1.27	4.398184	6887.791452	33130.088433
0.25	1.30	4.398035	6887.938536	33136.682597
0.23	1.24	4.397638	6721.994501	32352.640976
0.24	1.24	4.397985	6803.768223	32733.189395
0.26	1.24	4.398681	6973.708243	33524.041999
0.27	1.24	4.399029	7062.045482	33935.141924
0.28	1.24	4.399378	7152.749280	34357.257527

As presented in Table 5 and Fig. 6, p has very little effect within the tested range. When p rises from 1.15 to 1.30 (with $d_2 = 0.25$), cycle time remains constant at about 4.3986, inventory increases only from 6887.23 to 6887.94, and profit increases slightly from 33,103.84 to 33,136.01. On the other hand, d_2 exhibits a more notable impact. When d_2 increases from 0.23 to 0.28 (with $p = 1.24$), cycle time increases modestly

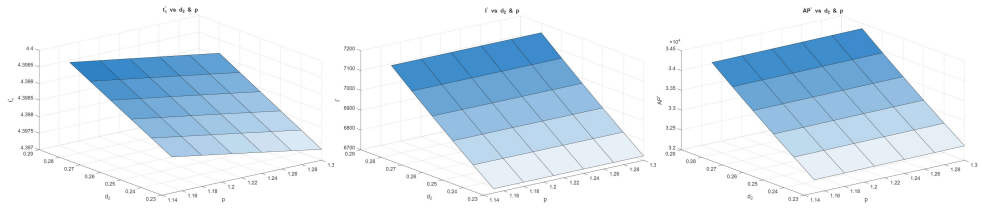


Fig. 6 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of d_2 and p.

from 4.3976 to 4.3994, inventory increases from 6721.99 to 7152.75, and profit increases from 32,352.64 to 34,357.26. This indicates that p is relatively unimportant and d_2 plays a constructive role in increasing profit.

Table 6 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of q and P.

q	P	t_1^*	I^*	AP^*
0.9360	60.0000	4.309877	6555.547915	29637.107078
0.9360	61.2000	4.346057	6705.153574	31091.419596
0.9360	62.4000	4.381273	6852.428793	32558.400416
0.9360	63.6000	4.415575	6997.485729	34037.518631
0.9360	64.8000	4.449010	7140.428050	35528.280310
0.9360	66.0000	4.481621	7281.351820	37030.224794
0.9000	62.4000	4.380980	6781.240227	32231.055440
0.9120	62.4000	4.381078	6804.886760	32339.788571
0.9240	62.4000	4.381176	6828.616183	32448.903147
0.9480	62.4000	4.381370	6876.324882	32668.281744
0.9600	62.4000	4.381467	6900.304743	32778.548443

Table 6 and Fig. 7 illustrate that q has a moderate effect and P strongly operates on the system performance. For instance, when q increases from 0.90 to 0.96 (with $P = 62.4$), inventory increases from 6781.43 to 6899.64, profit increases from 32,230.67 to 32,778.47, and cycle time remains steady near 4.386. By contrast, increasing P from 60 to 66 (with $q = 0.936$) leads to a slight increase in cycle time from 4.3099 to 4.4816, an increase in inventory from 6555.55 to 7281.35, and an increase in profit from 29,637.11 to 37,030.22. These results confirm that P is one of the most powerful levers for enhancing inventory and profit.

The sensitivity analysis with respect to the holding cost h, and purchase cost C reveals that both parameters have a negative effect on the inventory level (see Table 7). As C increases with fixed h, the optimal cycle time t_1^* consistently decreases, indicating shorter replenishment intervals at higher purchase costs. Simultaneously, the inventory level I^* and average profit AP^* decrease sharply, with profit decreasing from 38,065 at $C = 20$ to 28,638 at $C = 26$, highlighting the detrimental impact of increasing purchase cost. Variation in h with fixed C shows a similar but less effect. Also, higher holding costs reduce both inventory and profit while slightly decreasing

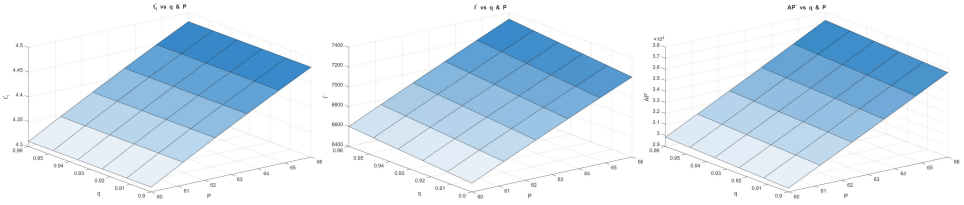


Fig. 7 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of q and P.

