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Complex Operator Growth in Dissipative Quantum Systems

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The universal operator-growth hypothesis (OGH) states that in a closed chaotic system the Lanczos coefficients grow linearly, $b_n \simeq \alpha n$, and that the rate α bounds the Lyapunov exponent, $\lambda_L \leq 2\alpha$. We ask how this structure is modified when the system is coupled to a Markovian environment, so that the generator becomes non-Hermitian. Applying the Arnoldi recursion to the vectorized Lindbladian in the infinite-temperature Wightman inner product, we organize the resulting complex growth rate $\alpha_{\mathbb{C}} = \alpha_R + i\alpha_I$ (the real sub-diagonal and complex diagonal Arnoldi coefficients) around three exact statements. First, the diagonal's real part is fixed exactly by $\mathrm{Re}\, a_n = -2\gamma\langle\hat{G}\rangle_n$, identifying the imaginary rate as the rate at which the growing operator accumulates weight in the dissipation channels; its magnitude σ_G is non-universal (set by the jump operators), but the generator symmetry $\mathcal{G}^\dagger(\gamma) = -\mathcal{G}(-\gamma)$ makes b_n even and $\mathrm{Re}\, a_n$ odd in the dissipation strength γ, so universally $\alpha_I = 2\gamma\sigma_G$ turns on linearly and α_R is renormalized only at $O(\gamma^2)$. Second, for the large-q SYK model we obtain the closed form $\alpha_{\mathbb{C}} = \mathcal{J} - 2i(q-2)\gamma$, whose imaginary part is fixed solely by the interaction range. Third, we argue—on the same footing as the OGH itself—that the bound extends as $\lambda_L \leq 2\alpha_R(\gamma)$ with $\alpha_R(\gamma) \leq \alpha_R(0) + O(\gamma)$, because the dissipator is a local bounded superoperator that narrows the autocorrelation's analyticity strip by at most $O(\gamma)$. Rather than merely acquiring a phase, the imaginary rate is an exact, parity-odd measure of dissipative operator spreading, while the complexity-growth rate is protected to $O(\gamma^2)$: dissipation does not suppress operator growth.

Introduction.—Operator growth—the progressive delocalization of an initially simple operator under Heisenberg evolution—underlies quantum chaos, thermalization, and the emergence of hydrodynamics [1]. The universal operator-growth hypothesis (OGH) [2] made this quantitative. Expanding the Heisenberg equation of motion in the Krylov basis generated by the Liouvillian $\mathcal{L}_0 = i[H, \cdot]$, one obtains a semi-infinite hopping chain whose amplitudes, the Lanczos coefficients b_n, grow linearly, $b_n \simeq \alpha n$ (with a logarithmic correction in one dimension). The single rate α controls the exponential growth of Krylov complexity [3, 4], upper-bounds a large class of complexity measures including the out-of-time-order correlator [5, 6], and yields $\lambda_L \leq 2\alpha$—a bound that is tighter than, and complementary to, the Maldacena–Shenker–Stanford bound $\lambda_L \leq 2\pi T$ [7].

Real systems are open. Under Markovian dissipation the generator of operator dynamics is the (adjoint) Lindbladian, which is *non-Hermitian*, and the symmetric Lanczos algorithm no longer applies: one must use the Arnoldi iteration [8] or a bi-Lanczos scheme [9–11], and the OGH has been examined for large-q SYK with weak dissipation [12]. These works establish the recursion machinery and compute the coefficients numerically; the one analytic large-q SYK rate [12] is extracted from the (real) off-diagonals and stays real, dissipation entering as a phase. The imaginary part, however, lives in the *diagonal* coefficients a_n, which prior open-system studies do not analyze. What has been missing is a *physical* and *analytic* account of it: what it measures, how it depends on dissipation, and whether the chaos bound survives.

Our advance is not that the coefficients are complex—they are, in any non-Hermitian recursion—but that the imaginary rate is pinned down *physically and analytically*: an exact identity identifies it with the operator's dissipation-grading weight; a generator symmetry fixes its linear-in-γ onset and protects α_R to $O(\gamma^2)$; large-q SYK yields the closed form $\alpha_{\mathbb{C}} = \mathcal{J} - 2i(q-2)\gamma$; and the chaos bound extends, on the OGH's own footing, to $\lambda_L \leq 2\alpha_R(\gamma)$. The unifying message is that dissipation does not slow operator growth—it rotates the rate into the complex plane, and the imaginary part is a new, environment-resolved observable.

