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Well-Possedness and Iterative Approximation for Elliptic Problems with Nonlinear Logarithmic Robin Boundary Conditions

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Abstract

In this paper, we investigate an elliptic boundary value problem involving a nonlinear logarithmic Robin boundary condition. The problem is formulated on a smooth bounded domain of $\mathbb{R}^2$ with mixed Dirichlet–Neumann–Robin boundary conditions. We introduce an iterative linearization scheme based on a sequence of linear problems and establish the the existence and uniqueness of the weak solution. A rigorous convergence analysis is provided using a local Lipshitz estimate of the logarithmic nonlinearity. We also study, qualitative properties of the solution and validate the theoretical results through numerical experiments based on finite element approximations.

Keywords: Elliptic equations, logarithmic nonlinearity, nonlinear Robin boundary condition, fixed point iteration, finite element method

Mathematics Subject Classification (2020): 35J25, 35J61, 65N30, 35B45.

1 Introduction

In this paper, we study a Laplace equation supplemented with a nonlinear Robin boundary condition involving a logarithmic nonlinearity. More precisely, we consider the following problem:

$$(P)\quad \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = \phi & \text{on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi(x) u \ln(e+u^2) = g & \text{on } \Gamma_R, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. The boundary is decomposed into three pairwise disjoint measurable parts, satisfying

$$\begin{aligned} &\partial\Omega = \Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N} \cup \overline{\Gamma_R}, \\ &\Gamma_D \cap \Gamma_N = \Gamma_D \cap \Gamma_R = \Gamma_N \cap \Gamma_R = \emptyset, \\ &\sigma(\Gamma_D) > 0,\ \sigma(\Gamma_N) > 0,\ \sigma(\Gamma_R) > 0, \end{aligned}$$

where σ denotes the surface measure on $\partial\Omega$.
The data are assumed to satisfy the following regularity conditions

$$\phi \in L^2(\Gamma_N), \quad \phi \not\equiv 0, \quad g \in L^2(\Gamma_R), \quad \varphi \in C^0(\overline{\Gamma_R}).$$

Boundary value problems with mixed boundary conditions arise in many physical, chemical and engineering applications. In the stationary case, such processes are typically governed by Laplace's equation, where the unknown represents, for example, a temperature distribution or an electrostatic potential.

The Dirichlet condition prescribes the state variable, while the Neumann condition describes a prescribed flux through Γ_N. The nonlinear Robin condition on Γ_R models an exchange mechanism between the system and its surrounding environment. In classical framework, Robin boundary conditions are usually linear, typically written as

$$\frac{\partial u}{\partial n} + \varphi(x)u = g.$$

Such problem have been extensively studied. In particular, in the context of corrosion detection in thin metallic plates, the Robin coefficient φ represents the corrosion intensity. Existence and uniqueness results, as well as inverse identification problems associated with this coefficient, have been investigated by Jaoue et al. [6, 7].

More generally, semilinear elliptic problems with mixed boundary conditions have recieved considerable attention. When the nonlinear boundary term has polynomial growth, the analysis can be carried out using Sobolev trace embeddings and classical variational methods.

Recently, nonlinear Robin boundary conditions have attracted increasing interest. Benameur et al. [2] considered a quadratic nonlinear Robin condition of the form

$$\frac{\partial u}{\partial n} + \varphi(x)u + \psi(x)u^2 = g,$$

and establishing existence and uniqueness in an appropriate Hilbert space $V \subset H^1(\Omega)$. These results were extended by Elhechmi [10] to a larger class of polynomial nonlinearities. In both works, the authors introduced iterative schemes based on sequence of linearized problems whose solutions converge to the solution of original nonlinear problem.

A more general nonlinear boundary interaction was investigated by Elhechmi et al. [3], where an exponential Robin condition was considered:

$$\frac{\partial u}{\partial n} + \varphi(x)(e^{\alpha u} - e^{(1-\alpha)u}) = g.$$

The convergence analysis in that framwork required refined estimates of Sobolev embedding constants and suitable control of the exponential growth.
However, in many pratical situations, boundary interactions cannot be accurately described by either linear, polynomial or exponnential laws. This motivates the study of more general nonlinearities. In particular, logarithmic terms arise in models involving saturation effects and moderate growth phenomena; see for instance [14, 16].

In the present work, we focus on the nonlinear Robin condition

$$\frac{\partial u}{\partial n} + \varphi(x)u\ln(e + u^2) = g,$$

The function

$$F(u) = u\ln(e + u^2)$$

has an intermediate growth between linear and polynomial behavior. Indeed,

$$u\ln(e+u^2)\sim u \quad \text{as } u\to 0, \qquad \text{while} \quad u\ln(e+u^2)\sim 2u\ln|u| \quad \text{as } |u|\to\infty.$$

Therefore, this nonlinearity grows slower than polynomial nonlinearities, but faster than linear law. This intermediate behaviour requires specific estimates in trace spaces and leads to analytical difficulties different from the classical semilinear framework.

Although logarithmic nonlinearities have been widely studied in other context, especially for nonlinear Schrödinger equations (see Cazenave [5]), most existing results concern nonlinear terms acting in the interior of the domain. To the best of our knowledge, elliptic problems involving logarithmic nonlinearities acting through Robin boundary conditions remain insufficiently studied.

The objective of this paper is to fill this gap by developing a complete analysis of problem (P). Our main contributions are summarized as follows:

- we establish a variational framework adapted to the logarithmic Robin boundary condition;
- we construct an iterative linearization scheme based on a sequence of linear problems and prove the existence and uniqueness of each iterate;
- we prove a rigorous convergence analysis using a local Lipschitz estimate of the logarithmic boundary operator;
- we investigate qualitative properties of the weak solution, including regularity and positivity;
- we validate the theoretical results through numerical experiments based on finite element approximations.

The remainder of this paper is organized as follows. Section 2 introduces the functional framework and preliminary estimates. Section 3 is devoted to construction of the iterative sequence and the well-posedness of the associated linearized problems. The convergence of the iterative scheme is established in section 4. In section 5, we investigate the qualitative properties of the weak solution. Numerical experiments illustrating the performance of the proposed method are presented in section 6. Finally, section 7 concludes the paper.