Table 7 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of h and C.

h	C	t_1^*	I^*	AP^*
1.5400	20.0000	4.579609	7700.771797	38065.431879
1.5400	21.2000	4.504910	7370.273846	36075.014212
1.5400	22.4000	4.431837	7054.729837	34138.400994
1.5400	23.6000	4.360213	6752.599513	32254.191555
1.5400	24.8000	4.289877	6462.530639	30421.188278
1.5400	26.0000	4.220681	6183.328341	28638.377786
1.3000	22.4000	4.497195	7336.601054	35395.685487
1.3800	22.4000	4.475126	7240.748088	34971.879741
1.4600	22.4000	4.453342	7146.813126	34552.813042
1.6200	22.4000	4.410604	6964.435028	33728.562459
1.7000	22.4000	4.389635	6875.868506	33323.218875

the cycle length. The three-dimensional surface plots in Fig. 8, with downward-facing slopes, confirm these results. This shows that minimizing purchase and holding costs is essential to maintain higher inventory levels and profitability.

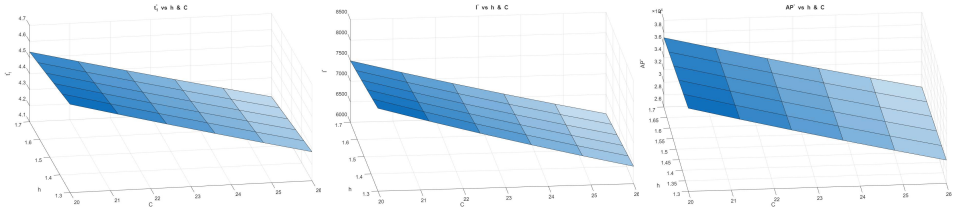


Fig. 8 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of h and C.

The sensitivity analysis for the demand scaling factor k and deterioration cost parameter c_d demonstrates adverse effects on the inventory level (see Table 8). When c_d increases at fixed k, both the cycle time and inventory decrease slightly, leading to a gradual fall from $33,660$ at $c_d = 0.6$ to $33,509$ at $c_d = 1.0$. In contrast, varying k with constant c_d produces a strong positive effect. Higher values of k significantly increase inventory levels and profits, while cycle time remains nearly unchanged. For example,

Table 8 Sensitivity analysis table of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of k and c_d.

k	c_d	t_1^*	I^*	AP^*
k	c_d	t_1^*	I^*	AP^*
680.00	0.600	4.405793	7014.686486	33660.909691
680.00	0.680	4.403345	7004.221743	33630.377073
680.00	0.760	4.400904	6993.797709	33599.900026
680.00	0.840	4.398470	6983.413360	33569.486343
680.00	0.920	4.396044	6973.068702	33539.133036
680.00	1.000	4.393625	6962.763542	33508.839244
500.00	0.840	4.388497	5144.430564	25011.280618
560.00	0.840	4.392543	5757.443139	27863.915424
620.00	0.840	4.395796	6370.435608	30716.660499
740.00	0.840	4.400706	7596.380015	36422.373119
800.00	0.840	4.402603	8209.338100	39275.306971

profit increases from $25,011$ at $k = 500$ to $39,275$ at $k = 800$, showing the dominance of demand growth in improving inventory outcomes. The sensitivity graphs in Fig. 9 illustrate these changes clearly, where inventory and profit surfaces slope upward with respect to k but downward with respect to c_d. This shows that maximizing profit requires reducing deterioration costs and achieving higher demand growth.

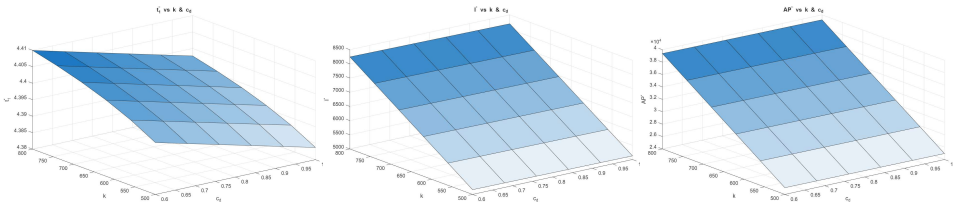


Fig. 9 Sensitivity graphs of t_1^* (Cycle time), I^* (Inventory) and AP^* (Average profit) in respect of k and c_d.