Setup.—We work at infinite temperature with the Wightman inner product $(A|B) = \mathrm{Tr}(A^\dagger B)/\dim$, under which the closed-system Liouvillian is Hermitian. As in all recursion-method studies of open systems, the construction is inner-product dependent; our results are for this standard choice, and their finite-temperature counterparts are left open. The Heisenberg generator of the Lindblad dynamics [13, 14] is

$$\mathcal{G}(O) = i[H, O] + \sum_\mu \left(L_\mu^\dagger O L_\mu - \tfrac{1}{2}\{L_\mu^\dagger L_\mu, O\} \right) = \mathcal{L}_0 + \mathcal{D}, \tag{1}$$

with $\mathcal{L}_0 = i[H, \cdot]$ anti-Hermitian and $\mathcal{D}$ the dissipator. Vectorizing O and orthonormalizing the Krylov sequence $\{\mathcal{G}^n O_0\}$ [15] by the Arnoldi iteration with full reorthogonalization produces an upper-Hessenberg matrix h whose sub-diagonal $b_n = h_{n+1,n} \in \mathbb{R}_{\geq 0}$ and diagonal $a_n = h_{n,n} \in \mathbb{C}$ define

$$\boxed{\alpha_{\mathbb{C}} \equiv \alpha_R + i\alpha_I, \quad b_n \to \alpha_R\, n, \quad -\mathrm{Re}\, a_n \to \alpha_I\, n.} \tag{2}$$

For $\gamma = 0$, $\mathcal{G} = \mathcal{L}_0$ is anti-Hermitian, the Hessen-

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berg matrix is tridiagonal, the diagonal is purely imaginary, and b_n reduce to the OGH Lanczos coefficients with $\alpha_I = 0$. Numerically we use a chaotic mixed-field Ising chain, $H = -\sum_i Z_i Z_{i+1} - g\sum_i X_i - h\sum_i Z_i$ with $(g, h) = (1.05, 0.5)$, under bulk dephasing $L_i = \sqrt{\gamma}\, Z_i$; details and further models are in the Supplemental Material (SM).

Pillar 1: the complex growth rate.—For dephasing the dissipator is diagonal in the Pauli-string basis, $\mathcal{D} = -2\gamma\hat{G}$, where $\hat{G}$ is the transverse-weight operator counting the sites on which an operator acts as X or Y. Decomposing the Arnoldi diagonal as $a_n = (v_n|\mathcal{L}_0|v_n) + (v_n|\mathcal{D}|v_n)$ and using that the first term is purely imaginary ($\mathcal{L}_0$ anti-Hermitian) while the second is real ($\mathcal{D}$ Hermitian) gives the exact identity

$$\boxed{\mathrm{Re}\, a_n = -2\gamma\, \langle \hat{G} \rangle_n,} \tag{3}$$

which is algebraic—it holds for every H, γ and n—and is confirmed numerically at the 10^{-15} level (SM). The imaginary part of the complex rate therefore *measures the rate at which the growing operator accumulates weight in the channels the environment couples to*, $\alpha_I = 2\gamma\,\sigma_G$ with $\sigma_G = \lim_n d\langle \hat{G} \rangle_n / dn$ [Fig. 1(a,b)].

The dependence on γ is fixed by an exact symmetry. Writing $\mathcal{G}(\gamma) = \mathcal{L}_0 + \gamma\tilde{\mathcal{D}}$ with $\tilde{\mathcal{D}}$ Hermitian, one has

$$\mathcal{G}^\dagger(\gamma) = -\mathcal{G}(-\gamma) \;\Rightarrow\; \mu_k(\gamma)^* = (-1)^k \mu_k(-\gamma), \tag{4}$$

for the moments $\mu_k = (O_0|\mathcal{G}^k|O_0)$. Propagated through the recursion (SM), Eq. (4) forces b_n to be *even* and $\mathrm{Re}\, a_n$ to be *odd* in γ, both exactly. Consequently

$$\alpha_R(\gamma) = \alpha_R(0) + c_2\gamma^2 + O(\gamma^4), \qquad \alpha_I(\gamma) = 2\kappa_0\gamma + O(\gamma^3), \tag{5}$$

so that at leading order $\alpha_I = 2\kappa_0\gamma$ is fixed entirely by the *closed-system* transverse-weight slope κ_0: measuring κ_0 on the $\gamma = 0$ Krylov chain predicts the leading-order α_I/γ with no fit parameter, in 0.2% agreement [Fig. 2(a)]; both $\alpha_R(0)$ and κ_0 saturate with system size up to $N = 10$, and α_R is basis-independent (Arnoldi and bi-Lanczos agree to $< 0.1\%$; SM). The complexity-growth rate α_R is renormalized only quadratically [Fig. 2(b)], so *for weak dissipation* the complex rate undergoes an approximately rigid rotation off the real axis, $\arg\alpha_\mathbb{C} \approx (2\kappa_0/\alpha_R)\gamma$ [Fig. 2(c)]; this small-angle picture holds to $\arg\alpha_\mathbb{C} \sim 15°$ at $\gamma = 0.4$ and breaks down once $\alpha_I \sim \alpha_R$. In parallel, the Krylov complexity saturates and the Wightman autocorrelation decays with γ [Fig. 1(e,f)], while the Liouvillian spectrum broadens from the imaginary axis into the left half-plane [Fig. 1(d)].