2 Functional framework and preliminary results

2.1 Variational setting

For $1\le q<\infty$, we denote by $W^{1,q}(\Omega)$ the Sobolev space

$$W^{1,q}(\Omega)=\{u\in L^q(\Omega);\ \nabla u\in (L^q(\Omega))^2\},$$

endowed with the norm

$$\|u\|_{W^{1,q}(\Omega)}=\big(\|u\|_{L^q(\Omega)}+\|\nabla u\|_{L^q(\Omega)}\big)^{1/q}.$$

In particular, $H^1(\Omega)=W^{1,2}(\Omega)$.

Since, problem (P) involves an homogeneous Dirichlet boundary condition on Γ_D, the natural functional framework is the space V defined by:

$$V=\{v\in H^1(\Omega);\ v=0 \text{ on } \Gamma_D\}.$$

The space V is a Hilbert space when endowed with the inner product

$$\langle u, v\rangle_V = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

Indeed, by the Poincaré inequality, the condition

$$\langle u, u\rangle_V = 0$$

implies that $u = 0$ in V, so that $\langle \cdot, \cdot \rangle_V$ defines an inner product on V.
The associated norm is given by

$$\|u\|_V = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2} = \|\nabla u\|_{L^2(\Omega)}, \qquad \forall u \in V.$$

Moreover, this norm is equivalent to the usual H^1-norm on V.

2.2 Trace estimates

Let τ denote the trace operator:

$$\begin{aligned} \tau : \quad V &\to \quad H^{\frac{1}{2}}(\partial\Omega), \\ u &\mapsto u_{|\partial\Omega}. \end{aligned}$$

This operator is linear and continuous with respect to $\|.\|_{L^2(\partial\Omega)}$.
The following trace embedding result will be used throughout the paper.

Lemma 2.1. *[9]. Let $\Omega \subset \mathbb{R}^2$ be a bounded and Lipschitz domain. Then*

$$H^1(\Omega) \hookrightarrow L^q(\partial\Omega), \qquad \forall q < \infty.$$

Moreover, there exists a constant C_q such that

$$\|u\|_{L^q(\partial\Omega)} \le C_q \|u\|_{H^1(\Omega)}, \qquad u \in H^1(\Omega). \tag{1}$$

Since the norm $\|.\|_V$ is equivalent to the H^1 norm on V, we deduce that there exists a constant $\beta_q > 0$ such that

$$\|u\|_{L^q(\partial\Omega)} \le \beta_q \|u\|_V, \quad \forall u \in V. \tag{2}$$

where

$$\beta_q = \inf \left\{ C > 0 \ ; \ \|u\|_{L^q(\partial\Omega)} \le C\|u\|_V, \ \forall u \in V \right\}.$$

Remark 2.2. In the particular case where $\Omega = D(0,1)$, the unit disk of $\mathbb{R}^2$, sharper estimates for the trace embedding constant are available. In particular, Elhechmi et al. [3] proved that

$$H^{1/2}(S(0,1)) \hookrightarrow L^q(S(0,1)), \qquad q \ge 2,$$

and

$$\|f\|_{L^q(S(0,1))} \le \lambda_q \|f\|_{H^{1/2}(S(0,1))}, \tag{3}$$

where, for $q > 2$,

$$\lambda_q = \left(C_1 \frac{q}{q-2} \, C_2^{q-2} \, q^{\frac{q-2}{2}} \right)^{\frac{1}{q}}. \tag{4}$$

Here C_1, C_2 are universal constants.

2.3 Admissible set and assumptions

In what follows we assume thatintroduce the following notations and assumptions:

- $\phi \in L^2(\Gamma_N)$, $\phi \not\equiv 0$ and $\phi \geq 0$.
- $g \in L^2(\Gamma_R)$ and $g \geq 0$.
- $M_0 = \beta_2 \left(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \right)$.
- The Robin coefficient is assumed to belong to the admissible set

$$\Phi_{ad} = \left\{ C^0(\overline{\Gamma_R}), \quad \varphi \geq 0, \quad \|\varphi\|_{L^\infty(\Gamma_R)} < \alpha \right\},$$

where

$$0 < \alpha \leq \frac{e}{2\beta_4^4 M_0^2}. \tag{5}$$

3 Construction of the iterative sequence

In this section, we introduce an iterative linearization procedure for problem (P). The main idea is to replace the nonlinear logarithmic Robin term by a sequence of linear boundary operators evaluated at the previous iteration.

Let $u_0 = 0$. For given $u_k \in V$, we define u_{k+1} as the solution of the following problem:

$$(P_k) \begin{cases} \Delta u_{k+1} & = 0 \quad \text{in } \Omega, \\ u_{k+1} & = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u_{k+1}}{\partial n} & = \phi \quad \text{on } \Gamma_N, \\ \dfrac{\partial u_{k+1}}{\partial n} + \varphi\, u_{k+1} \ln(e + u_k^2) & = g \quad \text{on } \Gamma_R. \end{cases}$$

Thus at each iteration, the logarithmic coeffichient is frozen using the previous approximation u_k.

The weak formulation of (P_k) is obtained by multiplying the equation by an arbitrary test function $u \in V$ and integrating by parts.

We obtain

$$a_k(u_{k+1}, v) = l(v), \qquad \forall v \in V,$$

where

$$a_k(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi(x) \ln(e + u_k^2) uv \, ds$$

and

$$l(v) = \int_{\Gamma_N} \phi v \, ds + \int_{\Gamma_R} gv \, ds,$$

It is clear that for $k = 0$, we recover the linear problem, which is well-posed and admits a unique solution $u_1 \in V$. Precisely, referring to [6], we have the following result.

Lemma 3.1. *The first linear problem (P_0) admits a unique solution $u_1 \in V$. Moreover, this solution satisfies the uniform bound*

$$\|u_1\|_V \leq M_0 \quad \textit{with} \quad M_0 = \beta_2 \Big(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \Big). \tag{6}$$

Hence, the sequence of problems (P_k) is well-defined.
Our main objectives are to prove that:

- for all $k \in \mathbb{N}$, the problem (P_k) admits a unique solution u_{k+1} in the space V,
- the sequence (u_k) converges to u which is the unique solution of the problem (P).

