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Coherence-to-Entanglement Conversion under Faulty CNOT Gates and Matched Idle Noise

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A controlled-NOT gate acting on a coherent control qubit converts single-qubit coherence into two-qubit entanglement, ideally only if the gate is perfect and the entangled state is used at once. We analyze how this conversion degrades when the gate carries a fault and the output then idles in a finite-temperature environment of relaxation, thermal excitation, and dephasing. Four representative fault classes are treated: coherent under- or over-rotation, a coherent control-phase error, a stochastic phase flip, and a stochastic bit flip. The gate faults and the idle environment are matched in a limited but explicit sense, referred to a common set of device parameters; a commutation argument identifies when an intra-gate noise process reduces exactly to one of the four classes, with the residual error quantified in the borderline cases. Within this deliberately minimal model we obtain closed-form expressions for the conversion efficiency of each fault class, for the entanglement surviving the idle period, and for the boundary at which entanglement disappears abruptly, recovering the known amplitude-damping threshold for sudden death as a special case. The classes leave distinct signatures: coherent calibration errors are insensitive to leading order and therefore mild, stochastic faults degrade the conversion in proportion to their probability, and the bit-flip class alone carries an angle-dependent signature separating it from a phase flip. Control-phase errors leave the entanglement measure unchanged and are invisible to this benchmark, a limitation we state plainly rather than present as a strength. All analytic results are verified against independent numerical simulation, and a diagnostic inferring the dominant fault class from angle-resolved measurements identifies strong stochastic faults reliably but cannot separate weak coherent errors from an ideal gate. The results are intended as transparent analytic benchmarks for a specific resource-conversion task, not as general device characterization.

I. INTRODUCTION

Quantum coherence and entanglement are the two elementary resources of quantum information processing, and the conversion of one into the other is among the simplest nontrivial operations a quantum processor performs [1–5]. In the minimal conversion protocol a single-qubit superposition

$$|\psi_A\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle \quad (1)$$

is applied to the control of a CNOT gate whose target is prepared in $|0\rangle$, producing $\cos\theta|00\rangle + e^{i\phi}\sin\theta|11\rangle$. The output negativity [6, 7]

$$N_0(\theta) = \sin(2\theta)/2 = C_{\ell_1}(\rho_A)/2 \quad (2)$$

equals exactly half the ℓ_1 coherence of the input [1], so the ideal protocol converts single-qubit coherence into two-qubit entanglement with unit efficiency, saturating the general bound on entanglement generated from coherence by incoherent operations [4]. This makes the protocol a natural *benchmark*: any deviation of the measured negativity from Eq. (2) quantifies imperfection of the entangling operation and of the subsequent storage.

Two idealizations, however, separate Eq. (2) from laboratory practice. First, the CNOT itself is faulty: control-amplitude miscalibration produces coherent under- or over-rotation of the conditional flip; residual ZZ coupling or ac-Stark shifts produce coherent controlled-phase errors; dephasing during the gate motivates stochastic

phase-flip (Z) faults; and driven, effectively bit-flip-like processes on the target (e.g., resonant heating under strong driving) motivate the stochastic bit-flip (X) class. Intra-gate amplitude damping itself, by contrast, reduces to *no* single Pauli class—a point quantified rather than glossed in Sec. IIIB—so the X class should be read as the bit-flip component of intra-gate noise, not as a stand-in for relaxation. Second, the entangled output is not consumed instantly but idles in the same physical environment—finite-temperature relaxation, thermal excitation, and dephasing—that also acted during the gate, and this post-gate evolution can destroy entanglement abruptly at finite time (entanglement sudden death, ESD) [8–12]. This protocol is not a competitor to randomized benchmarking [13, 14] or to process- and leakage-resolved characterization [15–18], which target average gate fidelity and full error structure, respectively. It probes a narrower question that those methods do not directly address—how specific fault classes degrade a specific resource-conversion task—and is intended as a complement whose value lies in complete analytic transparency rather than generality.

This paper develops a resource-resolved approach: a *matched* gate/post-gate noise model in which the idle channel is parametrized by measured idle-state T_1 , T_2 , p^{eq} , the gate-fault strengths are referred to the same rate scale (with the caveat, made precise in Sec. IIIB, that dissipation rates under strong driving need not equal their idle values), each fault class is paired with the post-gate mechanism of the same physical origin where the reduc-

tion criterion permits, and the single observable is the angle-resolved final negativity $N^f(\theta)$. How much microscopic content this pairing carries is analyzed rather than assumed: Sec. III B gives a commutation criterion under which the pairing is exact, and quantifies the error where it is not. All model assumptions are collected in Sec. III A. The framework is deliberately minimal—two qubits, one gate, one idle window—so that everything can be computed exactly, yet it is rich enough to assign each fault class a distinct closed-form signature. Our contributions are:

(i) *Exact gate-level benchmarks.* For all four fault classes we derive the conversion efficiency $\chi = N^{\text{gate}}/N_0$ in closed form [Table I], including the full two-block negativity of the stochastic- X class, Eq. (26), whose Φ - and Ψ -sector contributions exchange dominance at $q_X = 1/2$.

(ii) *Closed-form post-idle benchmarks.* We derive the exact final negativity of any X -state gate output under the matched thermal idle channel, Eq. (29). The two-block partial-transpose structure of X states used here is standard [9, 19, 20]; the contribution is its combination with the matched channel and the fault classes into closed-form benchmarks, namely the survival condition $\Lambda_{AB}^2 |\mathbf{c}_g|^2 > P_{01}^f P_{10}^f$ and the sudden-death boundaries [Table II]: the Z -fault boundary $q_Z^c(\theta)$ is given explicitly by Eq. (32), the X -fault boundary is the root of an explicit quadratic, Eq. (33), and the zero-temperature amplitude-damping limit recovers the Yu–Eberly threshold $\tan^2 \theta d_A d_B < 1$ [8].

(iii) *Sensitivity fingerprints and optimal angles.* Coherent calibration errors are quadratically harmless while stochastic faults are linearly harmful, and the bit-flip class carries the unique angle-dependent slope $\csc 2\theta$, Eq. (28). The noise-adapted optimal input angle is obtained in closed form, Eq. (37); its leading term $\cos 2\theta_{\text{opt}} \simeq -\Sigma_1/(\chi \Lambda_{AB})$ shows that the optimum shifts toward the ground state, by an amount inversely proportional to the surviving coherence, and reproduces direct numerical optimization to within 10^{-5} in θ_{opt}/π across the X -state classes [Table III].

(iv) *Algebraic consistency validation and a diagnostic protocol.* Every closed form is checked against independent numerical propagation of the same model, with maximal deviations at double-precision round-off, $\lesssim 10^{-15}$ [Table VI]—a validation of the algebra, not of the model’s fidelity to any device, which is addressed separately. A least-squares model-selection protocol on $N^f(\theta)$ data is analyzed for identifiability—which fault classes can and cannot be distinguished, and at what order—and demonstrated on synthetic data (Fig. 6).

(v) *Microscopic reduction and robustness.* A commutation criterion (Sec. III B) determines which intra-gate dissipation mechanisms reduce exactly to the four fault classes; two do (target bit flip, control dephasing), one reduces at first order with a quantified residual (target dephasing), and one does not reduce at all (intra-gate relaxation). The diagnostic is stress-tested under model mismatch—fault mixtures, SPAM-like contrast

loss, idle-parameter miscalibration, and unmodeled intra-gate relaxation—with both its graceful and its failing behaviors reported [Table IV]. A concrete five-setting Pauli scheme for estimating the negativity, with its caveats, is given in Sec. VIII C.

Section II fixes conventions and the ideal benchmark. Section III defines the matched noise model and establishes its physicality. Section IV derives the gate-level benchmarks, Sec. V the post-gate master formula, Sec. VI the sudden-death geometry, Sec. VII the optimal input angles, and Sec. VIII the diagnostic protocol, its robustness under model mismatch, and the measurement cost of the benchmark. Section IX details the numerical methods and validation. Technical derivations are collected in Appendixes A–D.

II. PROTOCOL, CONVENTIONS, AND THE IDEAL BENCHMARK

Qubit A is the control, qubit B the target, and the computational basis is ordered $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. We abbreviate

$$c \equiv \cos \theta, \quad s \equiv \sin \theta, \quad s_2 \equiv \sin 2\theta, \quad z \equiv \cos 2\theta, \quad (3)$$

with $\theta \in [0, \pi/2]$. The protocol input is $\rho_{\text{in}} = |\psi_A\rangle\langle\psi_A| \otimes |0\rangle\langle 0|_B$ with $|\psi_A\rangle$ of Eq. (1); its control coherence is $C_{\ell_1}(\rho_A) = 2|cs| = s_2$.

Entanglement is quantified by the negativity [6, 7, 21, 22],

$$N(\rho) = \sum_{\mu_i < 0} |\mu_i|, \quad (\rho^\Gamma)_{ab,cd} = \rho_{ad,cb}, \quad (4)$$

the sum of the negative eigenvalues of the partial transpose ρ^Γ over B . For two qubits the Peres–Horodecki criterion is necessary and sufficient [19, 20], so $N = 0$ characterizes separability exactly and ESD is the event $N \rightarrow 0$ at a finite parameter value.