Model dependence and large-q SYK.—The grading $\hat{G}$—and hence the sign and magnitude of α_I—is model-dependent, which is the non-universal content of the imaginary rate. For the SYK_q model [16, 17] with Majorana dephasing $L_i = \sqrt{2\gamma}\,\psi_i$, the dissipator reduces, in the size-parity sector of a single-Majorana seed, to the operator co-size: $\mathrm{Re}\, a_n = -2\gamma\,(N - \langle s \rangle_n)$, so a growing

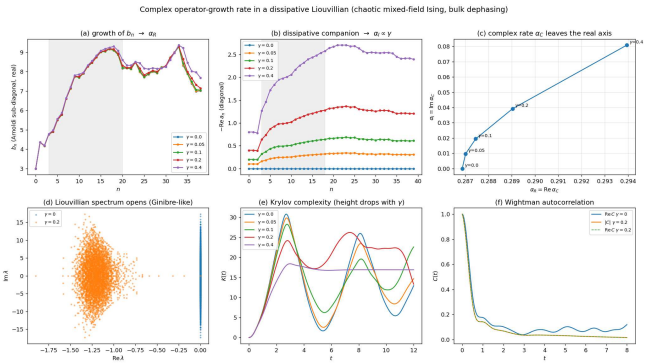


FIG. 1. Complex operator-growth rate for a chaotic mixed-field Ising chain ($N = 6$) under bulk dephasing. (a) Arnoldi sub-diagonal $b_n \to \alpha_R n$; (b) diagonal companion $-\mathrm{Re}\, a_n \to \alpha_I n$, fanning out $\propto \gamma$; (c) the complex rate leaves the real axis; (d) the Liouvillian spectrum opens from the imaginary axis into the complex plane; (e) Krylov complexity; (f) Wightman autocorrelation.

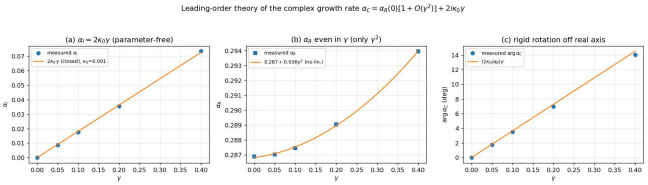


FIG. 2. Leading-order theory. (a) The parameter-free prediction $\alpha_I = 2\kappa_0\gamma$ with κ_0 measured on the closed-system Krylov chain (0.2% agreement); (b) α_R even in γ ($O(\gamma^2)$ only); (c) rigid rotation $\arg\alpha_\mathbb{C} \approx (2\kappa_0/\alpha_R)\gamma$.

operator becomes *less* dissipation-sensitive, opposite to the spin chain. The large-q size ladder is known—each interaction adds $q-2$ Majoranas, so $\langle s \rangle_n = 1 + (q-2)n$ [18] (we confirm the first two steps at finite q; SM)—giving $\sigma_G = -(q-2)$. With $\alpha_R = \mathcal{J}$ [2] we obtain the closed form

$$\boxed{\alpha_\mathbb{C}^{\mathrm{SYK}} = \mathcal{J} - 2i(q-2)\gamma,} \tag{6}$$

whose imaginary part is fixed *solely by the interaction range* q, independent of the coupling $\mathcal{J}$. Equations (3)–(6) constitute the first pillar of our results.

Pillar 2: a dissipative growth bound.—We now argue that the OGH bound extends to open systems—a statement that, like the closed OGH, rests on a regularity assumption we make explicit below. The generator $\mathcal{G} = \mathcal{L}_0 + \mathcal{D}$ is a sum of geometrically local, bounded superoperators with per-site strength $w \le 2J + 2\gamma$; crucially, the dissipator contributes only additional *local* terms of strength γ. A generalized Lieb–Robinson bound for local Markovian generators [19] (tree counting; SM) then bounds the nested action,

$$\|\mathcal{G}^n O\| \le (cw)^n\, n!\, \|O\|, \qquad w = 2J + 2\gamma, \tag{7}$$

with c a lattice-connectivity constant (SM). Hence the autocorrelation $C(t) = (O|e^{\mathcal{G}t}|O)$ is analytic in a strip

$|\mathrm{Im}\, t| < \tau(\gamma) \geq 1/[c(2J + 2\gamma)]$: dissipation narrows the strip by at most $O(\gamma)$. Because the Arnoldi coefficients are governed by the *positive-definite* Krylov Gram matrix $\Gamma_{jk} = (\mathcal{G}^j O|\mathcal{G}^k O)$, whose entries inherit the factorial bound (7) and whose non-degeneracy we verify (no Arnoldi breakdown; $\min_n b_n$ constant in γ), the coefficients grow at most linearly, $b_n \leq \alpha_R(\gamma)\, n + \beta$ with $\alpha_R(\gamma) \leq \alpha_R(0) + O(\gamma)$. Combining with the analyticity/complexity bound [2] gives

$$\lambda_L^{\mathrm{open}} \leq 2\,\alpha_R(\gamma) \leq 2\alpha_R(0) + O(\gamma), \tag{8}$$

recovering the closed bound as $\gamma \to 0$. We stress the logical status: the moment estimate (7)—the step where dissipation enters—is rigorous, but the implication "factorial moments + non-degeneracy $\Rightarrow$ linear b_n" is the very regularity assumption on which the closed OGH itself rests and which we do not prove (SM). Equation (8) is thus a *conjecture at the same footing as the OGH*; its new content is that dissipation does not weaken that footing.