3.1 Existence and Uniqueness of the iterarive sequence

Theorem 3.2. *For all $k \in \mathbb{N}$, the problem (P_k) admits a unique solution u_{k+1} in V. Moreover this solution satisfies,*

$$\|u_{k+1}\|_V \leq M_0 \quad \textit{with} \quad M_0 = \beta_2\Big(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)}\Big). \tag{7}$$

Remark 3.3. The problem (P_k) is easily shown to be equivalent to the following variational problem

$$(\tilde{P}_k) \begin{cases} \text{find } \ u \in V \text{ such that} \\ a_k(u,v) = l(v) \quad \forall v \in V, \end{cases}$$

where the bilinear form $a_k(\cdot,\cdot)$ and the linear form $l(.)$ are defined above.

In order to establish the existence and uniqueness of a solution to $(\tilde{P}_k)$, we apply the Lax–Milgram theorem. This requires proving that the bilinear form$a_k(\cdot,\cdot)$ is continuous and coercive on $V \times V$, and that the linear functional $l(\cdot)$ is continuous on V.

Proof. **Continuity of a_k:**

Using the Cauchy–Schwarz inequality, we have for all $u, v \in V$

$$\begin{aligned} |a_k(u,v)| &\leq \Big|\int_\Omega \nabla u \cdot \nabla v \, dx\Big| + \Big|\int_{\Gamma_R} \varphi(x)\ln(e+u_k^2)uv\, ds\Big| \\ &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + \|\varphi\|_{L^\infty(\Gamma_R)}\int_{\Gamma_R} |u\ln(e+u_k^2)v|\, ds. \end{aligned} \tag{8}$$

Since $\ln x \leq x-1$ for all $x>0$, we obtain $\ln(e+t) \leq 1+t$ for all $t \geq 0$.
By this inequality and along with the continuous Sobolev embeddings ($V \hookrightarrow L^4(\Gamma_R)$, we deduce:

$$\begin{aligned} \int_{\Gamma_R} |u\ln(e+u_k^2)v|\, ds &\leq \int_{\Gamma_R} |u|(1+|u_k|^2)|v|\, ds \\ &\leq \|u\|_{L^2(\Gamma_R)}\|v\|_{L^2(\Gamma_R)} + \|u_k\|^2_{L^4(\Gamma_R)}\|u\|_{L^4(\Gamma_R)}\|v\|_{L^4(\Gamma_R)} \\ &\leq \beta_2^2\|u\|_V\|v\|_V + \beta_4^4 M_0^2\|u\|_V\|v\|_V \\ &\leq \Big(\beta_2^2 + \beta_4^4 M_0^2\Big)\|u\|_V\|v\|_V. \end{aligned} \tag{9}$$

Combining equations (8) and (9), we get

$$\begin{aligned} |a_k(u,v)| &\leq \|u\|_V\|v\|_V + \|\varphi\|_{L^\infty(\Gamma_R)}\Big(\beta_2^2 + \beta_4^4 M_0^2\Big)\|u\|_V\|v\|_V \\ &\leq \Big(1 + \alpha(\beta_2^2 + \beta_4^4 M_0^2)\Big)\|u\|_V\|v\|_V. \end{aligned} \tag{10}$$

Thus

$$|a_k(u,v)| \leq \gamma\|u\|_V\|v\|_V,$$

with

$$\gamma = 1 + \alpha\,(\beta_2^2 + \beta_4^4 M_0^2). \tag{11}$$

Hence, a_k is continuous on $V \times V$.

Coercivity of a_k:

For any $u \in V$, we have

$$a_k(u,u) = \int_\Omega |\nabla u|^2 \, dx + \int_{\Gamma_R} \varphi \, \ln(e + u_k^2) \, u^2 \, ds.$$

Since $\varphi \geq 0$ and $\ln(e + u_k^2) \geq 1$, almost everywhere on Γ_R, the boundary integral is non-negative. Therefore

$$a_k(u,u) \geq \|u\|_V^2.$$

which proves coercivity.

Continuity of l:

Using the Cauchy-Schwarz inequality and the continuity of the trace mapping, we find for any $v \in V$

$$\begin{aligned} |l(v)| &\leq \Big| \int_{\Gamma_N} \phi \, v \, ds \Big| + \Big| \int_{\Gamma_R} g \, v \, ds \Big| \\ &\leq \|\phi\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \|v\|_{L^2(\Gamma_R)} \\ &\leq \beta_2 \Big(\|\phi\|_{L^2(\Gamma_N)} + \|g\|_{L^2(\Gamma_R)} \Big) \|v\|_V. \end{aligned}$$

Hence, the linear form $l(\cdot)$ is continuous on V.

Conclusion of the step: Since all requirements of the Lax-Milgram theorem's are satisfied, the variational problem $(\tilde{P}_k)$ admits a unique solution $u_{k+1} \in V$. □

3.2 Uniform Estimate

Taking $v = u_{k+1}$ as a test function in $(\tilde{P}_{k+1})$ yields:

$$\int_\Omega \nabla |u_{k+1}|^2 \, dx + \int_{\Gamma_R} \varphi \, u_{k+1}^2 \ln(e + u_k^2) \, ds = \int_{\Gamma_N} \phi \, u_{k+1} \, ds + \int_{\Gamma_R} g \, u_{k+1} \, ds.$$

Using again the non-negativity of the boundary term on the left-hand side, we get:

$$\begin{aligned} \|u_{k+1}\|_V^2 &\leq \int_{\Gamma_N} |\phi \, u_{k+1}| \, ds + \int_{\Gamma_R} |g \, u_{k+1}| \, ds \\ &\leq \beta_2 \Big(\|g\|_{L^2(\Gamma_R)} + \|\phi\|_{L^2(\Gamma_R)} \Big) \|u_{k+1}\|_V \\ &\leq M_0 \|u_{k+1}\|_V \end{aligned}$$

It follows directly that

$$\|u_{k+1}\|_V \leq M_0 \qquad \text{for all } \; g \geq 0.$$

Consequently, the sequence generated by the the iterative scheme remains bounded in the closed ball $B_v(0, M_0)$.

4 Convergence of the iterative section

In this section, we prove that the sequence $(u_k)_{k\in\mathbb{N}}$, converges in V to a function u, which is the unique solution of problem (P). We first establish a contraction estimate which is required to prove the convergence result.