Almost every state in this problem belongs to the X class,

$$\rho_X = \begin{pmatrix} P_{00} & 0 & 0 & \mathbf{c} \\ 0 & P_{01} & w & 0 \\ 0 & w^* & P_{10} & 0 \\ \mathbf{c}^* & 0 & 0 & P_{11} \end{pmatrix}, \quad (5)$$

with *outer* (Φ -sector) coherence $\mathbf{c} = \rho_{00,11}$ and *inner* (Ψ -sector) coherence $w = \rho_{01,10}$ [9]. Partial transposition exchanges the two coherences between the 2×2 blocks $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$, and diagonalizing the blocks gives the exact negativity (Appendix A)

$$\begin{aligned} N(\rho_X) = & \max \left[0, \sqrt{(P_{01} - P_{10})^2/4 + |\mathbf{c}|^2} \right. \\ & \left. - (P_{01} + P_{10})/2 \right] \\ & + \max \left[0, \sqrt{(P_{00} - P_{11})^2/4 + |w|^2} \right. \\ & \left. - (P_{00} + P_{11})/2 \right], \end{aligned} \quad (6)$$

with the block-determinant survival criteria

$$\Phi \text{ block: } |\mathfrak{c}|^2 > P_{01}P_{10}, \quad \Psi \text{ block: } |w|^2 > P_{00}P_{11}. \quad (7)$$

Equation (6) is the computational engine of the paper: the exchange of $\mathfrak{c}$ and w under partial transposition is the structural reason why odd-parity populations P_{01}, P_{10} kill Φ -type entanglement while even-parity populations kill Ψ -type entanglement, and both terms will be needed for the stochastic- X fault.

The ideal CNOT maps the input to $c|00\rangle + e^{i\phi}s|11\rangle$, an X state with $P_{00} = c^2$, $P_{11} = s^2$, $P_{01} = P_{10} = w = 0$, $\mathfrak{c} = cs e^{-i\phi}$. Equation (6) gives $N_0 = cs$, i.e., Eq. (2): the conversion law $N_0 = C_{\ell_1}/2$ is exact, and defines the benchmark against which every faulty gate is measured through

$$\chi \equiv N^{\text{gate}}/N_0. \quad (8)$$

III. MATCHED GATE AND POST-GATE NOISE MODEL

A. Scope and assumptions

The model is deliberately minimal, and all results below are results about this model. The assumptions, stated once here rather than scattered through the text, are: (i) two qubits, control A and target B , with the target prepared in $|0\rangle$; (ii) the gate is an ideal CNOT composed with exactly *one* phenomenological fault class at a time (Sec. IV); fault mixtures lie outside the fitted family and are treated as model mismatch in Sec. VIII B; (iii) idle noise is local, uncorrelated between the qubits, Markovian, and phase covariant, Eq. (13), with $T_2 \leq 2T_1$; (iv) the idle parameters T_1, T_2, p^{eq} are known (miscalibration is treated in Sec. VIII B); (v) there are no SPAM errors, leakage, crosstalk, correlated two-qubit (ZZ -type) stochastic faults, or control-qubit intra-gate faults beyond dephasing (a SPAM proxy is treated in Sec. VIII B); (vi) the negativity of the (X -form) output can be estimated; a five-setting measurement scheme and its caveats are given in Sec. VIII C. The extent to which the four fault classes capture microscopic intra-gate dissipation is itself analyzed—not assumed—in Sec. III B.

B. Matching principle and microscopic reduction

The evolution is split into a gate window t_g and an idle window t_{idle} ,

$$\rho_g^{(j)} = \mathcal{T} \exp \left[\int_0^{t_g} (\mathcal{L}_{\text{CNOT}}^{(j)} + \mathcal{L}_{\text{phys}}) dt \right] \rho_{\text{in}}, \quad (9)$$

$$\rho_f^{(j)} = \mathcal{T} \exp \left[\int_{t_g}^{t_g + t_{\text{idle}}} \mathcal{L}_{\text{phys}} dt \right] \rho_g^{(j)}, \quad (10)$$

with the modeling principle that $\mathcal{L}_{\text{phys}}$ describes the same physical environment during and after the gate, while the

intended control is active only during t_g . In the analytic treatment the gate window is compressed into an effective faulty operation drawn from four one-parameter classes. This compression is phenomenological, and we now establish exactly how much microscopic content it carries, rather than asserting it.

As a minimal analytic representative—not a model of any specific hardware pulse sequence—write the driven gate as $U(t) = e^{-iH_g t}$ with the single-pulse cross-resonance-type generator [23, 24]

$$H_g = [\pi/(2t_g)] |1\rangle\langle 1|_A \otimes (X_B - \mathbb{1}), \quad U(t_g) = U_{\text{CNOT}}, \quad (11)$$

and let a weight-one Lindblad dissipator $\gamma(L\rho L^\dagger - \rho)$, with unitary jump operator L , act throughout the gate. Hardware CNOTs are echoed, multipulse sequences; for those the same criterion must be applied to the noise operators in the toggling frame of the actual sequence, so the reductions established below are statements about this minimal generator, to be rederived per implementation. If $[L, H_g] = 0$, every stochastic fault insertion commutes through the remaining gate evolution, and the intra-gate channel reduces *exactly* to the corresponding post-gate stochastic fault with

$$q = (1 - e^{-2\gamma t_g})/2. \quad (12)$$

Checking the criterion against a Trotterized simulation of Eqs. (9) (800 Trotter steps): *Target bit flip*, $L = X_B$: $[X_B, H_g] = 0$, so intra-gate bit-flip noise *is* the stochastic- X class; simulated and reduced states agree to 3×10^{-14} for $\gamma t_g \leq 0.2$. *Control dephasing*, $L = Z_A$: $[Z_A, H_g] = 0$, and on the benchmark output family $c|00\rangle + e^{i\phi}s|11\rangle$ the actions of Z_A and Z_B coincide, so intra-gate control dephasing *is* the stochastic- Z class (agreement 9×10^{-14}). *Target dephasing*, $L = Z_B$: noncommuting; conjugation along the gate turns it into a mixture of Z_B - and $(XZ)_B$ -type faults, so the reduction holds only at first order in γt_g . A single-parameter stochastic- Z fit leaves a residual $\max_\theta |\Delta N| = 0.9, 1.5, 2.5 \times 10^{-2}$ at $\gamma t_g = 0.05, 0.1, 0.2$. *Intra-gate relaxation* (amplitude damping on B): noncommuting and irreducible; the closest single-class fit leaves $0.5\text{--}1.5 \times 10^{-2}$ over the same range—which is precisely why relaxation is assigned to the matched idle channel and kept out of the gate-fault set. Of the coherent classes, the controlled- R_x error is exact by construction (a scale error $H_g \rightarrow \lambda H_g$ produces $\tilde{U}_x(\lambda)$ identically), while the controlled-phase class is exact whenever the phase error accrues in gate segments that commute with the entangling generator (as in echoed implementations) and is otherwise first order.

The “matched” label therefore means two specific things: (a) *parameter-level* matching—the idle channel is parametrized by the independently measured idle-state T_1, T_2, p^{eq} , Eq. (14), and the gate-fault strengths are referred to the same rate scale via Eq. (12), with the caveat that dissipation rates under strong driving need not equal their idle values, so this identification is a calibration convention to be checked per device rather than

a physical identity; and (b) *microscopic* matching where the commutation criterion holds—exactly for two of the four classes, to first order with the quantified residuals above for the others. Readers should apply the closed forms below with this distinction in mind.

C. The physical idle channel

Each qubit $k \in \{A, B\}$ couples to an independent reservoir producing relaxation, thermal excitation, and dephasing. Solving the standard phase-covariant Lindblad equation [25, 26] with rates $\gamma_\downarrow, \gamma_\uparrow, \gamma_\varphi$ for a time t gives the single-qubit channel

$$\begin{aligned} \mathcal{E}_k : \quad \rho_{00} &\mapsto (1 - u_k)\rho_{00} + d_k\rho_{11}, \\ \rho_{11} &\mapsto u_k\rho_{00} + (1 - d_k)\rho_{11}, \quad \rho_{01} \mapsto k_k \rho_{01}, \end{aligned} \quad (13)$$

with the thermal parametrization

$$\begin{aligned} u_k &= (1 - e^{-t/T_{1,k}})p_k^{\text{eq}}, \quad d_k = (1 - e^{-t/T_{1,k}})(1 - p_k^{\text{eq}}), \\ k_k &= e^{-t/T_{2,k}}, \end{aligned} \quad (14)$$

where $1/T_1 = \gamma_\downarrow + \gamma_\uparrow$, $p^{\text{eq}} = \gamma_\uparrow/(\gamma_\downarrow + \gamma_\uparrow)$, and $1/T_2 = 1/(2T_1) + \gamma_\varphi$. A useful identity is

$$u_k + d_k = 1 - r_k, \quad r_k \equiv e^{-t/T_{1,k}}, \quad (15)$$

so r_k plays the role of a population “memory” factor. The map (13) is completely positive if and only if $|k_k|^2 \leq (1 - u_k)(1 - d_k)$; with Eq. (14) this holds at *all* idle times precisely when $T_2 \leq 2T_1$, i.e., the familiar laboratory bound is exactly the physicality condition of the model (Appendix B, where an explicit Kraus decomposition as generalized amplitude damping followed by pure dephasing is also given).