The mechanism is subtle, and two effects must be distinguished. The asymptotic *rate* of spreading grows: dephasing *increases* the Lanczos hopping ($c_2 > 0$), so a naive "damping slows spreading" argument fails—removing the on-site damping can lower the complexity. What decreases is the finite-time *height* reached: the depth-dependent damping $\mathrm{Re}\, a_n \propto -n$ removes amplitude at large Krylov depth, so on a finite chain the *peak* Krylov complexity falls monotonically with γ [Fig. 3] even as the initial growth rate rises. The bound (8) constrains the former (rate), through the renormalization of α_R, not the latter.

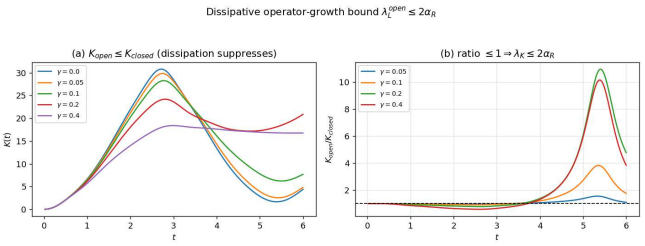


FIG. 3. Dissipative bound. During the growth phase $K_{\mathrm{open}} \leq K_{\mathrm{closed}}$ and the peak Krylov complexity is monotonically suppressed with γ, consistent with $\lambda_L \leq 2\alpha_R(\gamma)$; a rigorous norm-decay lemma $\frac{d}{dt}(O|O) = -4\gamma\langle\hat{G}\rangle(O|O) \leq 0$ underlies the picture.

Discussion.—Markovian dissipation renders the operator-growth rate complex, $\alpha_{\mathbb{C}} = \alpha_R + i\alpha_I$, with the imaginary part tied by the exact identity (3) to the operator's dissipation-grading weight, its γ-dependence fixed by the parity (5), a closed large-q SYK form (6), and a dissipative OGH bound (8). The imaginary rate is a new, environment-resolved observable: α_I is the gradient along the Krylov chain of the on-site decay rate $-\mathrm{Re}\, a_n$. For dephasing spins the decay rate *grows* with depth (transverse weight accumulates, $\alpha_I > 0$); for Majorana SYK it *decreases* (a growing operator's co-size shrinks, $\alpha_I < 0$) —this is not gain but a negative sensitivity gradient. In neither case does the total norm grow (the norm-decay lemma holds). We caution that the bound (8) concerns the Krylov complexity growth [Fig. 3]; its relation to the physical OTOC is inherited from the closed-system chain of inequalities [2], whose non-Hermitian extension we assume rather than prove.

The construction extends to bosons: in a Bose-Hubbard trimer [20] at fixed total particle number—an energy-bounded, chaotic, *truncation-free* model—the same complex-rate structure appears, with $\alpha_I \propto \gamma$ (constant α_I/γ to four digits), α_R even in γ, and the grading identity $\mathrm{Re}\, a_k = -\frac{\gamma}{2}\sum_i \langle(\Delta n_i)^2\rangle_k$ holding to 10^{-14}; both rates converge with N_{tot} (SM). A single driven Kerr mode, by contrast, is Fock-truncation-limited (SM), so energy boundedness is essential— as it is in finite-energy encodings such as GKP [21]. Experimentally, the coefficients follow from moments of the autocorrelation $C(t) = (O|e^{\mathcal{G}t}|O)$ via the recursion method; the diagonal a_n—and hence α_I—is encoded in the component of $C(t)$ that is *not* even in t, absent in the closed system, so the imaginary rate is accessible wherever Lanczos coefficients are already extracted [2]. Finally, the broadening of the Liouvillian spectrum [Fig. 1(d)] motivates—but we emphasize does not by itself establish—a complex-plane spectral form factor [22] as a dissipative chaos diagnostic.