Let $w_k = u_{k+1} - u_k$, where u_k and u_{k+1} are the respective solutions of problems (P_{k-1}) and (P_k).
Then the error term w_k satisfies the following problem:

$$\begin{cases} \Delta w_k & = 0 \quad \text{in } \Omega, \\ w_k & = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial w_k}{\partial n} & = 0 \quad \text{on } \Gamma_N, \\ \dfrac{\partial w_k}{\partial n} + \varphi[\ln(e+u_k^2)u_{k+1} - \ln(e+u_{k-1}^2)u_k] & = 0 \quad \text{on } \Gamma_R. \end{cases}$$

4.1 Contraction estimate

Proposition 4.1. *There exists a positive constant $\rho < 1$ such that, for all $k \in \mathbb{N}^*$, we have:*

$$\|w_k\|_V \le \rho \, \|w_{k-1}\|_V.$$

Proof. Applying Green's formula, the variational formulation satisfied by w_k is given by:

$$\int_\Omega \nabla w_k \cdot \nabla v \, dx + \int_{\Gamma_R} \varphi \Big[u_{k+1} \ln(e+u_k^2) - u_k \ln(e+u_{k-1}^2) \Big] v \, ds = 0, \quad \forall v \in V, \tag{12}$$

we rewrite the nonlinear boundary term as follows:

$$u_{k+1} \ln(e+u_k^2) - u_k \ln(e+u_{k-1}^2) = w_k \ln(e+u_k^2) + u_k \Big(\ln(e+u_k^2) - \ln(e+u_{k-1}^2) \Big).$$

Choosing $v = w_k$ as the test function, we obtain:

$$\int_\Omega |\nabla w_k|^2 \, dx + \int_{\Gamma_R} \varphi \Big[w_k \ln(e+u_k^2) + u_k \Big(\ln(e+u_k^2) - \ln(e+u_{k-1}^2) \Big) \Big] w_k \, ds = 0. \tag{13}$$

Rearranging the terms and keeping the non-negative trace terms on the left-hand side yields:

$$\int_\Omega |\nabla w_k|^2 \, dx + \int_{\Gamma_R} \varphi w_k^2 \ln(e+u_k^2) \, ds \le \int_{\Gamma_R} \Big| \varphi w_k u_k \big[\ln(e+u_k^2) - \ln(e+u_{k-1}^2) \big] \Big| \, ds. \tag{14}$$

Since $\varphi \ge 0$ and the map $t \mapsto \ln(e+t)$ is $\dfrac{1}{e}$-Lipschitz continuous on $[0,\infty)$, we obtain the pointwise estimate:

$$\big| \ln(e+u_k^2) - \ln(e+u_{k-1}^2) \big| \le \frac{1}{e} \big| u_k^2 - u_{k-1}^2 \big| \le \frac{1}{e} \big| u_k + u_{k-1} \big| \big| w_{k-1} \big|. \tag{15}$$

Substituting this into our integral inequality (14) produces:

$$\|w_k\|_V^2 \int_\Omega |\nabla w_k|^2 \, dx \le \frac{1}{e} \|\varphi\|_{L^\infty(\Gamma_R)} \int_{\Gamma_R} |u_k| \big| u_k + u_{k-1} \big| |w_k| |w_{k-1}|.$$

Applying the Holder inequality on the boundary domain Γ_R leads to:

$$\|w_k\|_V^2 \leq \frac{1}{e}\|\varphi\|_{L^\infty(\Gamma_R)}\|u_k\|_{L^4(\Gamma_R)}\|u_k + u_{k-1}\|_{L^4(\Gamma_R)}\|w_k\|_{L^4(\Gamma_R)}\|w_{k-1}\|_{L^4(\Gamma_R)}. \tag{16}$$

Using the triangle inequality : $\|u_k + u_{k-1}\|_{L^4(\Gamma_R)} \leq \|u_k\|_{L^4(\Gamma_R)} + \|u_{k-1}\|_{L^4(\Gamma_R)}$, The continuous Sobolev embedding trace theorem, along with the estimation (7), we obtain:

$$\|w_k\|_V^2 \leq \frac{2}{e}\|\varphi\|_{L^\infty(\Gamma_R)}\beta_4^4 M_0^2\|w_k\|_V\|w_{k-1}\|_V.$$

Simplifying by $\|w_k\|_V$, we conclude:

$$\|w_k\|_V \leq \rho\|w_{k-1}\|_V,$$

where the contraction constant is explicitly defined by:

$$\rho = \frac{2}{e}\,\beta_4^4\,M_0^2\|\varphi\|_{L^\infty(\Gamma_R)}.$$

Under the assumption that $\|\varphi\|_{L^\infty(\Gamma_R)}$ satisfies the condition (5), we have $\rho < 1$, which complete the proof. □

4.2 Existence of a solution to problem (P)

Theorem 4.2. *The sequence $(u_k)_{k\in\mathbb{N}}$ converges in V to u, which is the unique solution of problem (P).*

Proof. By the contraction estimate 4.1:

$$\|w_k\|_V \leq \rho\|w_{k-1}\|_V \quad \forall k \geq 1 \quad \text{with } \rho < 1,$$

we obtain

$$\|u_{k+1} - u_k\|_V \ \leq \rho^k\|u_1 - u_0\|_V \ \leq \rho^k M_0.$$

Then the series $\displaystyle\sum_{k=2}^{\infty} w_k$ converges in the Hilbert space V to an element $S \in V$. Consequently, the sequence $(u_k)_{k\in\mathbb{N}}$ converges to an element u in V.
Specifically, we have

$$u = \lim_{k\to\infty} u_{k+1} = S + u_1 \quad \text{ in } V.$$

It remains to prove that u is the unique solution to the original problem (P).