D. Two-qubit action: sector decomposition

Because $\mathcal{E}_k$ maps diagonal single-qubit elements to diagonal ones and off-diagonal to off-diagonal, the sixteen two-qubit matrix elements of $\mathcal{E}_A \otimes \mathcal{E}_B$ decompose into four invariant sectors: (1) *populations*, which evolve by the product of column-stochastic transfer matrices,

$$P_{ab}^f = \sum_{a'b'} [M_A]_{aa'} [M_B]_{bb'} P_{a'b'}^g, \quad M_k = \begin{pmatrix} 1 - u_k & d_k \\ u_k & 1 - d_k \end{pmatrix}, \quad (16)$$

(2) the *X sector* $\{\rho_{00,11}, \rho_{01,10}\}$, whose elements are simply multiplied by

$$\Lambda_{AB} \equiv k_A k_B = e^{-t_{\text{idle}}(1/T_{2,A} + 1/T_{2,B})}; \quad (17)$$

(3) the *A-coherence sector* $\{\rho_{00,10}, \rho_{01,11}\}$, which mixes internally as $k_A M_B$; and (4) the *B-coherence sector* $\{\rho_{00,01}, \rho_{10,11}\}$, which mixes as $k_B M_A$. In particular

the X form (5) is preserved, with $\mathbf{c}_f = \Lambda_{AB} \mathbf{c}_g$ and $w_f = \Lambda_{AB} w_g$, so Eq. (6) applies directly to the final state. For gate outputs supported on $\{|00\rangle, |11\rangle\}$ with weights (c^2, s^2) , Eq. (16) evaluates to

$$\begin{aligned} P_{01}^f &= (1 - u_A)u_B c^2 + d_A(1 - d_B) s^2, \\ P_{10}^f &= u_A(1 - u_B) c^2 + (1 - d_A)d_B s^2, \end{aligned} \quad (18)$$

which reads physically: the odd-parity populations that kill Φ -type entanglement are fed by *excitation out of the* $|00\rangle$ *branch* ($u_k c^2$ terms) and *relaxation out of the* $|11\rangle$ *branch* ($d_k s^2$ terms). This single observation will explain both the sudden-death geometry (Sec. VI) and the direction of the optimal-angle shift (Sec. VII).

IV. GATE-LEVEL BENCHMARKS: EXACT CONVERSION EFFICIENCIES

We now derive N^{gate} and χ for the four fault classes. The closed forms are compiled in Table I and plotted in Fig. 1.

A. Coherent controlled- R_x calibration error

Amplitude miscalibration of the conditional drive implements

$$\tilde{U}_x(\lambda) = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes R_x(\lambda\pi), \quad (19)$$

$R_x(\alpha) = \cos(\alpha/2)I - i\sin(\alpha/2)X$, with $\lambda = 1$ the ideal point [there $\tilde{U}_x$ equals the CNOT up to a conditional phase, i.e., a local unitary on A , under which the negativity is invariant]. Writing $C_\lambda = \cos(\lambda\pi/2)$, $S_\lambda = \sin(\lambda\pi/2)$, the gate output is the pure state

$$|\Psi_\lambda\rangle = c|00\rangle + e^{i\phi} s C_\lambda |10\rangle - i e^{i\phi} s S_\lambda |11\rangle. \quad (20)$$

For any two-qubit pure state the negativity equals the product of the two Schmidt coefficients, $N = \sigma_1 \sigma_2 = \sqrt{\det \rho_A}$ [7, 27, 28]. Here $|\Psi_\lambda\rangle = c|0\rangle|0\rangle + e^{i\phi} s|1\rangle|v\rangle$ with target overlap $\langle 0|v\rangle = C_\lambda$, so $\det \rho_A = c^2 s^2 (1 - C_\lambda^2) = c^2 s^2 S_\lambda^2$ and

$$N_{R_x}^{\text{gate}} = cs |S_\lambda| \implies \chi_{R_x}(\lambda) = |\sin(\lambda\pi/2)|, \quad (21)$$

independent of θ : gate error and input coherence decouple exactly. Near the ideal point, $\lambda = 1 + \epsilon$,

$$\chi_{R_x} = |\cos(\pi\epsilon/2)| = 1 - (\pi^2/8)\epsilon^2 + O(\epsilon^4); \quad (22)$$

a coherent calibration error detunes a smooth maximum and is therefore only *quadratically* harmful, in the precise sense that the first-order sensitivity vanishes, $d\chi_{R_x}/d\epsilon|_{\epsilon=0} = 0$, with leading curvature $-\pi^2/4$; we use “quadratically harmless” below as shorthand for this vanishing first-order sensitivity.

B. Coherent controlled-phase error

Residual ZZ coupling or Stark shifts implement $\tilde{U}_\delta = CZ_\delta U_{\text{CNOT}}$ with $CZ_\delta = \text{diag}(1, 1, 1, e^{i\delta})$, producing $c|00\rangle + e^{i(\phi+\delta)}s|11\rangle$. Populations and coherence *magnitude* coincide with the ideal output; only $\arg \mathbf{c}_g$ shifts. Since the idle channel multiplies $\mathbf{c}_g$ by the positive scalar Λ_{AB} and evolves populations independently of that phase, Eq. (6) is blind to δ at every idle time:

$$\chi_\delta = 1 \quad \text{and} \quad N_\delta^f(\theta; t_{\text{idle}}) = N_{\text{ideal}}^f(\theta; t_{\text{idle}}) \quad \forall t_{\text{idle}}. \quad (23)$$

Controlled-phase miscalibration is thus exactly passive for this negativity-only benchmark—not merely at the gate level but through the entire matched evolution; it is emphatically not passive physically, remaining fully visible to computation, Bell-basis interference, and logical phase fidelity. This result should be read in two ways at once. As a consistency check it is unsurprising: Eq. (6) depends only on $|\mathbf{c}|$, so any phase-covariant benchmark must be blind to δ . As a statement about the protocol it is a *structural limitation*: an entire coherent error class is invisible to any diagnostic built on this scalar, phase-insensitive entanglement measure. The one operational consequence is negative but useful: a measured deficit of N^f certifies a non-phase fault, while agreement with the ideal curve certifies nothing about δ (Sec. VIII).

C. Stochastic Z fault

Dephasing during the gate is modeled by $\tilde{C}_Z(\rho) = (1 - q_Z)U\rho U^\dagger + q_Z Z_B U\rho U^\dagger Z_B$ with $U = U_{\text{CNOT}}$. Conjugation by Z_B fixes all populations and flips the sign of the Φ -sector coherence, $|00\rangle\langle 11| \mapsto -|00\rangle\langle 11|$, so the gate output is the ideal X state with $\mathbf{c}_g = (1 - 2q_Z)cs e^{-i\phi}$. Equation (6) gives

$$N_Z^{\text{gate}} = |1 - 2q_Z|cs \implies \chi_Z = |1 - 2q_Z|, \quad (24)$$

linearly harmful with universal slope -2 , independent of θ ; the kink at $q_Z = 1/2$ (complete Φ -sector dephasing) and the revival beyond it are exact. The revival must be read correctly: $q_Z = 1$ is not maximal noise but a *deterministic* Z_B frame error—a unitarily correctable relabeling—and $q_Z = 1/2$ is the maximally stochastic point. The revival is the approach to an almost-deterministic wrong Pauli frame, not an entanglement recovery caused by stronger noise—indeed the $q_Z = 1$ state is local-unitarily equivalent to the ideal output; the same reading applies to the Ψ -block revival of the X class below.

D. Stochastic X fault

Bit-flip-type faults on the target, $\tilde{C}_X(\rho) = (1 - q)U\rho U^\dagger + q X_B U\rho U^\dagger X_B$ (write $q \equiv q_X$), act qualitatively differently: X_B maps $|00\rangle\langle 11| \mapsto |01\rangle\langle 10|$, i.e., it *transfers outer coherence into inner coherence*, populating both blocks of Eq. (6). The gate output is the X state

$$P^g = \begin{pmatrix} (1-q)c^2 & qc^2 \\ qs^2 & (1-q)s^2 \end{pmatrix}, \quad \mathbf{c}_g = (1-q)cs e^{-i\phi}, \quad w_g = qcs e^{-i\phi}, \quad (25)$$

where P^g collects $(P_{00}, P_{01}; P_{10}, P_{11})$. Substituting into Eq. (6) with $c^2 - s^2 = z$, $2cs = s_2$ gives the exact two-block negativity

$$N_X^{\text{gate}} = \max(0, -\mu_-) + \max(0, -\nu_-), \quad \mu_- = \left[(1-q) - \sqrt{(1-q)^2 z^2 + q^2 s_2^2} \right] / 2, \quad \nu_- = \left[q - \sqrt{q^2 z^2 + (1-q)^2 s_2^2} \right] / 2. \quad (26)$$

The block criteria (7) evaluate to $(1-q)^2 \geq q^2$, so for every $\theta \in (0, \pi/2)$ the Φ block carries the entanglement for $q < 1/2$, the Ψ block for $q > 1/2$, and the state is exactly separable at $q = 1/2$. At the Bell angle $\theta = \pi/4$ ($z = 0$) Eq. (26) collapses to $N_X^{\text{gate}} = |1 - 2q|/2$, i.e.,

$$\chi_X(\pi/4, q) = |1 - 2q| = \chi_Z(q) : \quad (27)$$

a Bell-point-only benchmark cannot distinguish bit-flip from phase-flip faults. The discrimination lives entirely in the angle dependence: expanding Eq. (26) for small q ,

$$\chi_X(\theta, q) = 1 - q(1 + \csc 2\theta) + O(q^2), \quad (28)$$

a linear fingerprint whose slope steepens as $\csc 2\theta$ toward the separable endpoints, because the leakage populations qc^2, qs^2 overwhelm the shrinking coherence $\propto s_2$. Equation (28) is the analytic basis of the angle-resolved diagnostic of Sec. VIII.