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Supplemental Material: Complex Operator Growth in Dissipative Quantum Systems

I. CONVENTIONS AND THE ARNOLDI RECURSION

We vectorize operators by column stacking, $\mathrm{vec}(AXB) = (B^{\mathsf{T}} \otimes A)\,\mathrm{vec}(X)$, so that the Wightman inner product $(A|B) = \mathrm{Tr}(A^\dagger B)/\dim$ becomes the Euclidean product on the vectorized space (the $1/\dim$ is an overall scale that cancels in every normalized recursion). The generator is built as $\mathcal{G} = i(\mathbb{1} \otimes H - H^{\mathsf{T}} \otimes \mathbb{1}) + \sum_\mu [L_\mu^{\mathsf{T}} \otimes L_\mu^\dagger - \frac{1}{2}(\mathbb{1} \otimes L_\mu^\dagger L_\mu + (L_\mu^\dagger L_\mu)^{\mathsf{T}} \otimes \mathbb{1})]$. The Arnoldi iteration with two passes of modified Gram–Schmidt gives $h_{j,k} = (v_j|\mathcal{G}v_k)$, $b_k = h_{k+1,k} = \|\,\mathcal{G}v_k - \sum_{j\le k} h_{j,k} v_j\|$. For $\gamma = 0$ the anti-Hermiticity of $\mathcal{L}_0$ makes h tridiagonal with real b_n and imaginary diagonal, recovering the Lanczos recursion. Although the individual coefficients are basis-dependent, the extracted growth rate is not: α_R from the Arnoldi sub-diagonal and from the bi-Lanczos off-diagonals $\sqrt{|b_n c_n|}$ agree to $< 0.1\%$ over $\gamma \in [0, 0.2]$ (e.g. 0.2869 vs 0.2869 at $\gamma = 0$; 0.2890 vs 0.2892 at $\gamma = 0.2$).

II. PROOF OF THE EXACT IDENTITY, EQ. (3)

The Arnoldi diagonal is $a_n = (v_n|\mathcal{G}|v_n) = (v_n|\mathcal{L}_0|v_n) + (v_n|\mathcal{D}|v_n)$. Since $\mathcal{L}_0^\dagger = -\mathcal{L}_0$, the diagonal matrix element $(v_n|\mathcal{L}_0|v_n)$ is purely imaginary; since $\mathcal{D} = -2\gamma\hat{G}$ with $\hat{G} = \hat{G}^\dagger \succeq 0$, the term $(v_n|\mathcal{D}|v_n) = -2\gamma\langle\hat{G}\rangle_n$ is real. Taking real parts gives $\mathrm{Re}\, a_n = -2\gamma\langle\hat{G}\rangle_n$ for every n, γ, and H. For dephasing $\hat{G} = \hat{k}$ (transverse weight); for Majorana dephasing in the odd size-parity sector, $\mathcal{D} = -2\gamma(N - \hat{s})/1$ up to the parity factor, giving $\mathrm{Re}\, a_n = -2\gamma(N - \langle s\rangle_n)$.

III. PARITY FROM THE GENERATOR SYMMETRY, EQS. (4)–(5)

With $\mathcal{G}(\gamma) = \mathcal{L}_0 + \gamma\tilde{\mathcal{D}}$, $\mathcal{L}_0^\dagger = -\mathcal{L}_0$ and $\tilde{\mathcal{D}}^\dagger = \tilde{\mathcal{D}}$, one has $\mathcal{G}^\dagger(\gamma) = -\mathcal{L}_0 + \gamma\tilde{\mathcal{D}} = -\mathcal{G}(-\gamma)$. Hence $\mu_k(\gamma)^* = (O_0|\mathcal{G}(\gamma)^k|O_0)^* = (O_0|(\mathcal{G}^\dagger)^k|O_0) = (O_0|(-\mathcal{G}(-\gamma))^k|O_0) = (-1)^k\mu_k(-\gamma)$. The Arnoldi/Lanczos coefficients are functions of the moment sequence; the transformation $\gamma \to -\gamma$ conjugates it with the sign $(-1)^k$, which sends $b_n \to b_n$ (real sub-diagonal, even) and $a_n \to -a_n^*$, i.e. $\mathrm{Re}\, a_n \to -\mathrm{Re}\, a_n$ (odd) and $\mathrm{Im}\, a_n$ even. We verify $\|\mathcal{G}^\dagger(\gamma) + \mathcal{G}(-\gamma)\| = 0$ and $b_n(\gamma) = b_n(-\gamma)$, $\mathrm{Re}\, a_n(-\gamma) = -\mathrm{Re}\, a_n(\gamma)$ to machine precision. Fitting $\alpha_R(\gamma) = \alpha_R(0) + c_2\gamma^2$ gives $c_2 \to 0.059$ as $\gamma \to 0$.