Since $u_k \to u$ strongly in V, and each u_k solves the linear problem (P_k), we can pass to the limit both inside the domain Ω and on the boundary parts Γ_D and Γ_N. More precisely, since the differential and trace operators are continuous, we have

$$\begin{aligned}
\Delta u_k = 0 \text{ in } \Omega \quad &\Longrightarrow \quad \Delta u = 0 \text{ in } \Omega,\\
u_k = 0 \text{ on } \Gamma_D \quad &\Longrightarrow \quad u = 0 \text{ on } \Gamma_D,\\
\frac{\partial u_k}{\partial n} = 0 \text{ on } \Gamma_N \quad &\Longrightarrow \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N.
\end{aligned}$$

It now remains to identify the limit in the nonlinear Robin boundary condition. We need to prove that

$$\lim_{k\to\infty} \|u_k \ln(e + u_{k-1}^2) - u\ln(e + u^2)\|_{L^1(\Gamma_R} = 0.$$

First, we split the terms as follows:

$$u_k \ln(e+u_{k-1}^2) - u\ln(e+u^2) = (u_k-u)\ln(e+u_{k-1}^2) + u\big(\ln(e+u_{k-1}^2) - \ln(e+u^2)\big).$$

Taking the $L^1(\Gamma_R)$-norm and applying the triangle inequality yields

$$\begin{aligned}\|u_k \ln(e+u_{k-1}^2) - u\ln(e+u^2)\|_{L^1(\Gamma_R)} &= \int_{\Gamma_R} \Big|u_k \ln(e+u_{k-1}^2) - u\ln(e+u^2)\Big|\, ds \\ &\le \int_{\Gamma_R} \Big|(u_k-u)\ln(e+u_{k-1}^2)\Big|\, ds \qquad (17)\\ &+ \int_{\Gamma_R} \Big|u\big(\ln(e+u_{k-1}^2) - \ln(e+u^2)\big)\Big|\, ds \qquad (18)\end{aligned}$$

Let us denote these two integrals by

$$I_1 = \int_{\Gamma_R} \Big|(u_k-u)\ln(e+u_{k-1}^2)\Big|\, ds \quad \text{and} \quad I_2 = \int_{\Gamma_R} \Big|u\big(\ln(e+u_{k-1}^2) - \ln(e+u^2)\big)\Big|\, ds.$$

Using the real analytic inequality $\ln(e+t) \le 1+t$ for all $t \ge 0$, Cauchy-Schwarz inequality, and the continuous Sobolev trace embedding along with the estimation (7), we obtain:

$$\begin{aligned} I_1 &\le \int_{\Gamma_R} |u_k-u|\big(1+u_{k-1}^2\big)\, ds \\ &\le \int_{\Gamma_R} |u_k-u|\, ds + \int_{\Gamma_R} |u_k-u| u_{k-1}^2\, ds \\ &\le \sqrt{\sigma(\Gamma_R)}\, \|u_k-u\|_{L^2(\Gamma_R)} + \|u_k-u\|_{L^2(\Gamma_R)} \|u_{k-1}\|_{L^4(\Gamma_R)}^2 \\ &\le \|u_k-u\|_{L^2(\Gamma_R)} \Big(\sqrt{\sigma(\Gamma_R)} + \beta_4^2 M_0^2\Big) \\ &\le \beta_2 \Big(\sqrt{\sigma(\Gamma_R)} + \beta_4^2 M_0^2\Big) \|u_k-u\|_V \qquad (19)\end{aligned}$$

where β_2, β_4 are the continuous embedding constant into $L^2(\Gamma_R)$ and $L^4(\Gamma_R)$ respectively. Since the map $t \mapsto \ln(e+t^2)$ is $\dfrac{1}{\sqrt{e}}$ Lipschitz continuous on $\mathbb{R}$, the second integral I_2 can be bounded by

$$\begin{aligned} I_2 &\le \frac{1}{\sqrt{e}} \int_{\Gamma_R} |u||u_{k-1}-u|\, ds \\ &\le \frac{1}{\sqrt{e}} \|u\|_{L^2(\Gamma_R)} \|u_{k-1}-u\|_{L^2(\Gamma_R)} \\ &\le \frac{1}{\sqrt{e}}\, \beta_2^2 \|u\|_V \|u_{k-1}-u\|_V \\ &\le \frac{1}{\sqrt{e}}\, \beta_2^2 M_0 \|u_{k-1}-u\|_V \qquad (20)\end{aligned}$$

Combining the equations (17), (19) and (20), we obtain:

$$\|u_k \ln(e+u_{k-1}^2) - u\ln(e+u^2)\|_{L^1(\Gamma_R)} \le \Big[\beta_2\Big(\sqrt{\sigma(\Gamma_R)} + \beta_2^2 M_0^2\Big) + \frac{1}{\sqrt{e}}\, \beta_2^2 M_0\Big] \|u_{k-1}-u\|_V$$

Since the sequence (u_k) converges strongly to u in the Hilbert space V, taking the limit as $k \to \infty$ yields

$$\lim_{k\to\infty} \|u_k \ln(e+u_{k-1}^2) - u\ln(e+u^2)\|_{L^1(\Gamma_R)} = 0.$$

Thus, passing to the limit in the nonlinear Robin boundary condition establishes that

$$\frac{\partial u}{\partial n} + \varphi(x) u \ln(e+u^2) = g \quad \text{ on } \Gamma_R.$$

We conclude that u is a weak solution of problem (P). □

4.3 uniqueness result

Let u and v be two solutions to problem (P), and set $w = u - v$. By proceeding as in equation (12), we obtain:

$$\int_\Omega \nabla w \cdot \nabla z \, dx + \int_{\Gamma_R} \varphi \Big[u \ln(e + u^2) - v \ln(e + v^2) \Big] z \, ds = 0, \quad \forall z \in V,$$

Testing this formulation with $z = w$ gives

$$\int_\Omega |\nabla w|^2 \, dx + \int_{\Gamma_R} \varphi \Big[u \ln(e + u^2) - v \ln(e + v^2) \Big] (u - v) \, ds = 0, \quad \forall z \in V.$$

Define

$$G(t) = t \ln(e + t^2).$$

Then $G'(t) = \ln(e + t^2) + \dfrac{2t^2}{e + t^2} > 0$ for all $t \in \mathbb{R}$.

Hence G is strictly increasing.
Therefore, since $\varphi \geq 0$, the second integrand is non negative.
Thus, we get

$$\|w\|_V^2 = \int_\Omega |\nabla w|^2 \, dx = 0,$$

Consequently, we conclude that $w = u - v = 0$, which proves the uniqueness of the solution to problem (P).

We are now in position to state the main result.