V. POST-GATE SURVIVAL: THE TWO-BLOCK MASTER FORMULA

Combining the sector decomposition of Sec. III with Eq. (6) yields the computational backbone of the analysis. The two-block structure itself is the standard X -state bookkeeping of Ref. [9] (built on Refs. [19, 20]); the content of this section is its closed-form combination with the matched channel and the fault classes. For any X -state gate output—exactly the ideal, phase, Z , and X classes, and the R_x class to $O(\epsilon^2)$ (Appendix D)—the final negativity after the matched idle channel is

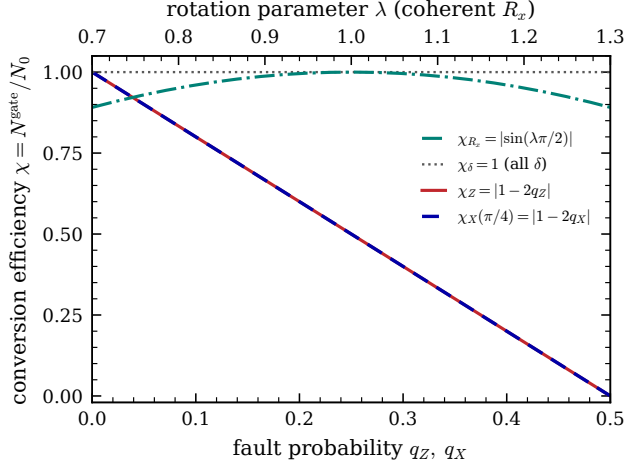


FIG. 1. Gate conversion efficiencies $\chi = N^{\text{gate}}/N_0$ for the four fault classes [Eqs. (21), (23), (24), (26)]. Stochastic classes are plotted against the bottom axis (fault probability q_Z, q_X); the coherent R_x class against the top axis (rotation parameter λ). Coherent controlled-phase errors are passive for this negativity-only benchmark; the coherent R_x error is quadratically flat at its maximum; stochastic faults fall linearly, with the X class evaluated at $\theta = \pi/4$, where it coincides with the Z class [dashed over solid, Eq. (27)].

TABLE I. Closed-form gate-level benchmarks for the four faulty CNOT classes. $N_0 = \sin(2\theta)/2$; $\epsilon = \lambda - 1$; $s_2 = \sin 2\theta$. “Harm order” is the leading order at which the fault reduces χ near the ideal point.

Fault class	Implementation	Gate output	Exact $\chi = N^{\text{gate}}/N_0$	Near-ideal expansion	Harm order; θ dep.
Coherent R_x	$ 0\rangle\langle 0 \otimes I + 1\rangle\langle 1 \otimes R_x(\lambda\pi)$	pure, non-X	$ \sin(\lambda\pi/2) $	$1 - (\pi^2/8)\epsilon^2 + O(\epsilon^4)$	quadratic; none
Coherent phase	$CZ_\delta U_{\text{CNOT}}$	pure X	1	1 (exact, all δ)	passive; none
Stochastic Z	$(1 - q_Z)\rho' + q_Z Z_B \rho' Z_B$	X (ϵ only)	$ 1 - 2q_Z $	$1 - 2q_Z + O(q_Z^2)$	linear; none
Stochastic X	$(1 - q_X)\rho' + q_X X_B \rho' X_B$	X (ϵ and w)	N_X^{gate}/N_0 , Eq. (26)	$1 - q_X(1 + \csc 2\theta) + O(q_X^2)$	linear; slope $\propto \csc 2\theta$

$$N^f = \max\left[0, \left(\sqrt{(P_{01}^f - P_{10}^f)^2 + 4\Lambda_{AB}^2 |\mathbf{c}_g|^2} - P_{01}^f - P_{10}^f\right)/2\right] + \max\left[0, \left(\sqrt{(P_{00}^f - P_{11}^f)^2 + 4\Lambda_{AB}^2 |w_g|^2} - P_{00}^f - P_{11}^f\right)/2\right], \quad (29)$$

with final populations from Eq. (16) and $\Lambda_{AB} = k_A k_B$. The Φ block survives iff

$$\Lambda_{AB}^2 |\mathbf{c}_g|^2 > P_{01}^f P_{10}^f, \quad (30)$$

with ESD exactly at equality; the Ψ block survives iff $\Lambda_{AB}^2 |w_g|^2 > P_{00}^f P_{11}^f$. The first term of Eq. (29) is the master formula quoted in the abstract; the second is required whenever $w_g \neq 0$, i.e., for the stochastic- X class, where it carries *all* the entanglement for $q_X > 1/2$.

For the Z class the populations are q_Z independent and only $|\mathbf{c}_g| \rightarrow |1 - 2q_Z| s_2/2$ changes. For the X class the final populations are *exactly linear* in q , with the memory factor r_B of Eq. (15) appearing because the fault flips

qubit B (Appendix C):

$$\begin{aligned} P_{01}^f(q) &= P_{01}^f(0) + q r_B [(1 - u_A)c^2 - d_A s^2], \\ P_{10}^f(q) &= P_{10}^f(0) + q r_B [(1 - d_A)s^2 - u_A c^2], \end{aligned} \quad (31)$$

where $P_{01}^f(0), P_{10}^f(0)$ are the ideal-gate values (18); the analogous relations hold for P_{00}^f, P_{11}^f . Equations (29)–(31) reproduce direct simulation of all fault classes and idle parameters at double-precision round-off (Table VI). Figure 2 shows the resulting $N^f(\theta)$ curves for representative fault strengths under the realistic idle-parameter set.

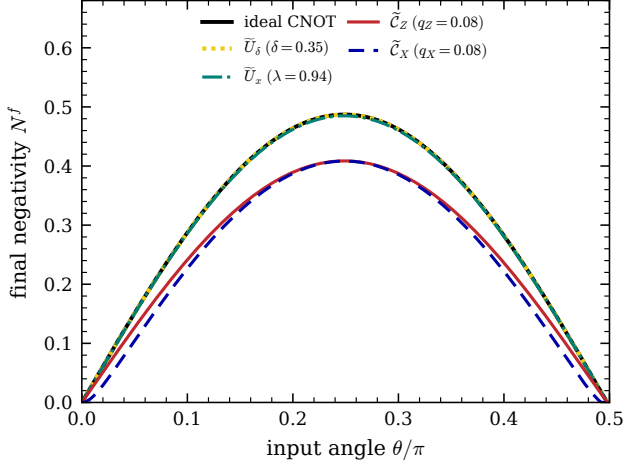


FIG. 2. Final negativity $N^f(\theta)$ for the four fault classes followed by the same matched idle environment ($T_1 = 40/35 \mu\text{s}$, $T_2 = 28/24 \mu\text{s}$, $p^{\text{eq}} = 1.5\%/2.0\%$ for A/B , $t_{\text{idle}} = 250 \text{ ns}$), from Eq. (29). The phase-error curve (dotted) coincides exactly with the ideal one and is drawn over it, Eq. (23).

VI. SUDDEN-DEATH GEOMETRY

A. Closed-form death boundaries

Because the Z -fault populations do not depend on q_Z , the survival condition (30) solves explicitly for the boundary in the (q_Z, θ) plane:

$$q_Z^c(\theta) = \left[1 - 2\sqrt{P_{01}^f P_{10}^f / (\Lambda_{AB} \sin 2\theta)} \right] / 2, \quad (32)$$

with $P_{01,10}^f$ from Eq. (18). Equation (32) is overlaid on Fig. 3 and is indistinguishable from the simulated $N^f = 0$ contour; entanglement is possible at all only in the angle window where the bracket is positive, $\Lambda_{AB} \sin 2\theta > 2\sqrt{P_{01}^f P_{10}^f}$, so the finite-temperature environment closes the landscape near $\theta = 0, \pi/2$ even for a perfect gate.

For the X class, inserting the linear populations (31) into Eq. (30) makes the Φ -block boundary the smaller positive root of an explicit quadratic, $\mathcal{A}q^2 + \mathcal{B}q + \mathcal{C} = 0$, with

$$\begin{aligned} \mathcal{A} &= \Lambda_{AB}^2 s_2^2 / 4 - r_B^2 \alpha_{01} \alpha_{10}, \\ \mathcal{B} &= -\Lambda_{AB}^2 s_2^2 / 2 - r_B [\alpha_{01} P_{10}^f(0) + \alpha_{10} P_{01}^f(0)], \\ \mathcal{C} &= \Lambda_{AB}^2 s_2^2 / 4 - P_{01}^f(0) P_{10}^f(0), \end{aligned} \quad (33)$$

where $\alpha_{01} = (1 - u_A)c^2 - d_A s^2$ and $\alpha_{10} = (1 - d_A)s^2 - u_A c^2$; the Ψ block obeys the mirror condition and can revive entanglement at large q_X . The different functional forms—explicit linear expression (32) versus quadratic root (33)—are the analytic origin of the visibly different landscape shapes in Figs. 3 and 4. Under the aggressive idle-parameter set of those figures, at $\theta = 0.3\pi$ the

boundaries evaluate to $q_Z^c = 0.38293$ and $q_X^c = 0.35368$; bracketing simulations confirm $N^f > 0$ at $q^c - 10^{-4}$ and $N^f = 0$ at $q^c + 10^{-4}$ for both (Table VI). The bit-flip class dies earlier at this angle because it attacks the state twice, degrading the coherence *and* feeding the killing populations.