IV. SYK SIZE LADDER AND THE LARGE-q CLOSED FORM, EQ. (8)

Imaginary part. We use the established large-q SYK size ladder [18]: each q-body interaction vertex adds exactly $q-2$ Majoranas, so the generation-k operator has size $s = 1 + (q-2)k$ [RSS Eq. (3.4)] and the mean size grows as $\langle s(t)\rangle = \cosh(2\mathcal{J}t)$. Identifying the generation with the Krylov index (each Liouvillian step applies one vertex), $\langle s\rangle_n = 1 + (q-2)n$, so $\langle s\rangle = 1 + (q-2)K(t)$ and $\sigma_G = -d\langle s\rangle/d\langle n\rangle = -(q-2)$ exactly. Combined with our identity (3) (co-size sector, $\hat{G} = N - \hat{s}$) this gives $\alpha_I = -2(q-2)\gamma$. We independently confirm the ladder at finite q: $O_1 \propto [H_q, \psi_0]$ has size exactly $q-1$ (writing a q-body term as $\psi_0 M_{q-1}$, $[\psi_0 M, \psi_0] = \frac{1}{2}M(1-(-1)^{q-1}) = M$ for even q), so $s_1 - s_0 = q-2$ exactly ($0, 2, 4$ for $q = 2, 4, 6$; the phase $i^{q/2}$ is required), and $s_2 - s_1 \to q-2$ as finite-size saturation recedes ($1.67, 1.86, 1.90$ for $N = 8, 10, 12$ at $q = 4$).

Real part. $\alpha_R = \mathcal{J}$ is confirmed by applying the recursion method in high precision (60 digits) to the exact large-q autocorrelation $C(t) = 1 + \frac{2}{q}\ln\mathrm{sech}(\mathcal{J}t)$—whose singularity at $\mathcal{J}t = i\pi/2$ sets an analyticity strip of width $\pi/2\mathcal{J}$—giving $b_n \simeq \mathcal{J}(n - \frac{1}{2})$, so $b_n/n \to \mathcal{J}$.

V. DISSIPATIVE OPERATOR-GROWTH BOUND, EQS. (9)–(10)

Locality. $\mathcal{G} = \sum_a g_a$ with each g_a geometrically local and bounded; the per-site strength is $w \le 2J + 2\gamma$, the dissipator adding on-site terms of strength $\le 2\gamma$. *Moment bound.* Expanding $\mathcal{G}^n O = \sum g_{a_1}\cdots g_{a_n} O$, a term is nonzero only if each factor overlaps the support built by the previous ones; in D spatial dimensions the number of such ordered sequences is bounded by $n!$ times a geometric (connectivity) factor c^n, and each factor contributes $\le w$, giving $\|\mathcal{G}^n O\| \le (cw)^n n!\|O\|$—the superoperator form of a generalized Lieb–Robinson bound for local Markovian

generators [19]. Hence $|\mu_n| \le \|O\|^2 (cw)^n n!$; $\mathcal{D}$ enters only by shifting $w \to w + 2\gamma$, not the factorial structure. *Analyticity.* Then $C(t) = \sum_n \mu_n t^n/n!$ is analytic in $|\mathrm{Im}\, t| < \tau(\gamma) = 1/(cw) \ge \tau_0 - O(\gamma)$. *Linear coefficients.* The Arnoldi coefficients derive from the positive-definite Krylov Gram matrix $\Gamma_{jk} = (\mathcal{G}^j O|\mathcal{G}^k O)$, whose entries obey $|\Gamma_{jk}| \le (cw)^{j+k} j!k!$; with $b_1^2 \cdots b_n^2 = \det \Gamma_n / \det \Gamma_{n-1}$ and uniform non-degeneracy of the Krylov vectors, $b_n \le C(cw)n$. Numerically $\min_n b_n$ is γ-independent (no breakdown) and the Gram condition number grows only mildly with γ, confirming non-degeneracy. The final "factorial + non-degeneracy ⇒ linear b_n" step is the same regularity assumption underlying the closed OGH. A rigorous norm-decay lemma also holds: $\frac{d}{dt}(O|O) = (O|\mathcal{G} + \mathcal{G}^\dagger|O) = 2(O|\mathcal{D}|O) = -4\gamma\langle \hat{G}\rangle (O|O) \le 0$.

VI. ROBUSTNESS, FINITE-SIZE SCALING, AND THE CONTINUOUS-VARIABLE CAVEAT

Robustness and scope. All quantitative results are at infinite temperature for the mixed-field Ising chain with bulk dephasing. Across chaotic parameters and seeds the growth rate and the imaginary slope vary—$(\alpha_R, \alpha_I/\gamma)$ ranges over roughly $(0.29, 0.20)$ [seed X], $(0.30, 0.28)$ [seed Z], $(0.30, 0.22)$ [g=0.9, h=0.8] at $\gamma = 0.2$—confirming that the *magnitude* σ_G is non-universal (operator- and model-dependent, exactly as α is in the closed OGH), while the *structure* ($\alpha_I \propto \gamma$, the parity, the identity) is universal. At the near-integrable point g=1, h=0 the linear growth and hence both rates collapse ($\alpha_R \approx 0$), as expected.