Theorem 4.3. *The original problem (P) admits a unique solution $u \in V$ such that*

$$\|u\|_V \leq M_0. \tag{21}$$

5 Qualitative Properties of the Solution

In this section, we study some qualitative properties of the weak solution of problem (P), we establish regularity results and a positivity property under suitable assumption.

5.1 Regularity of the solution

In order to establish the regularity properties of the weak solution, we introduce the following assumptions:

(H1) $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$.

(H2) The interface set

$$\Sigma = \big(\overline{\Gamma}_D \cap \overline{\Gamma}_N\big) \cup \big(\overline{\Gamma}_D \cap \overline{\Gamma}_R\big) \cup \big(\overline{\Gamma}_N \cap \overline{\Gamma}_R\big)$$

is assumed to be locally flat or locally smooth.

(H3) The boundary data satisfy

$$\phi \in L^p(\Gamma_N), \quad g \in L^p(\Gamma_R), \qquad p > 2.$$

(H4) The Robin coefficient satisfies

$$\varphi \in \Phi_{ad} = \{C^0(\overline{\Gamma}_R),\ \varphi \geq 0,\ \|\varphi\|_{L^\infty(\Gamma_R)} < \alpha\}.$$

Theorem 5.1. *Assume that hypotheses* **(H1)**–**(H4)** *hold. Then the weak solution u of problem (P) satisfies*

$$u \in W^{1,q}(\Omega), \qquad 2 < q < 4.$$

Moreover,

$$u \in C^{0,\gamma}(\overline{\Omega}), \quad \text{where} \quad \gamma = 1 - \frac{2}{q}$$

Proof. Let $u \in V$ be the weak solution of problem (P):

$$(P) \quad \begin{cases} \Delta u & = 0 \quad \text{in } \Omega, \\ u & = 0 \quad \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} & = \phi \quad \text{on } \Gamma_N, \\ \dfrac{\partial u}{\partial n} + \varphi(x) u \ln(e + u^2) & = g \quad \text{on } \Gamma_R. \end{cases}$$

We rewrite the Robin Boundary condition as

$$\frac{\partial u}{\partial n} = g - \varphi(x) u \ln(e + u^2) \qquad \text{on } \Gamma_R.$$

Since $u \in V \subset H^1(\Omega)$ and $\Omega \subset \mathbb{R}^2$ is bounded with Lipschitz boundary, the trace theorem implies

$$u_{|\partial\Omega} \in L^q(\partial\Omega) \quad \forall q < \infty.$$

Moreover, the function

$$F : t \mapsto t \ln(e + t^2)$$

has at most cubic growth, indeed

$$|F(t)| = |t| \ln(e + t^2) \leq C(1 + |t|^3)$$

for some constant $C > 0$. Therefore,

$$u \ln(e + u^2) \in L^p(\Gamma_R), \qquad \forall\, p < \infty.$$

Since $\varphi \in L^\infty(\Gamma_R)$ and by assumption **(H3)** $g \in L^p(\Gamma_R)$, we deduce that

$$g - \varphi(x) u \ln(e + u^2) \in L^p(\Gamma_R) \qquad \forall\, p < \infty.$$

Therefore, The Neumann-type boundary data on each boundary component belong to L^p space with $p > 2$.

Applying the regularity theory of elliptic equations with mixed boundary conditions (see Grisvard [12]) we deduce that

$$u \in W^{1,q}(\Omega), \qquad 2 < q < 4.$$

We note that the condition $q < 4$ is due to the possible singularities generated at the junction points Dirichlet, Neumann and Robin boundary parts.

Finally, since $\Omega \subset \mathbb{R}^2$ and $q > 2$, the Sobolev embedding theorem gives

$$W^{1,q}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega}), \qquad \text{with} \qquad \gamma = 1 - \frac{2}{q}.$$

Consequently,

$$u \in C^{0,\gamma}(\overline{\Omega}).$$

□

5.2 Positivity of the solution

For the positivity result, we assume in addition the following condition:

(H5) The boundary portions Γ_D, Γ_N and Γ_R are of class C^2 in a neighborhood of their relative interior points.

Theorem 5.2. *Assume that*

$$\phi \geq 0 \quad \text{on} \quad \Gamma_N, \qquad g \geq 0 \quad \text{on} \quad \Gamma_R.$$

Then the weak solution of problem (P) satisfies

$$u(x) \geq 0 \qquad \forall x \in \Omega.$$

Proof. Since $\Delta u = 0$ in Ω, the function u is harmonic in Ω. By the weak maximum principle [11]

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Then the minimum is attained at some point $x_0 \in \partial\Omega$:

$$u(x_0) = \min_{\overline{\Omega}} u.$$

Assume, by contradiction that,

$$u(x_0) < 0.$$

We examine the possible location of x_0.

(i) The point x_0 belongs to Γ_D
By the Dirichlet condition,

$$u = 0 \qquad \text{on } \Gamma_D,$$

Hence,

$$u(x_0) = 0$$

which contradicts the assumption that the minimum is negative.

(ii) The point x_0 belongs to Γ_N
Since u atteints its minimum at $x_0 \in \Gamma_N$, Hopf's boundary point lemma (see [11]) implies

$$\frac{\partial u}{\partial n}(x_0) < 0.$$

In other hand,

$$\frac{\partial u}{\partial n} = \phi \geq 0 \quad \text{on} \quad \Gamma_N$$

which is impossible.

(iii) The point x_0 belongs to Γ_R
Assume that $u(x_0) < 0$. Again, by Hopf's lemma $\frac{\partial u}{\partial n}(x_0) < 0$.
using the Robin boundary condition,

$$\frac{\partial u}{\partial n} + u\varphi \ln(e + u^2) = g \quad \text{on } \Gamma_R,$$

we deduce

$$g(x_0) < \varphi(x_0)u(x_0)\ln\big(e + u(x_0)^2\big).$$

Since

$$\varphi(x_0) \geq 0, \quad \ln(e + u(x_0)^2) > 0, \quad u(x_0) < 0,$$

the right-hand side is negative, and consequently

$$g(x_0) < 0,$$

which contradicts the assumption $g \geq 0$ on Γ_R.
Therefore, a negative minimum cannot occur any part of the boundary.
Consequently,