The overall sensitivity analysis demonstrates that the proposed inventory model is generally stable with respect to minor parameter fluctuations. Indeed, the inventory levels and profits are considerably more sensitive than the cycle time. The deterioration rate ξ and replenishment rate r come up as more important than the other parameters. These two have significant impacts on profitability and inventory performance. Also, the unit revenue P and the demand scaling factor play critical roles in enhancing stock levels and returns, without significantly altering cycle length. In contrast, purchase cost, holding cost, and deterioration cost negatively affect inventory and profits, though their effects are relatively moderate compared to demand and replenishment factors. These findings highlight two key managerial priorities. First is controlling deterioration through preservation technologies and efficient cost management. Second is to enhance demand growth and optimize replenishment policies. By focusing on these high-impact levers, the industry can increase profitability, enhance

system stability, and ensure more efficient inventory management under diverse market and operational conditions.

OPTIMIZE_AP_ROOT Find t1*, I*, AP*

```
function [t1_star, I_star, AP_star, TP_star] = optimize_AP_root()
% OPTIMIZE_AP_ROOT  Find t1*, I*, AP* by solving t1* TP'(t1) - TP(t1) = 0
% Uses boxed equation and parameters taken from your example (f(t)=673*t).
%
% Usage:
%   [t1_star, I_star, AP_star, TP_star] = optimize_AP_root()

%% ---------- Parameters (from your example) ----------
alpha = 53; beta = 3;
d = 2.23; r = 2.83;
xi = 0.073; a = xi/2;
d1 = 0.153; d2 = 0.253; p = 1.23; q = 0.93;
da = (1 - d1)^(-p);
db = (1 - d2)^(-q);

C = 23; P = 63;
h = 1.53; cd = 0.83;
o = 123;       % ordering cost per cycle
A_cost = 203;   % advertisement cost per cycle

% sales function
k = 673; f = @(t) k.*t;
fr = f(r);

% Precompute integrals that don't depend on t1
RelTol = 1e-9; AbsTol = 1e-12;
I_part = integral(@(t) f(t), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol);      % ∫_d^r f(t) dt = k/2 (r^2 - d^2)
A_int  = integral(@(s) f(s).*exp(a*s.^2), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol); % ∫_d^r f(s)e^{a s^2} ds
Jd_neg = integral(@(t) exp(-a.*t.^2), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol);    % ∫_d^r e^{-a t^2} dt

% constants for convenience
const_da_part = da*(alpha*d + 0.5*beta*d^2);

% define function handles used below
Eint_pos = @(t1) integral(@(s) exp(a.*s.^2), r, t1, 'RelTol',RelTol, 'AbsTol',AbsTol); % ∫_r^{t1} e^{a s^2} ds
Jr_neg = @(t1) integral(@(t) exp(-a.*t.^2), r, t1, 'RelTol',RelTol, 'AbsTol',AbsTol);    % ∫_r^{t1} e^{-a t^2} dt

% Bridge: I(t1)
I_of = @(t1) const_da_part + db.*exp(-a.*d.^2).*( A_int + fr.*Eint_pos(t1) );

% SR (revenue)
Q_of = @(t1) const_da_part + db.*( I_part + fr.*(t1 - r) );
SR_of = @(t1) P .* Q_of(t1);

% PC
PC_of = @(I) C .* I;

% DC
DC_of = @(I,t1) cd .* ( I - db.*I_part - db.*fr.*(t1 - r) );

% Holding cost H(t1): numeric via piecewise I(t) integrated 0..t1