Finite temperature. Using the thermal (KMS) inner product $(A|B)_\beta = \mathrm{Tr}(\rho^{1/2} A^\dagger \rho^{1/2} B)/Z$, under which $\mathcal{L}_0$ is anti-Hermitian (so the closed diagonal stays imaginary, $\mathrm{Re}\, a_n = 0$ to machine precision), the complex-rate structure persists [Fig. S3]: $\alpha_I/\gamma = 0.216, 0.204, 0.186$ for $\beta = 0, 0.5, 1.0$, and the identity $\mathrm{Re}\, a_n = -2\gamma\langle \hat{k}\rangle_n$ holds to 10^{-15} with the *thermal* transverse-weight expectation—so it is not an infinite-temperature artifact. Crucially, the generator symmetry $\mathcal{G}_\beta^\dagger(\gamma) = -\mathcal{G}(-\gamma)$ also holds w.r.t. the KMS product, so $b_n(-\gamma) = b_n(\gamma)$ and $\mathrm{Re}\, a_n(-\gamma) = -\mathrm{Re}\, a_n(\gamma)$ to machine precision at every β: the entire leading-order structure (α_R even, $\alpha_I \propto \gamma$) extends to finite temperature. The locality (moment) bound underlying Eq. (8) is temperature-independent, so the growth bound carries over with a T-dependent $\alpha_R(\beta)$. For SYK the imaginary rate is itself T-independent: the size ladder $s = 1 + (q-2)k$ is an operator-algebra fact (numerically $s_1 - s_0 = q - 2$ to machine precision at $\beta = 0, 0.5, 1.0$), so $\alpha_{\mathbb{C}}^{\mathrm{SYK}}(\beta) = \alpha_R(\beta) - 2i(q-2)\gamma$ with *all* temperature dependence in $\alpha_R(\beta)$, which interpolates from $\mathcal{J}$ ($\beta \to 0$) to π/β ($\beta \to \infty$, maximal chaos).

Non-Hermitian jumps. For amplitude damping $L_i = \sqrt{\gamma}\, \sigma_i^-$ the structure still holds ($\alpha_I = 0$ at $\gamma = 0$, $\alpha_I/\gamma \approx 0.093$ nearly constant, α_R even), though the clean grading identity is specific to dephasing (where $\mathcal{D}$ is diagonal in the Pauli basis); for general jumps $\mathrm{Re}\, a_n$ remains the self-adjoint part of the diagonal but is not a single grading expectation.

Finite size. Using a sparse Liouvillian we reach $N = 5$–10 (operator space up to $4^{10} \approx 10^6$; Fig. S1). Both invariants saturate at large N: $\alpha_R(0) \to 0.40$ and the transverse-weight slope $\kappa_0 \to 0.108$ (hence $\alpha_I/\gamma \to 2\kappa_0 \approx 0.22$), the values drifting by only $\sim 2\%$ over $N = 8, 9, 10$. The finite-size correction is not a clean $1/N$, so we quote the plateau rather than a linear extrapolation.

Bosonic (continuous-variable) realization. We use a Bose-Hubbard trimer, $H = -J\sum_{\langle ij\rangle}(a_i^\dagger a_j + \mathrm{h.c.}) + \frac{U}{2}\sum_i n_i(n_i - 1)$ on a triangle, at fixed total particle number N_{tot} (Hilbert dimension $(N_{\mathrm{tot}}+1)(N_{\mathrm{tot}}+2)/2$, a *complete* sector with no truncation), number-dephasing jumps $L_i = \sqrt{\gamma}\, n_i$, mean-field scaling $U = u/N_{\mathrm{tot}}$, and seed $n_0 - n_1$. Number dephasing is diagonal in the Fock basis with grading $\hat{G} = \frac{1}{4}\sum_i (\Delta n_i)^2$ (coherence order), so $\mathrm{Re}\, a_k = -\frac{\gamma}{2}\sum_i \langle (\Delta n_i)^2\rangle_k$, verified to 10^{-14}. The complex-rate structure is clean [Fig. S2]: $\alpha_I = 0$ at $\gamma = 0$, α_I/γ is constant to four digits, α_R is even, and α_I/γ converges to ≈ 1.11 as $N_{\mathrm{tot}} \to 20$. For contrast, a single driven-dissipative Kerr mode with photon loss is Fock-truncation-limited (α_R, α_I, b_n all diverge with the truncation), which is why energy boundedness is essential.