$$u \geq 0 \quad \text{in } \Omega.$$

□

6 Numerical validation

In this section, we present several numerical experiments designed to validate the accuracy, robustness, and convergence properties of the proposed iterative linearization scheme for solving the elliptic problem with a nonlinear logarithmic Robin boundary condition.
To evaluate the performance of the method, we consider the exact quadratic solution

$$u(x, y) = x^2 - y^2 + 3,$$

which satisfies $\Delta u = 0 \quad \text{in } \Omega.$

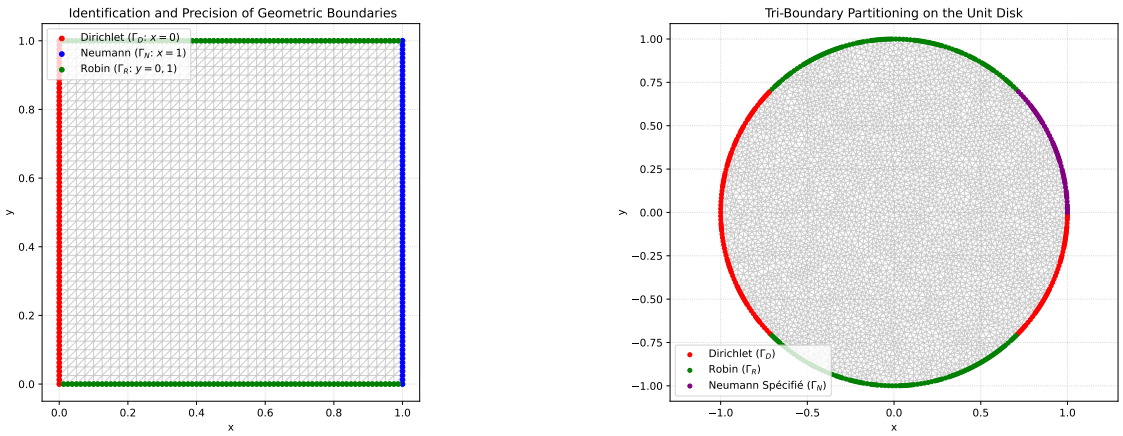


Figure 1: Geometric partitioning of the boundaries for the square and disk domains.

The numerical implementation is carried out in the **FEniCS** environment using quadratic Lagrange finite element (P_2). In order to demonstrate the robustness of the algorithm, the numerical study is performed on two distinct geometries:

1. **Polygonal domain (unit square):** $\Omega = (0, 1)^2$.
   The boundary $\partial\Omega$ is divided into flat segments. This configuration serves as a primary check of the treatment of Dirichlet, Neumann, and Robin boundary conditions.

2. **Curved boundary domain (unit disk):** $\Omega = \{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 < 1\}$.
   The unit outward unit normal vector is given by: $\vec{n} = (x, y)^t$.
   This geometry allows us to evaluate the capability of the iterative scheme to accurately approximate nonlinear boundary terms on curved interfaces without loss of precision.

The geometric partition of the boundary conditions with the associated computational meshes are illustrated in Figure 1.

6.1 Iterative scheme and convergence analysis

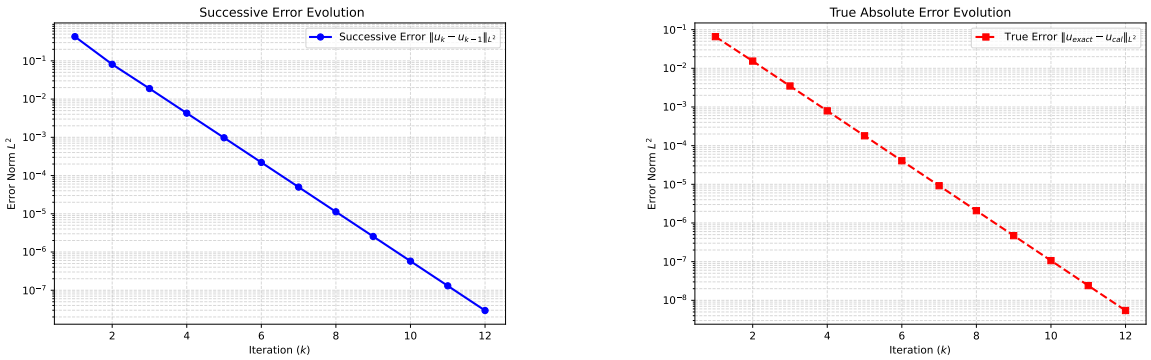
The originality of our approch lies in the construction of the sequence of linearized subproblems (P_k). At each iteration, the nonlinear term: $\ln(e + u^2)$ is evaluated using the solution obtained at the previous iteration $(k - 1)$.
Consequently, the nonlinearity is incorporated directly into the Robin Boundary bilinear form.
This strategy avoids the use of global nonlinear solvers, such as Newton-Raphson methods, wich are often sensitive to the choice of the initial approximation u_0.
In the present computations, a simple constant initialization of u_0 is sufficient to ensure convergence. The stopping criterion of the fixed-point algorithm is based on the successive error

$$E_{succ} = \frac{\|u_k - u_{k-1}\|_{L^2(\Omega)}}{\|u_k\|_{L^2(\Omega)}},$$

with tolerance $\varepsilon = 10^{-7}$.

6.2 Discussion of numerical results

To evaluate the efficiency of the method, we present both the sucvessive residual error and the absolute error with respect to the exact solution. The numerical results demonstrate the performance of the proposed linearization scheme.
For both geometries, the fixed fixed-point algorithm converges within approximately 12 iterations to satisfy the prescribed stopping criterion.
The evolution of the errors with respect to the iteration number is represented on semi-logarithmic scales in Figure 2 (square domain) and Figure 3 (disk domain).