B. Critical idle time and a finite-temperature no-go

Fixing the gate and increasing t_{idle} , the left side of Eq. (30) decays as $\Lambda_{AB}^2 \propto e^{-2t(1/T_{2,A} + 1/T_{2,B})}$ while $P_{01}^f P_{10}^f$ approaches the thermal value $(1 - p_A^{\text{eq}})p_B^{\text{eq}} p_A^{\text{eq}}(1 - p_B^{\text{eq}}) > 0$ whenever both equilibrium excited populations are nonzero. Hence at any finite temperature, for every input angle and every gate with $\chi > 0$, there exists a finite critical time t_c solving $\Lambda_{AB}(t_c) \chi s_2 = 2\sqrt{P_{01}^f(t_c) P_{10}^f(t_c)}$ beyond which $N^f = 0$: *entanglement sudden death is unavoidable at finite temperature*. For the aggressive parameter set at $\theta = \pi/4$ the ideal-gate critical time is $t_c = 7.997 \mu\text{s}$, confirmed by bracketing simulations at $t_c \pm 1 \text{ ns}$ (Table VI).

C. Recovery of the Yu–Eberly threshold

As an external consistency check, specialize to zero temperature and pure amplitude damping: $u_k = 0$ and no excess dephasing, $k_k = \sqrt{1 - d_k}$ (the $T_2 = 2T_1$ limit). Then $P_{01}^f = d_A(1 - d_B)s^2$, $P_{10}^f = (1 - d_A)d_B s^2$, $\Lambda_{AB}^2 = (1 - d_A)(1 - d_B)$, and the ideal-gate survival condition (30) collapses to

$$\tan^2 \theta d_A d_B < 1. \quad (34)$$

Since $d_A d_B < 1$ at all finite times, inputs with $\theta \leq \pi/4$ never suffer sudden death—their negativity decays only asymptotically—whereas inputs with $\theta > \pi/4$ die at the finite time where $d_A(t)d_B(t) = \cot^2 \theta$. This is precisely the Yu–Eberly dichotomy for Φ -type states under two-sided spontaneous emission [8, 10], here obtained as a two-line corollary of Eq. (29).

VII. NOISE-ADAPTED OPTIMAL INPUT ANGLES

The Bell angle $\theta = \pi/4$ maximizes the ideal negativity, but under the matched channel the optimum shifts. For any X -state class with ideal-form populations and coherence $|c_g| = \chi s_2/2$ (ideal, phase, Z ; the X and R_x classes are handled below), write the final populations (18) as affine functions of $z = \cos 2\theta$,

$$P_{01}^f + P_{10}^f = \Sigma_0 + \Sigma_1 z, \quad P_{01}^f - P_{10}^f = \Delta_0 + \Delta_1 z, \quad (35)$$

with coefficients given in Appendix C; in particular $\Sigma_1 \simeq [(u_A + u_B) - (d_A + d_B)]/2 < 0$ in any cold environment. The Φ -block negativity is $N^f(z) = [R(z) - \Sigma_0 -$

TABLE II. Closed-form post-idle benchmarks under the matched thermal channel. $\Lambda_{AB} = k_A k_B$; $s_2 = \sin 2\theta$; $P_{01}^f(0), P_{10}^f(0)$ are the ideal-gate final populations, Eq. (18); $r_B = e^{-t_{\text{idle}}/T_{1,B}}$; α_{01}, α_{10} as in Eq. (33). All entries follow from the two-block master formula, Eq. (29); in the pure-AD row the two population entries are P_{01}^f and P_{10}^f , respectively.

Class	Surviving coherences	Final populations	Φ -block survival	ESD boundary
Ideal/phase	$ \mathbf{c}_f = \Lambda_{AB} s_2 / 2$	Eq. (18)	$\Lambda_{AB}^2 s_2^2 > 4P_{01}^f P_{10}^f$	t_c from Eq. (30)
Stochastic Z	$ \mathbf{c}_f = 1 - 2q_Z \Lambda_{AB} s_2 / 2$	Eq. (18), q_Z indep.	$(1 - 2q_Z)^2 \Lambda_{AB}^2 s_2^2 > 4P_{01}^f P_{10}^f$	$q_Z^c(\theta)$, Eq. (32)
Stochastic X	$ \mathbf{c}_f = (1 - q_X) \Lambda_{AB} s_2 / 2$, $ w_f = q_X \Lambda_{AB} s_2 / 2$	Eq. (31), linear in q_X	$(1 - q_X)^2 \Lambda_{AB}^2 s_2^2 > 4P_{01}^f P_{10}^f$ (+ Ψ -block mirror cond.)	smaller root of $\mathcal{A}q^2 + \mathcal{B}q + \mathcal{C} = 0$, Eq. (33)
Pure AD, $T=0$	$ \mathbf{c}_f = \sqrt{(1-d_A)(1-d_B)} s_2 / 2$	$d_A(1-d_B)s^2, (1-d_A)d_B s^2$	$\tan^2 \theta d_A d_B < 1$	Yu–Eberly [8], Eq. (34)

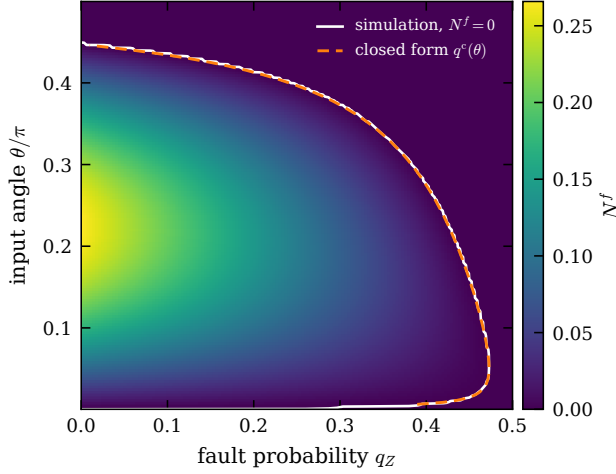


FIG. 3. Survival landscape $N^f(q_Z, \theta)$ for the stochastic-Z class under the aggressive idle set ($T_1 = 20/18 \mu\text{s}$, $T_2 = 12/10 \mu\text{s}$, $p^{\text{eq}} = 2.0\%/2.5\%$, $t_{\text{idle}} = 2.5 \mu\text{s}$). White solid line: simulated $N^f = 0$ boundary; dashed line: closed form $q_Z^c(\theta)$, Eq. (32). The two are indistinguishable on this scale.

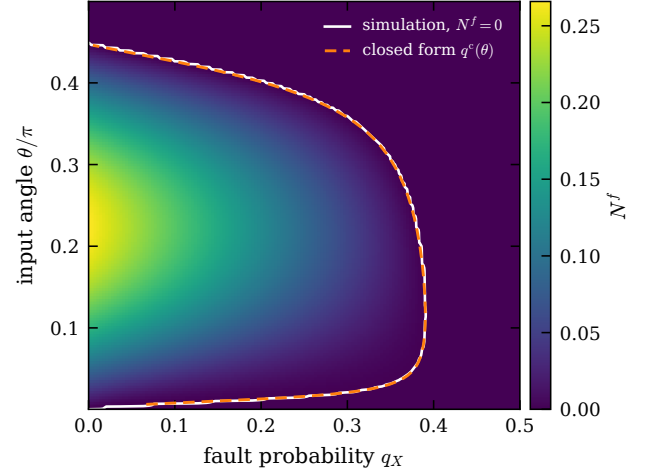


FIG. 4. Survival landscape for the stochastic-X class under the same idle set. White solid line: simulated $N^f = 0$ boundary; dashed line: the quadratic root of Eq. (33). The different shape relative to Fig. 3 reflects the fault's simultaneous attack on coherence and populations.

$\Sigma_1 z]/2$ with $R(z) \equiv \sqrt{(\Delta_0 + \Delta_1 z)^2 + \chi^2 \Lambda_{AB}^2 (1 - z^2)}$, and $dN^f/dz = 0$ gives the exact stationarity condition

$$(\Delta_0 + \Delta_1 z) \Delta_1 - \chi^2 \Lambda_{AB}^2 z = \Sigma_1 R(z). \quad (36)$$

Since the noise scale $\eta \equiv \max_k(u_k, d_k)$ satisfies $\eta \ll 1$ for realistic idle windows, solving Eq. (36) iteratively about $z = 0$ gives the closed-form optimum

$$\begin{aligned} z_* &= (\Delta_0 \Delta_1 - \Sigma_1 \chi \Lambda_{AB}) / (\chi^2 \Lambda_{AB}^2) \\ &= -\Sigma_1 / (\chi \Lambda_{AB}) + O(\eta^2), \quad \theta_{\text{opt}} \simeq \pi/4 - z_*/2, \end{aligned} \quad (37)$$

Two structural conclusions follow. *Direction*: $\Sigma_1 < 0$ implies $z_* > 0$, i.e., $\theta_{\text{opt}} < \pi/4$ —the optimal input is biased toward $|0\rangle$. The physics is read off Eq. (18): the killing populations are fed dominantly by relaxation out of the $|11\rangle$ branch ($d_k s^2$ terms), so reducing s^2 below $1/2$ trades a second-order loss of coherence for a first-order suppression of those populations. *Fault dependence*: the shift scales as $1/(\chi \Lambda_{AB})$ —weaker gates and longer idles

push the optimum *further* from the Bell angle, because the protective population bias becomes relatively more valuable as the surviving coherence shrinks.