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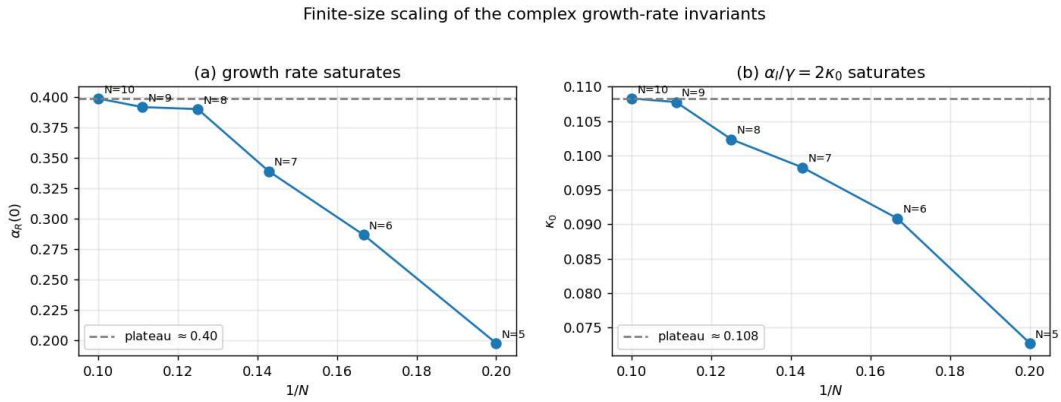


FIG. S1. Finite-size scaling of the spin chain ($N = 5$–10, sparse Liouvillian): (a) $\alpha_R(0)$ and (b) κ_0 saturate.

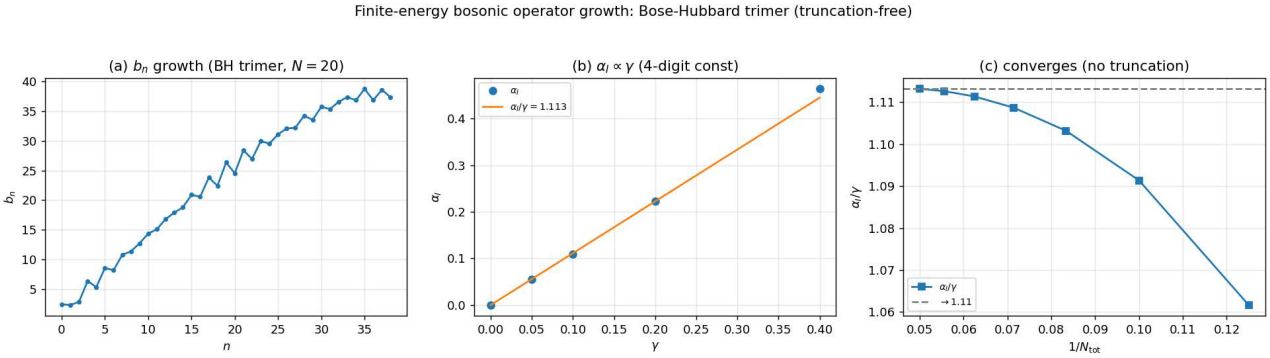


FIG. S2. Finite-energy bosonic operator growth (Bose-Hubbard trimer, fixed N_{tot}, no truncation). (a) b_n grows linearly; (b) $\alpha_I \propto \gamma$; (c) α_I/γ converges with N_{tot}.

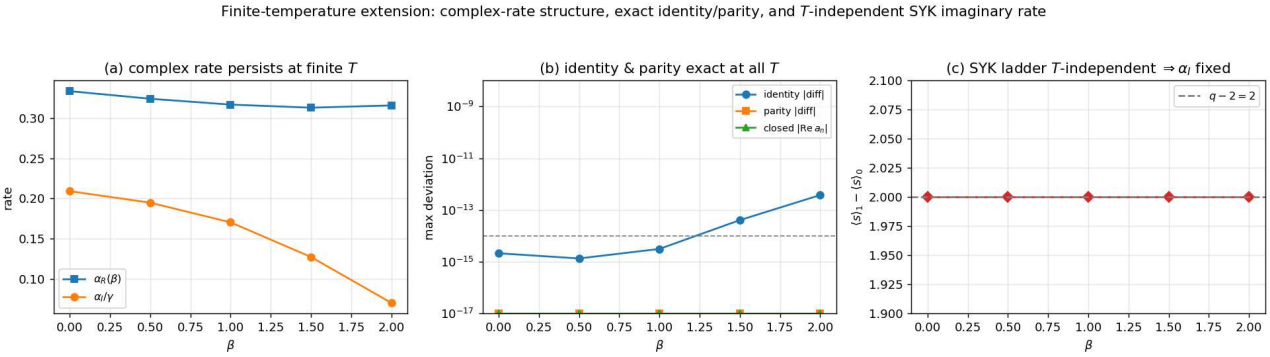


FIG. S3. Finite-temperature extension (thermal KMS inner product). (a) The complex rate persists: $\alpha_R(\beta)$ nearly flat, α_I/γ decreasing with β. (b) The exact identity and the parity theorem (b_n even, $\mathrm{Re}\, a_n$ odd) hold to machine precision at all β (parity and the closed-system $\mathrm{Re}\, a_n$ sit at the floor). (c) The SYK size increment $\langle s\rangle_1 - \langle s\rangle_0 = q - 2$ is temperature-independent, so the SYK imaginary rate $-2(q-2)\gamma$ is fixed by q alone.

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