TI_piece = @(tt, t1, Ireq, alpha_loc, beta_loc) ( ...
    (tt <= d) .* ( Ireq - da.*( alpha_loc.*tt + 0.5.*beta_loc.*tt.^2 )) + ...
    (tt > d & tt <= r) .* ( (Ireq - da.*( alpha_loc.*d + 0.5.*beta_loc.*d.^2 )) .* exp(-a.*(tt.^2 - d.^2)) ...
        - db .* arrayfun(@(tth) integral(@(s) f(s).*exp(a.*(s.^2 - tth.^2)), d, tth, 'RelTol',RelTol, 'AbsTol',AbsTol), tt) ) + ...
    (tt > r) .* ( db .* fr .* arrayfun(@(tth) integral(@(s) exp(a.*(s.^2 - tth.^2)), tth, t1, 'RelTol',RelTol, 'AbsTol',AbsTol), tt) ) ...
);

H_of = @(Ireq, t1, alpha_loc, beta_loc) h .* integral(@(tt) TI_piece(tt, t1, Ireq, alpha_loc, beta_loc), 0, t1, 'RelTol',1e-8, 'AbsTol',1e-11);

% H'(t1) closed form from your derivation:
Hprime_of = @(t1) h .* db .* fr .* ( ...
    exp(a.*(t1.^2 - d.^2)).*( d + exp(a.*d.^2).*Jd_neg ) + exp(a.*t1.^2).*Jr_neg(t1) ...
);

% TP(t1)
TP_of = @(t1) SR_of(t1) - ( PC_of(I_of(t1)) + H_of(I_of(t1), t1, alpha, beta) + DC_of(I_of(t1), t1) + o + A_cost );

% TP'(t1) analytic form (component derivatives substituted)
% I'(t1) = db * fr * exp(a*(t1^2 - d^2))
Iprime_of = @(t1) db .* fr .* exp(a.*(t1.^2 - d.^2));
TPprime_of = @(t1) db .* fr .* ( P + cd - (C + cd).*exp(a.*(t1.^2 - d.^2)) ) - Hprime_of(t1);

% The final equation (boxed): F(t1) = t1*TP'(t1) - TP(t1) = 0
F_of = @(t1) t1 .* TPprime_of(t1) - TP_of(t1);

%% ---------- Robust bracketing then fzero ----------
t1_left = r + 1e-6;
t1_right = r + 20;
Nscan = 400;
tgrid = linspace(t1_left, t1_right, Nscan);
Fvals = nan(size(tgrid));
for ii = 1:numel(tgrid)
  try
    Fvals(ii) = F_of(tgrid(ii));
  catch
    Fvals(ii) = NaN;
  end
end

% find sign changes
idx = find(~isnan(Fvals));
bracket_found = false;
t1_star = NaN;

for ii = 1:(length(idx)-1)
  i1 = idx(ii); i2 = idx(ii+1);
  if Fvals(i1)*Fvals(i2) < 0
    left = tgrid(i1); right = tgrid(i2);
    try
      t1_star = fzero(F_of, [left right], optimset('TolX',1e-8,'Display','off'));
      bracket_found = true;
      break;
    catch
      % try next bracket
    end
  end
end

% if no sign-change bracket found, try fzero with single initial guess near best F (max abs)
if ~bracket_found
  % try a guess at location of min absolute F
  [~, imin] = min(abs(Fvals));
  guess = tgrid(imin);
  try
    t1_star = fzero(F_of, guess, optimset('TolX',1e-8,'Display','off'));
    bracket_found = true;
  catch
    % last resort: try fzero on the full interval (may fail)
    try
      t1_star = fzero(F_of, [t1_left, t1_right], optimset('TolX',1e-8,'Display','off'));
      bracket_found = true;
    catch ME
      warning('Root finding failed: %s', ME.message);
      t1_star = NaN;
    end
  end
end

%% ---------- Compute I*, TP*, AP* if root found ----------
if isnan(t1_star)
  I_star = NaN; TP_star = NaN; AP_star = NaN;
  fprintf('No root found for F(t1). Try expanding search interval or adjust tolerances.\n');
else
  I_star = I_of(t1_star);
  TP_star = TP_of(t1_star);
  AP_star = TP_star ./ t1_star;

  % print results
  fprintf('t1* = %.6f\n', t1_star);
  fprintf('I*  = %.6f\n', I_star);
  fprintf('AP* = %.6f\n', AP_star);
end

end
```