For the stochastic-X class the same construction applies with $\chi \rightarrow 1 - q_X$ and the q -linear population corrections of Eq. (31) added to the affine coefficients ($\Sigma_1 \rightarrow \Sigma_1 + q_X r_B (d_A - u_A)$, $\Delta_1 \rightarrow \Delta_1 + q_X r_B$; Appendix C); the two corrections partially compensate the smaller χ , which is why the X-class optimum in Table III sits much closer to the ideal one than the Z-class optimum at equal fault probability. For the R_x class the prediction uses the X-truncated state of Appendix D and inherits its $O(\epsilon^2)$ systematic.

We state the status of the R_x row explicitly: the R_x results are *exact* at the gate level, Eq. (21), but only approximate—at $O(\epsilon^2)$ —in any post-idle treatment that assumes X form, including both the truncated optimum used here and the five-setting estimate of Sec. VIII C. Table III compares Eq. (37) with direct numerical maximization of the full model: the closed form reproduces the

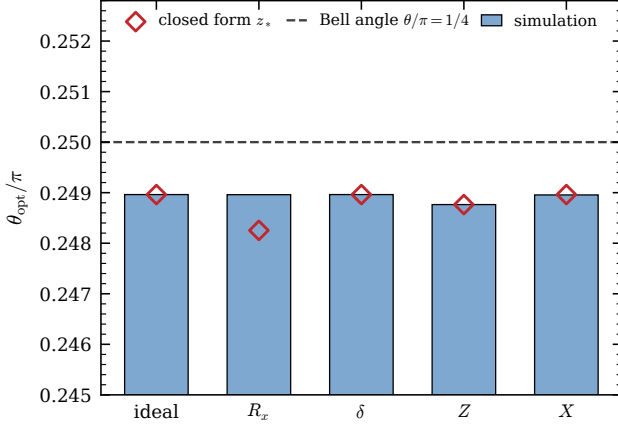


FIG. 5. Noise-adapted optimal input angles under the realistic idle environment. Bars: numerical maximization of the full model; open diamonds: closed form Eq. (37); dashed line: Bell angle $\theta/\pi = 1/4$. The residual for the R_x class is the $O(\epsilon^2)$ truncation systematic of Appendix D; see Table III.

simulated optimum to within 4×10^{-8} in θ_{opt}/π for the uniform-coherence classes (ideal, phase, Z) and 8×10^{-6} for the X class; for the R_x class the residual 7×10^{-4} is the expected $O(\epsilon^2)$ truncation systematic at $\epsilon = -0.06$. Figure 5 displays the corresponding bar chart.

VIII. DIAGNOSTIC PROTOCOL, ROBUSTNESS, AND MEASUREMENT COST

A. Estimator and identifiability

Given measured negativities $N_{\text{exp}}(\theta_m)$, $m = 1, \dots, M$, the dominant fault class is identified by nonlinear least squares over each one-parameter family followed by model selection on the residual,

$$\hat{j} = \arg \min_j \min_{\epsilon_j} \sum_{m=1}^M [N_{\text{exp}}(\theta_m) - N_j^f(\theta_m; \epsilon_j)]^2. \quad (38)$$

TABLE III. Noise-adapted optimal input angles under the realistic idle set: closed-form prediction Eq. (37) versus direct numerical maximization of the full density-matrix model. Fault strengths as in Fig. 2. The R_x row inherits the $O(\epsilon^2)$ truncation systematic of Appendix D.

Model	χ	θ_{opt}/π theory	θ_{opt}/π sim.	$\max N^f$
Ideal	1	0.2489623	0.2489623	0.487115
Phase $\delta = 0.35$	1	0.2489623	0.2489623	0.487115
Z , $q_Z = 0.08$	0.84	0.2487646	0.2487647	0.408650
X , $q_X = 0.08$	0.92	0.2489628	0.2489547	0.408413
R_x , $\lambda = 0.94$	0.99556	0.2482537	0.2489603	0.484947

Equation (38) is stated in unweighted form for transparency; in practice it should be applied with uncertainty weights, $\mathcal{S}_j = \sum_m [N_{\text{exp}}(\theta_m) - N_j^f(\theta_m; \epsilon_j)]^2 / \sigma_m^2$, with per-angle uncertainties σ_m propagated from the counting statistics of the five Pauli settings of Sec. VIII C. Parameter confidence intervals then follow from likelihood profiling ($\Delta\chi^2 = 1$) or a nonparametric bootstrap over the M angles, and selection among the families requires a complexity penalty (we use the Akaike criterion in Sec. VIII B), because the parameterized classes contain the ideal curve as an interior point and would otherwise absorb noise at the expense of the phase/passive hypothesis. The RMSE values quoted in this paper are point summaries, not calibrated statistics, and we claim no statistical optimality for the estimator.

The closed forms derived above determine exactly what is identifiable:

(i) *Phase faults are unidentifiable in principle*: by Eq. (23), $N_\delta^f \equiv N_{\text{ideal}}^f$ for all δ , θ , and t_{idle} . The protocol can report “phase/passive” only as a class, never estimate δ ; conversely, any statistically significant deficit of N^f below the ideal curve certifies a non-phase fault.

(ii) *Bit-flip and phase-flip faults are identifiable from curve shape but not from a single angle*: both are linear in their parameter, and at the Bell point they coincide exactly, Eq. (27); but the Z class rescales the whole curve uniformly while the X class tilts it by $\csc 2\theta$, Eq. (28), and additionally shifts the populations by Eq. (31). Two well-separated angles suffice in the noiseless limit.

(iii) *Coherent R_x and stochastic Z faults are exactly degenerate at the gate level along the matching $|\sin(\lambda\pi/2)| = |1 - 2q_Z|$* , since both act as uniform rescalings of $N_0(\theta)$. The matched idle channel lifts the degeneracy only at second order in the calibration error—through the $O(\epsilon^2)$ population leakage $s^2 C_\lambda^2$ into $|10\rangle$ and the sector-3/4 coherences of Appendix D—so the discrimination power between these two classes grows as ϵ^2 and small- ϵ assignments should be reported with that caveat.

Figure 6 demonstrates the protocol on synthetic data generated from a hidden stochastic- Z fault with $q_Z = 0.12$ plus Gaussian measurement noise $\sigma = 0.006$ at $M = 18$ angles: the fit recovers the Z class with $\hat{q}_Z$ at the injected value, with the residuals of the competing classes ordered as predicted by (i)–(iii). This in-family test establishes only that the estimator works when the data come from the fitted family; the harder and more important question is what it does when they do not.

B. Robustness to model mismatch

Table IV applies the unmodified estimator (38) to data generated *outside* the fitted family (same $M = 18$ angles, $\sigma = 0.006$). The outcomes are as follows: (a) *Fault mixture* [$Z(0.08)$ followed by $X(0.05)$]: the estimator selects a *single* class (X , $\hat{q} = 0.125$) and the best/runner-up residual gap collapses (RMSE 0.0065 vs 0.0083). Single-class selection reports the dominant angular signature,

not the composition; a compressed residual margin is the practical warning sign, and resolving mixtures requires extending the fitted family to two parameters. (b) *SPAM proxy* (5% uniform contrast loss on true Z , $q_Z = 0.12$): the class label survives, but the parameter estimate is biased upward, $\hat{q}_Z = 0.138$ —contrast loss masquerades as extra dephasing. Parameter estimates from this protocol inherit SPAM error directly. (c) *Idle miscalibration* (T_1 , T_2 20% shorter than the fit model assumes): class and parameter remain essentially correct here ($\hat{q}_Z = 0.123$), because a modest uniform extra decay is largely absorbed without changing the curve shape; larger or asymmetric miscalibrations will not be so benign. (d) *Unmodeled intra-gate relaxation* ($\gamma t_g = 0.1$, no injected Pauli fault): the estimator reports a spurious weak fault with the Z and R_x hypotheses statistically tied (RMSE 0.00654 vs 0.00654)—the Z - R_x degeneracy of point (iii) surfacing exactly where predicted. The classifier fails gracefully in these directions (labels stable or degeneracy-flagged), but it does not fail loudly: none of the mismatch scenarios is rejected outright, so a small residual should never be read as confirmation that the model family is correct.

Beyond single realizations, Table V reports a Monte Carlo confusion matrix: 400 noisy data sets per true class ($\sigma = 0.006$, $M = 18$), classified by an Akaike-penalized version of Eq. (38) (penalty $2k$ with $k = 1$ for the parameterized families and $k = 0$ for phase/ideal; without the penalty the nested families absorb noise and the phase hypothesis is never selected). Strong stochastic faults are identified essentially always (100% for both Z and X at $q = 0.12$) and the passive class is recovered in 92% of trials. The weak coherent calibration error $\lambda = 0.94$ ($\chi_{R_x} = 0.9956$), by contrast, is classified as phase/ideal in 56% of trials and correctly in only 31%: its curve deficit, $\lesssim 0.2\%$ of N^f , sits below the noise, so the same second-order flatness that makes such errors harmless renders them statistically invisible to this benchmark. Halving the noise to $\sigma = 0.003$ raises the R_x recovery to 83%; doubling it to $\sigma = 0.012$ lowers it to 14% while Z and X remain above 94%. As a practical rule suggested by the compressed margins of Table IV, a class label whose best and runner-up residuals differ by less than the noise floor $\sim \sigma/\sqrt{M}$ should be reported as ambiguous rather than selected.