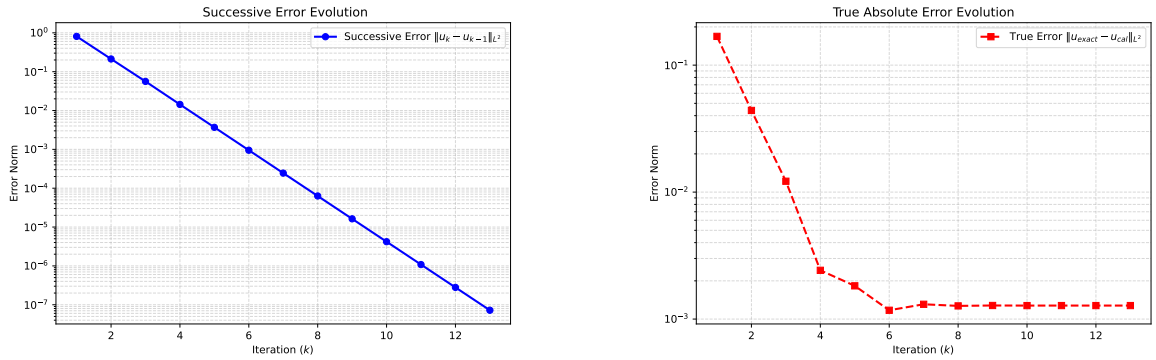


(a) Semi-log plot of the successive error.

(b) Semi-log plot of the exact error.

Figure 2: Error evolution for the square domain.

The linear decay observed on the semi-logarithmic plots provides strong numerical evidence of the geometric convergence of the iterative linearization scheme.



(a) Semi-log plot of the successive error.

(b) Semi-log plot of the exact.

Figure 3: Error evolution for the disk domain.

We observe that while the successive fixed-point error $\|u_k - u_{k+1}\|_{L^2}$ falls below the prescribed tolerence of 10^{-7} for both geometries, the true absolute error $\|u - u_k\|_{L^2}$ exhibits a differnt behavior.
On the square domain, it reaches the expected precision, whereas on the unit disk, the error stagnates around $\approx 2 \cdot 10^{-3}$.
This behavior is classical geometric artefact in finite element methods: a geometric error of order $\mathcal{O}(h^2)$ is naturally introduced due to the approximation of curved boundaries. This error acts as an irreducible lower band, reflecting the geometric approximation of the domain rather than a limitation of the fixed-pont iterative scheme.

Next, to more evaluate the accuracy of the method at the nonlinear interface, the numerical and exact solutions are compared along the entire boundary of the domain.
The solution profiles shown in the figure 4 show an almost excellent agreement between the calculated solution and the exact solution over the whole boundary. Furthermore, the smooth transitions observed near the junctions between the different boundary portions (for example, between Γ_N and Γ_R) confirm the consistency of the variational formulation and the absence of numerical artifacts. Moreover, we focus on the Robin interface Γ_R, where the

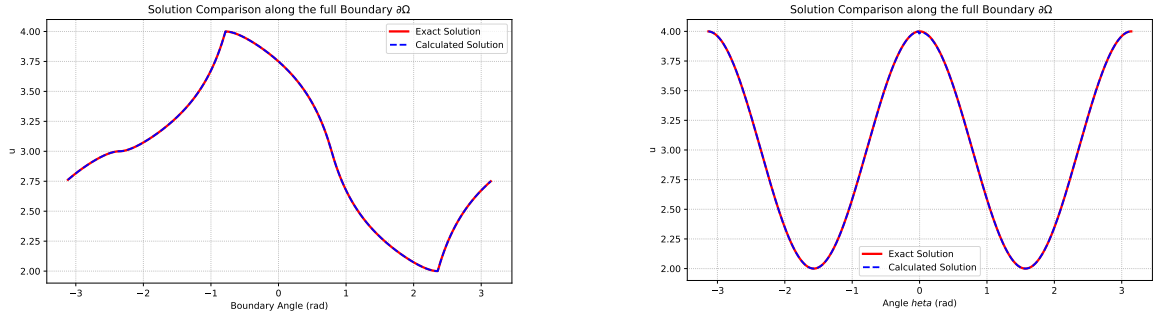


Figure 4: Comparison between the exact and computed solutions along the whole boundary.

nonlinear boundary condition is imposed.
The profiles extracted along the Robin boundary (Figure 5) reveal an excellent agreement between the exact and computed solutions.

Finally, we conclude that the numerical experiments confirm that the proposed iterative method is stable, accurate and efficient for slving elliptic problems with nonlinear logarithmic Robin boundary conditions.

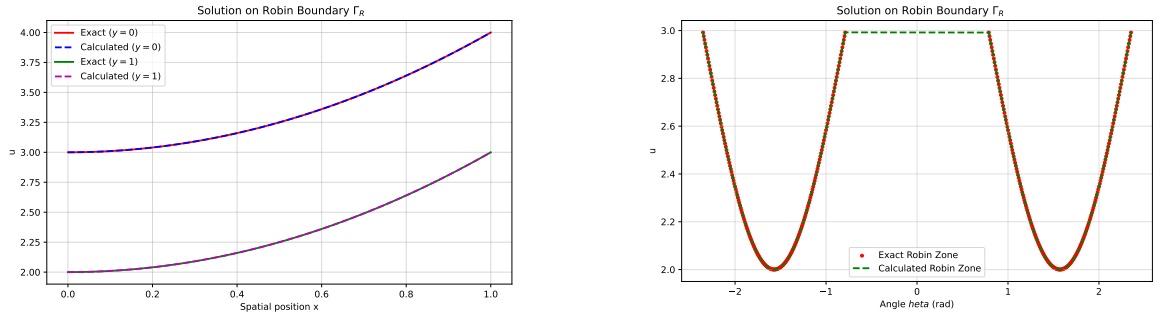


Figure 5: Comparison between the exact and computed solutions along the Robin boundary Γ_R.

7 Conclusion

In this paper, we have studied an elliptic boundary value problem with a nonlinear logarithmic Robin boundary condition. After estblishing an appropriate variational framework, we proved the existence and uniqueness of a weak solution by introducing an iterative linearization procedure. The well-posedness of each linearized problem was obtained via the Lax–Milgram theorem, and the convergence of the iterative sequence was established under a suitable admissibility condition on the Robin coefficient.

We also investigated qualitative proporties of the solution. In particular, regularity results were derived using elliptic regularity theory for mixed boundary value problem, and a positivity property was established under natural assumptions on the boundary data.

Finally, numerical experiments performed with the finite element methodin the FEniCS environment confirmed the theoretical analysis. The proposed iterative scheme exhibits fast convergence, numerical stability, and excellent agreement with the exact solution on both polygonal and curved domains.

Future work will focus on inverse identification problems involving logarithmic Robin coefficient.

Conflicts of Interest

The author declare no conflicts of interest.

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