Sensitivity Analysis

```
function ex1_sensitivity_alpha_beta()
    % ---------- Base Parameters ----------
    d = 2.23; r = 2.83;
    xi = 0.073; a = xi/2;
    d1 = 0.153; d2 = 0.253; p = 1.23; q = 0.93;
    da_const = (1 - d1)^(-p);
    db = (1 - d2)^(-q);

    C = 23; P = 63;
    h = 1.53; cd = 0.83;
    o = 123;       % ordering cost per cycle
    A_cost = 203;   % advertisement cost per cycle

    % sales function
    k = 673; f = @(t) k.*t;
    fr = f(r);

    % Precompute fixed integrals
    RelTol = 1e-9; AbsTol = 1e-12;
    I_part = integral(@(t) f(t), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol);
    A_int  = integral(@(s) f(s).*exp(a*s.^2), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol);
    Jd_neg = integral(@(t) exp(-a.*t.^2), d, r, 'RelTol',RelTol, 'AbsTol',AbsTol);

    Eint_pos = @(t1) integral(@(s) exp(a.*s.^2), r, t1, 'RelTol',RelTol, 'AbsTol',AbsTol);
    Jr_neg = @(t1) integral(@(t) exp(-a.*t.^2), r, t1, 'RelTol',RelTol, 'AbsTol',AbsTol);

    % Alpha-beta range (tight range around previous example)
    alpha_vals = linspace(50, 56, 7);
    beta_vals  = linspace(2.5, 3.5, 7);

    % Prepare result arrays
    t1_star_mat = nan(length(alpha_vals), length(beta_vals));
    I_star_mat  = nan(size(t1_star_mat));
    AP_star_mat = nan(size(t1_star_mat));

    % ---------- Sensitivity Loop ----------
    for i = 1:length(alpha_vals)
        for j = 1:length(beta_vals)
            alpha = alpha_vals(i);
            beta  = beta_vals(j);

            % constants that depend on alpha, beta
            const_da_part = da_const*(alpha*d + 0.5*beta*d^2);

            % I(t1)
            I_of = @(t1) const_da_part + db.*exp(-a.*d.^2).*( A_int + fr.*Eint_pos(t1) );

            % Q(t1)
            Q_of = @(t1) const_da_part + db.*( I_part + fr.*(t1 - r) );
            SR_of = @(t1) P .* Q_of(t1);

            % Costs
            PC_of = @(I) C .* I;
            DC_of = @(I,t1) cd .* ( I - db.*I_part - db.*fr.*(t1 - r) );

            % Inventory profile
            TI_piece = @(tt, t1, Ireq, alpha_loc, beta_loc) ( ...
                (tt <= d) .* ( Ireq - da_const.*( alpha_loc.*tt + 0.5.*beta_loc.*tt.^2 )) + ...
                (tt > d & tt <= r) .* ( (Ireq - da_const.*( alpha_loc.*d + 0.5.*beta_loc.*d.^2 )) .* exp(-
a.*(tt.^2 - d.^2)) ...
                    - db .* arrayfun(@(tth) integral(@(s) f(s).*exp(a.*(s.^2 - tth.^2)), d, tth,
'RelTol',RelTol, 'AbsTol',AbsTol), tt) ) + ...
                (tt > r) .* ( db .* fr .* arrayfun(@(tth) integral(@(s) exp(a.*(s.^2 - tth.^2)), tth, t1,
'RelTol',RelTol, 'AbsTol',AbsTol), tt) ) ...
                );
            H_of = @(Ireq, t1, alpha_loc, beta_loc) h .* integral(@(tt) TI_piece(tt, t1, Ireq,
alpha_loc, beta_loc), 0, t1, 'RelTol',1e-8, 'AbsTol',1e-11);

            % H'(t1)
            Hprime_of = @(t1) h .* db .* fr .* ( ...
                exp(a.*(t1.^2 - d.^2)).*( d + exp(a.*d.^2).*Jd_neg ) + exp(a.*t1.^2).*Jr_neg(t1) ...
            );

            % TP(t1)
            TP_of = @(t1) SR_of(t1) - ( PC_of(I_of(t1)) + H_of(I_of(t1), t1, alpha, beta) +
DC_of(I_of(t1), t1) + o + A_cost );

            % TP'(t1)
            Iprime_of = @(t1) db .* fr .* exp(a.*(t1.^2 - d.^2));
            TPprime_of = @(t1) db .* fr .* ( P + cd - (C + cd).*exp(a.*(t1.^2 - d.^2)) ) - Hprime_of(t1);

            % F(t1)
            F_of = @(t1) t1 .* TPprime_of(t1) - TP_of(t1);

            % Bracket search
            t1_left = r + 1e-6;
            t1_right = r + 10;
            Nscan = 200;
            tgrid = linspace(t1_left, t1_right, Nscan);
            Fvals = arrayfun(@(t) F_of(t), tgrid);

            idx = find(~isnan(Fvals));
            root_found = false; t1_star = NaN;
            for ii = 1:(length(idx)-1)
                if Fvals(idx(ii)) * Fvals(idx(ii+1)) < 0
                    try
                        t1_star = fzero(F_of, [tgrid(idx(ii)), tgrid(idx(ii+1))]);
                        root_found = true; break;
                    catch
                    end
                end
            end

            if root_found
                I_star = I_of(t1_star);
                TP_star = TP_of(t1_star);
                AP_star = TP_star / t1_star;

                t1_star_mat(i,j) = t1_star;
                I_star_mat(i,j)  = I_star;
                AP_star_mat(i,j) = AP_star;
            end
        end
    end
    % ---------- Numerical Table ----------
    fprintf('\nSensitivity Analysis Table:\n');
    fprintf('%8s %8s %12s %12s %12s\n', 'Alpha', 'Beta', 't1_star', 'I_star', 'AP_star');
    fprintf('%s\n', repmat('-', 1, 60));
    for i = 1:length(alpha_vals)
        for j = 1:length(beta_vals)
            fprintf('%8.3f %8.3f %12.6f %12.6f %12.6f\n', ...
                alpha_vals(i), beta_vals(j), ...
                t1_star_mat(i,j), I_star_mat(i,j), AP_star_mat(i,j));
        end
    end
    % ---------- Plots ----------
    [Agrid, Bgrid] = meshgrid(beta_vals, alpha_vals);

    figure;
    surf(Agrid, Bgrid, t1_star_mat);
    xlabel('\beta'); ylabel('\alpha'); zlabel('t1^*');
    title('Sensitivity of t1^*');

    figure;
    surf(Agrid, Bgrid, I_star_mat);
    xlabel('\beta'); ylabel('\alpha'); zlabel('I^*');
    title('Sensitivity of I^*');

    figure;
    surf(Agrid, Bgrid, AP_star_mat);
    xlabel('\beta'); ylabel('\alpha'); zlabel('AP^*');
    title('Sensitivity of AP^*');
end
```