C. Estimating the negativity

Negativity is not a directly measured observable, and the cost of estimating it must be stated. For the X-form states produced by this protocol (all classes exactly, the R_x class up to $O(\epsilon)$ corrections), Eq. (6) requires only the four populations and the two complex coherences, which are fixed by *five* of the nine local Pauli settings: the ZZ

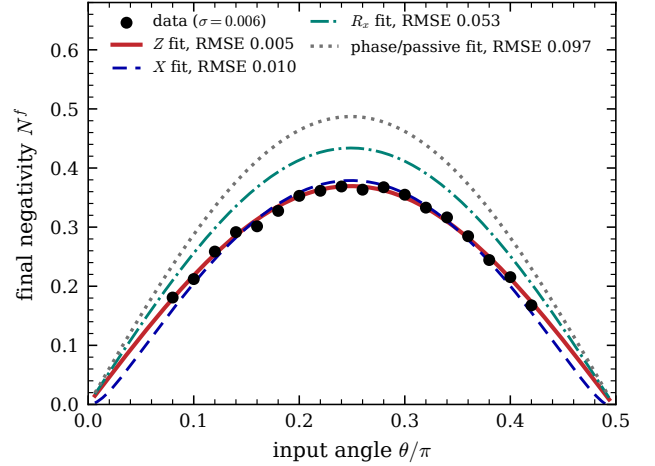


FIG. 6. Diagnostic model selection from synthetic $N^f(\theta)$ data (hidden model: stochastic Z , $q_Z = 0.12$; $\sigma = 0.006$). The estimator (38) recovers the injected class and parameter ($\hat{q}_Z = 0.12$, RMSE = 0.005); competing fits are shown with their residuals, and the phase/passive curve is excluded because the data lie significantly below the ideal curve [point (i)].

basis gives P_{ab} , and

$$\begin{aligned} \text{Re } \mathbf{c} &= (\langle XX \rangle - \langle YY \rangle)/4, & \text{Re } w &= (\langle XX \rangle + \langle YY \rangle)/4, \\ \text{Im } \mathbf{c} &= -(\langle XY \rangle + \langle YX \rangle)/4, & \text{Im } w &= (\langle XY \rangle - \langle YX \rangle)/4, \end{aligned} \quad (39)$$

verified against the exact negativity to 1.1×10^{-16} across the fault grid (Table VI). This is cheaper than the nine settings of full two-qubit tomography but is not free, and it carries three caveats stated plainly: (i) Eqs. (39) and (6) are exact *only* for X-form states. For the controlled- R_x class the output is not X form: the reconstruction returns the X projection of the state, and the number obtained is neither an upper nor a lower bound on N in general. It is trustworthy only when the off-X coherences are independently bounded—the remaining four Pauli settings suffice for this—and found negligible, i.e., $|\epsilon| \ll 1$; (ii) the estimates inherit SPAM error directly, with the induced parameter bias quantified in Table IV; (iii) statistical noise propagates nonlinearly through the $\max(0, \cdot)$ in Eq. (6), biasing small negativities upward near the sudden-death boundary. Any experimental use of these benchmarks must budget for the five settings and these systematics; we make no claim that the protocol is tomography-free beyond this concrete accounting.

IX. NUMERICAL METHODS AND VALIDATION

The checks in this section are *algebraic consistency* checks: they certify that the closed forms are exact consequences of the stated model, by comparison against

TABLE IV. Diagnostic behavior under model mismatch. The estimator (38) is applied unmodified to synthetic data generated outside the fitted one-fault family ($M = 18$ angles, $\sigma = 0.006$). RMSE in negativity units; the first row is the in-family reference.

Scenario (truth)	Selected	$\hat{q}$	RMSE (best)	Runner-up (RMSE)
In-family reference: $Z(0.12)$	Z	0.122	0.0040	X (0.0094)
Mixture: $Z(0.08)$ then $X(0.05)$	X	0.125	0.0065	Z (0.0083)
SPAM proxy: 5% contrast loss, $Z(0.12)$	Z	0.138	0.0054	X (0.0122)
Idle miscalibration -20% , $Z(0.12)$	Z	0.123	0.0055	X (0.0105)
Unmodeled intra-gate AD, $\gamma t_g=0.1$	Z	0.022	0.00654	R_x (0.00654)

an independent numerical propagation of the same equations. They do not, by themselves, certify the model’s fidelity to any physical device—that separate question is addressed by the microscopic-reduction analysis of Sec. III B and the mismatch tests of Sec. VIII B. With that scope fixed, all analytics are validated against direct simulation of the full model with no intermediate approximations: 4×4 density matrices are propagated exactly, the channel (13) is applied element-wise via its action on the operator basis $|a\rangle\langle c| \otimes |b\rangle\langle d|$, and the negativity is obtained from the eigenvalues of the numerically partial-transposed matrix, Eq. (4). Two parameter sets are used throughout: the *realistic* set $T_1 = (40, 35) \mu\text{s}$, $T_2 = (28, 24) \mu\text{s}$, $p^{\text{eq}} = (1.5, 2.0)\%$, $t_{\text{idle}} = 250$ ns for Figs. 2 and 5, and the *aggressive* set $T_1 = (20, 18) \mu\text{s}$, $T_2 = (12, 10) \mu\text{s}$, $p^{\text{eq}} = (2.0, 2.5)\%$, $t_{\text{idle}} = 2.5 \mu\text{s}$ for Figs. 3 and 4, chosen so that the survival boundaries fall inside the plotted windows. Both sets satisfy $T_2 \leq 2T_1$, hence complete positivity at all times (Sec. III).

Table VI summarizes the checks. The closed-form gate negativities, the two-block master formula, and the population linearity (31) agree with simulation at double-precision round-off, confirming that these results are exact rather than approximate; the fingerprint slope (28), the ESD boundaries (32)–(33), the critical time, the Yu–Eberly reduction, and the R_x truncation order are confirmed at their stated asymptotic orders.

TABLE V. Monte Carlo confusion matrix for Akaike-penalized model selection: percentage of 400 trials per row assigned to each class ($\sigma = 0.006$, $M = 18$; true models as in Fig. 2). Strong stochastic faults are always identified; the weak coherent R_x error is more often indistinguishable from the ideal gate than recovered.

True model	phase/ideal	R_x	Z	X
Phase, $\delta = 0.35$	92.0	6.5	0.2	1.2
R_x , $\lambda = 0.94$	55.5	31.0	3.0	10.5
Z , $q_Z = 0.12$	0	0	100	0
X , $q_X = 0.12$	0	0	0	100

X. CONCLUSIONS

Within a restricted but analytically transparent two-qubit model, we have obtained closed-form negativity benchmarks for four representative CNOT fault classes under matched thermal idle noise: conversion efficiencies, survival conditions, sudden-death boundaries, sensitivity fingerprints, and noise-adapted optimal input angles (Tables I–III), all verified as exact consequences of the model (Table VI). The physical picture within that scope is simple. Coherent calibration errors detune a smooth maximum and are quadratically harmless, Eq. (22); coherent phase errors are exactly invisible to the negativity—a structural limitation of conversion-based diagnostics that any user of this protocol must weigh, Eq. (23); stochastic faults are linearly harmful, with the bit-flip class uniquely tagged by the $\csc 2\theta$ slope, Eq. (28), and by its two-block structure. The post-gate environment enters through exactly two objects—the coherence attenuation Λ_{AB} and the population transfer matrices $M_A \otimes M_B$ —whose competition produces the closed-form death boundaries, an unavoidable finite-temperature ESD, the Yu–Eberly threshold as a limit, and an optimal angle biased toward the ground state by $-\Sigma_1/(\chi\Lambda_{AB})$.

Two analyses delimit how far these results reach beyond the model. The commutation criterion of Sec. III B shows that the “matched” pairing is exact for intra-gate target bit-flip noise and control dephasing, first order with residuals of order 10^{-2} for target dephasing, and unavailable for intra-gate relaxation; all reduction statements are tied to the minimal generator of Eq. (11) and must be rederived in the toggling frame of any actual pulse sequence. The mismatch tests of Sec. VIII B show that the diagnostic’s class labels are robust to modest SPAM-like contrast loss and idle miscalibration, but that fault mixtures are absorbed into single effective classes and unmodeled intra-gate relaxation produces a spurious weak-fault detection with the Z – R_x ambiguity predicted by the identifiability analysis. The Monte Carlo confusion analysis (Table V) sharpens this: strong stochastic faults are identified essentially always at realistic measurement noise, whereas a weak coherent calibration error is more often classified as ideal than recovered—the same second-order flatness that makes such errors harmless makes them invisible here. These closed-form fin-

TABLE VI. Analytic results versus direct density-matrix simulation. “Max. dev.” is the maximum absolute deviation over the stated grid; entries at $\sim 10^{-16}$ saturate double-precision round-off. Boundary checks bracket the predicted value; the R_x truncation check confirms the $O(\epsilon^2)$ error order of Appendix D.