6 Conclusion

In the paper, we introduced and demonstrated an EOQ-type inventory model for deteriorating items under a realistic setting where demand, deterioration, and marketing effects interact dynamically. Our model clearly includes the impact of advertising by considering a nonzero initial demand that increases linearly in the early phase, whereas traditional models assume zero initial demand. After deterioration begins, the demand is represented by a general ramp-type function that stabilizes over time. This demonstrates the natural life cycle of many consumer goods and deteriorating items. In addition, the deterioration of items has been represented as proportional to time, showing the gradual loss of item quality during storage. To address practical pricing strategies, separate discounting policies have been applied in the pre-deterioration and post-deterioration phases. This offers flexibility for the industry to stimulate demand and reduce inventory losses.

The model is formulated using closed-form expressions for the inventory dynamics for a single-item system without shortages over an infinite horizon. To validate the analytical findings, the model is examined numerically through a series of computational experiments. A numerical example is provided to illustrate the model. Also, graphical sensitivity analyses are conducted to investigate the effects of key parameters, including the deterioration rate (ξ), demand growth parameters (α, β), discount levels (d_1, d_2), and advertising-driven initial demand. The sensitivity results confirmed that each of these parameters has a significant impact on the optimal cycle length, total cost, and average profit.

The numerical investigations show several important insights. For instance, an increase in advertisement-driven initial demand significantly improves profitability by reducing the average cycle cost and inventory holding period. The dual discounting policy effects in both stimulating early demand and preventing losses during the deterioration phase. Moreover, sensitivity graphs demonstrated that higher deterioration rates shorten the optimal cycle length. This leads the industry to adopt faster replenishment policies. Similarly, adjustments in the discount parameters provided a practical lever for balancing profit with sales acceleration. These findings highlight the importance of jointly considering advertising, deterioration, and pricing techniques when formulating inventory models.

Overall, the study extends classical ramp-type demand models by integrating advertising effects, a two-phase demand structure, time-proportional deterioration, and dual-discount techniques into a single model. The combination of analytical derivations, numerical validation, and sensitivity analysis provides both theoretical contribution and practical guidance. Future research may extend the present work by incorporating shortages, trade credit, or stochastic demand patterns, as well as multi-item or supply-chain settings. This will further enhance the applicability of the model in complex real-world environments.

Acknowledgements. The first author Rina Khatun thanks MHRD, India for the junior research fellowship through IIEST Shibpur.

Declarations

Authors' contributions

Conceptualization, R.K. K.H.A.; formal analysis, R.K., K.H.A., and S.A.; investigation, K.H.A., and S.A.; supervision, S.A.; software, R.K., and K.H.A.; writing original draft preparation, R.K.; writing review and editing, K.H.A., and S.A. All authors have reviewed and approved the final version of the manuscript.

Funding

This research was conducted without external funding.

Conflict of interest

The authors declare no competing interests.

Availability of data and materials

The data used to support the findings are within the manuscript.

References

[1] Manna, S. K., & Chaudhuri, K. S. (2006). An EOQ model with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages. European Journal of Operational Research, 171(2), 557–566.

[2] Skouri, K., Konstantaras, I., Manna, S. K., & Chaudhuri, K. S. (2011). Inventory models with ramp type demand rate, time dependent deterioration rate, unit production cost and shortages. Annals of Operations Research, 191(1), 73–95.

[3] Wu, C. F., & Zhaob, Q. H. (2015). An inventory model for deteriorating items with inventory-dependent and linear trend demand under trade credit. Scientia Iranica Transactions E: Industrial Engineering, 22(6), 2558–2570.

[4] Rabiu, N., & Musa, A. (2024). Inventory Ordering Policies of Delayed Deteriorating Items with Ramp Type Demand and Price Discount. Sule Lamido University Journal of Science & Technology, 9(1,2), 74–84.

[5] Ghare, P. M., & Schrader, G. F. (1963). A model for exponentially decaying inventories. Journal of Industrial Engineering, 14, 238–243.

[6] Sharma, P., Sharma, A., & Jain, S. (2019). Inventory model for deteriorating items with price and time-dependent seasonal demand. International Journal of Procurement Management, 12(4), 363–375.

[7] Dey, D., Kar, S., & Bhattacharya, D. K. (2024). Analytical solution of a modified single item continuous production inventory model under constant deterioration, goal levels and penalties. Opsearch, 61(1), 399–420.

[8] Aliyu, I., & Sani, B. (2018). An Inventory Model for Deteriorating Items with Generalised Exponential Decreasing Demand, Constant Holding Cost and Time-Varying Deterioration Rate. American Journal of Operations Research, 8, 1–16.

[9] Rao, A. L., & Bora, S. (2023). Inventory model for Exponential Deterioration Items with Power Dependent Demand. International Journal of Information Technology, Research and Applications, 2(3), 31–38. https://doi.org/10.59461/ijitra.v2i3.70

[10] Kaushik, J., & Sharma, A. (2021). The preservative technology in the inventory model for the deteriorating items with Weibull deterioration rate. In Data Science and Security: Proceedings of IDSCS 2021 (pp. 348-355). Singapore: Springer Singapore.

[11] Hatibaruah, A., & Saha, S. (2023). An inventory model for two-parameter Weibull distributed ameliorating and deteriorating items with stock and advertisement frequency dependent demand under trade credit and preservation technology. Opsearch, 60(2), 951–1002.

[12] Chung K. J., & Lin, C. N. (2001). Optimal inventory replenishment models for deteriorating items taking account of time discounting. Computers & Operations Research, 28(1), 67–83.

[13] Benkherouf, L., Boumenir, A., & Aggoun, L. (2003). A diffusion inventory model for deteriorating items. Applied Mathematics and Computation, 138(1), 21–39.

[14] Giri, B. C., Chakrabarty, T. & Chaudhuri, K. S. (2000). A note on a lot sizing heuristic for deteriorating items with time-varying demands and shortages. Computers & Operations Research, 27(6), 495–505.

[15] Teng, J. T., Chang, H. J., Dye, C. Y. & Hung, C. H. (2002). An optimal replenishment policy for deteriorating items with time-varying demand and partial backlogging. Operations Research Letters, 30, 387–393.

[16] Halim, M. A., Paul, A., Mahmoud, M., Alshahrani, B., Alazzawi, A. Y., & Ismail, G. M. (2021). An overtime production inventory model for deteriorating items with nonlinear price and stock dependent demand. Alexandria Engineering Journal, 60(3), 2779–2786.

[17] Kaushik, J. (2025). An inventory model for deteriorating items with ramp type demand pattern: a stock-dependent approach. Cogent Business & Management, 12(1), 2442547.

[18] Das, S. C., Zidan, A. M., Manna, A. K., Shaikh, A. A., & Bhunia, A. K. (2020). An application of preservation technology in inventory control system with price dependent demand and partial backlogging. Alexandria Engineering Journal, 59(3), 1359–1369.

[19] Hill, R. M. (1995). Inventory model for increasing demand followed by level demand. The Journal of the Operational Research Society, 46, 1250–1259.

[20] Chauhan, E., Shrivastav, R. K., & Gupta, S. (2025). Inventory Optimization with Ramp-Type Demand, Time-Dependent Holding Costs, and Discounted Backorders under Inflation. Communications on Applied Nonlinear Analysis, 32(6s), 418–426.

[21] Shah, N. H., Soni, H. N., & Patel, K. A. (2013). Optimizing inventory and marketing policy for non-instantaneous deteriorating items with generalized type deterioration and holding cost rates. Omega, 41(2), 421–430.

[22] Palanivel, M., & Uthayakumar, R. (2015). Finite horizon EOQ model for non-instantaneous deteriorating items with price and advertisement dependent demand and partial backlogging under inflation. International Journal of Systems Science, 46(10), 1762–1773.

[23] Manna, A. K., Dey, J. K., & Mondal, S. K. (2017). Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand. Computers & Industrial Engineering, 104, 9-22.

[24] Saha, S., & Chakrabarti, T. (2018). Two-echelon supply chain model for deteriorating items in an imperfect production system with advertisement and stock dependent demand under trade credit. International Journal of Supply and Operations Management, 5(3), 207–217.

[25] Rathore, H. (2019). An inventory model with advertisement dependent demand and reliability consideration. International Journal of Applied and computational mathematics, 5(2), 33.

[26] Shaikh, A. A., Cárdenas–Barrón, L. E., Bhunia, A. K., & Tiwari, S. (2019). An inventory model of a three parameter Weibull distributed deteriorating item with variable demand dependent on price and frequency of advertisement under trade credit. RAIRO-Operations Research, 53(3), 903–916.

[27] Udayakumar, R., Geetha, K. V., & Sana, S. S. (2021). Economic ordering policy for non-instantaneous deteriorating items with price and advertisement dependent demand and permissible delay in payment under inflation. Mathematical Methods in the Applied Sciences, 44(9), 7697–7721.

[28] Shaikh, A. A., Khan, M. A. A., Panda, G. C., & Konstantaras, I. (2019). Price discount facility in an EOQ model for deteriorating items with stock-dependent

demand and partial backlogging. International Transactions in Operational Research, 26(4), 1365–1395.

[29] Samanta, A. (2022). Inventory Model for Non-instantaneous Deteriorating Items with Time-dependent Deterioration Rate and Price Discounts. Calcutta Statistical Association Bulletin, 74(1), 62–72.

[30] Kharvi, S., & Pakkala, T. (2020). An optimal inventory policy for subsequent price reduction problem. International journal of operations and quantitative management, 26(2), 147–161.

[31] Buyukdag, N., Soysal, A. N., & Kitapci, O. (2020). The effect of specific discount pattern in terms of price promotions on perceived price attractiveness and purchase intention: An experimental research. Journal of Retailing and Consumer Services, 55, 102112.

[32] Padiyar, S. V. S., Kuraie, V. C., Makholia, D., Singh, S. R., Singh, V., & Joshi, N. (2024). An imperfect production inventory model for instantaneous deteriorating items with preservation investment under inflation on time value of money. Contemporary Mathematics, 2271–2295.

[33] Ma, S., Jemai, Z., & Bai, Q. (2022). Optimal pricing and ordering decisions for a retailer using multiple discounts. European Journal of Operational Research, 299(3), 1177–1192.

[34] Raafat, F. (1991). Survey of literature on continuously deteriorating inventory model. The Journal of the Operational Research Society, 42, 27–37.

[35] Li, R., Lan, H., & Mawhinney, J. R. (2010). A review on deteriorating inventory study. Journal of Service Science and Management, 3(1), 117–129.

[36] Karimi, M. (2025). Inventory control of deteriorating items: A review. International Journal of Industrial Engineering Computations, 16(1), 93–128.