Result	Test	Outcome
$N_{R_x}^{\text{gate}} = (s_2/2) \sin(\lambda\pi/2) $, Eq. (21)	25×21 grid in (θ, λ)	max. dev. 5.0×10^{-16}
$N_{\delta}^{\text{gate}} = s_2/2$ (passivity), Eq. (23)	25×11 grid in (θ, δ)	max. dev. 1.1×10^{-16}
$N_Z^{\text{gate}} = 1 - 2q_Z s_2/2$, Eq. (24)	25×26 grid in (θ, q_Z)	max. dev. 1.1×10^{-16}
N_X^{gate} two-block formula, Eq. (26)	25×41 grid, $q_X \in [0, 1]$	max. dev. 2.4×10^{-16}
Two-block master formula, Eq. (29)	phase, Z, X classes + idle, 25×21 grid	max. dev. 2.2×10^{-16}
Population linearity, Eq. (31)	3×4 grid in (θ, q_X)	max. dev. 1.1×10^{-16}
Fingerprint slope, Eq. (28)	$q_X = 10^{-4}$, $\theta = 0.3$	residual $1.1 \times 10^{-8} = O(q_X^2)$
$q_Z^c(0.3\pi) = 0.382928$, Eq. (32)	bracketing $\pm 10^{-4}$	$N^f = 6.0 \times 10^{-5}$ / 0 below/above
$q_X^c(0.3\pi) = 0.353675$, Eq. (33)	bracketing $\pm 10^{-4}$	$N^f = 6.3 \times 10^{-5}$ / 0 below/above
Critical time $t_c(\pi/4) = 7.997 \mu\text{s}$	bracketing ± 1 ns	$N^f = 2.7 \times 10^{-5}$ / 0 below/above
Yu–Eberly threshold, Eq. (34)	$\theta = \pi/6$, $d_A d_B \rightarrow 0.9999$	survival margin positive throughout
Optimal angles, Eq. (37)	vs. maximization in θ_{opt}/π (Table III)	dev. 4×10^{-8} (ideal/phase/ Z); 8×10^{-6} (X)
R_x truncation order (Appendix D)	$\epsilon = 0.1 \rightarrow 0.0125$	$ N_{R_x}^f - N_{R_x}^{\text{tr}} /\epsilon^2 \rightarrow 0.556$ (const.)
Exact intra-gate reduction (X_B, Z_A)	Trotterized gate vs. Eq. (12), $\gamma t_g \leq 0.2$	max. state dev. 9×10^{-14}
Five-setting negativity, Eq. (39)	X-state grid in (θ, q_X)	max. dev. 1.1×10^{-16}

gerprints may therefore serve as diagnostics when the assumptions of Sec. III A—local Markovian idle noise, negligible SPAM and leakage, a single dominant fault, and X-form outputs—are approximately valid, and the five-setting scheme of Sec. VIII C fixes the measurement cost of testing them.

The limitations point to concrete extensions: two-parameter (mixture) fits, which the compressed residual margins of Table IV directly motivate; correlated ZZ -type stochastic faults and control-line errors, which enlarge the fault algebra beyond weight one; leakage and SPAM incorporated into the model rather than treated as mismatch; and non-Markovian idle environments [17], for which the sector decomposition of Sec. III no longer closes. Each extension breaks a specific assumption of Sec. III A while leaving the X-state machinery and the matched calibration principle intact.

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DATA AVAILABILITY

The numerical code and data supporting the findings of this study are not publicly deposited; they are available from the author upon reasonable request.

Appendix A: Partial-transpose spectrum of X states

Applying the convention $(\rho^\Gamma)_{ab,cd} = \rho_{ad,cb}$ entry by entry to Eq. (5): diagonal entries are fixed; the outer coherence moves to the inner block, $(\rho^\Gamma)_{01,10} = \rho_{00,11} = \mathfrak{c}$; the inner coherence moves to the outer block, $(\rho^\Gamma)_{00,11} = \rho_{01,10} = w$; and every zero entry of the X pattern is mapped to another zero entry. Hence

$$\rho_X^\Gamma \cong \begin{pmatrix} P_{00} & w \\ w^* & P_{11} \end{pmatrix} \oplus \begin{pmatrix} P_{01} & \mathfrak{c} \\ \mathfrak{c}^* & P_{10} \end{pmatrix} \quad (\text{A1})$$

in the split $\{|00\rangle, |11\rangle\} \oplus \{|01\rangle, |10\rangle\}$. Each Hermitian block $\begin{pmatrix} p & x \\ x^* & q \end{pmatrix}$ has eigenvalues $(p + q)/2 \pm \sqrt{(p - q)^2/4 + |x|^2}$; the + roots are nonnegative, and the − root is negative iff the block determinant $pq - |x|^2$ is negative. Summing the magnitudes of the (at most two) negative roots gives Eqs. (6) and (7). For two qubits negativity vanishes iff the state is separable [19, 20], so the block-determinant conditions are exact separability boundaries.

Appendix B: Physicality of the idle channel

The Choi matrix of the map (13) is $C = \sum_{ij} |i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)$; using $\mathcal{E}(|0\rangle\langle 0|) = \text{diag}(1 - u, u)$, $\mathcal{E}(|1\rangle\langle 1|) = \text{diag}(d, 1 - d)$, $\mathcal{E}(|0\rangle\langle 1|) = k|0\rangle\langle 1|$,

$$C = \begin{pmatrix} 1 - u & 0 & 0 & k \\ 0 & u & 0 & 0 \\ 0 & 0 & d & 0 \\ k^* & 0 & 0 & 1 - d \end{pmatrix}, \quad (\text{B1})$$

which is positive semidefinite iff $u, d \geq 0$ and $|k|^2 \leq (1-u)(1-d)$. With the thermal parametrization (14), $(1-u)(1-d) = r + ud \geq r = e^{-t/T_1}$ while $|k|^2 = e^{-2t/T_2}$, so $T_2 \leq 2T_1$ guarantees complete positivity at all t . An explicit Kraus form is the composition of generalized amplitude damping,

$$\begin{aligned} K_1 &= \sqrt{1-p^{\text{eq}}} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{r} \end{pmatrix}, \quad K_2 = \sqrt{1-p^{\text{eq}}} \begin{pmatrix} 0 & \sqrt{1-r} \\ 0 & 0 \end{pmatrix}, \\ K_3 &= \sqrt{p^{\text{eq}}} \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1 \end{pmatrix}, \quad K_4 = \sqrt{p^{\text{eq}}} \begin{pmatrix} 0 & 0 \\ \sqrt{1-r} & 0 \end{pmatrix}, \end{aligned} \quad (\text{B2})$$

which yields the population map of Eq. (13) with coherence factor $\sqrt{r} = e^{-t/2T_1}$, followed by pure dephasing $\mathcal{D}_\kappa(\rho) = [(1+\kappa)\rho + (1-\kappa)\sigma_z\rho\sigma_z]/2$ with $\kappa = k/\sqrt{r} = e^{-t(1/T_2 - 1/2T_1)} \in (0, 1]$ under the same bound.

Appendix C: Affine population coefficients and the optimal-angle expansion

With $c^2 = (1+z)/2$, $s^2 = (1-z)/2$ in Eq. (18), the affine coefficients of Eq. (35) are

$$\begin{aligned} \Sigma_0 &= [(1-u_A)u_B + u_A(1-u_B) \\ &\quad + d_A(1-d_B) + (1-d_A)d_B]/2, \\ \Sigma_1 &= [(1-u_A)u_B + u_A(1-u_B) \\ &\quad - d_A(1-d_B) - (1-d_A)d_B]/2, \\ \Delta_0 &= [(u_B - u_A) + (d_A - d_B)]/2, \\ \Delta_1 &= [(u_B - u_A) - (d_A - d_B)]/2. \end{aligned} \quad (\text{C1})$$

To leading order in η , $\Sigma_1 \simeq [(u_A + u_B) - (d_A + d_B)]/2$, which is negative whenever relaxation dominates excitation. With $R(z)$ as defined above Eq. (36) and noting $R = \chi\Lambda_{AB} + O(\eta, z^2)$ near $z = 0$, one iteration of Eq. (36) gives Eq. (37); the retained subleading term $\Delta_0\Delta_1/(\chi\Lambda_{AB})^2$ is what brings the prediction to the 10^{-8} level of Table III for the uniform-coherence classes.

For the stochastic- X class the fault moves diagonal weight qc^2 from $|00\rangle$ to $|01\rangle$ and qs^2 from $|11\rangle$ to $|10\rangle$; propagating the difference through Eq. (16) and using $[M_B]_{11} - [M_B]_{10} = [M_B]_{00} - [M_B]_{01} = 1 - u_B - d_B = r_B$ yields Eq. (31). In affine form the corrections are $\Sigma \rightarrow \Sigma + qr_B[r_A + z(d_A - u_A)]$ and $\Delta \rightarrow \Delta + qr_Bz$, i.e., $\Sigma_1 \rightarrow \Sigma_1 + qr_B(d_A - u_A)$ and $\Delta_1 \rightarrow \Delta_1 + qr_B$, with $\chi = 1 - q$; inserting these into Eq. (37) produces the X -class row of Table III.

Appendix D: Controlled- R_x class after the idle: exact structure and truncation order

The output (20) is not an X state: besides the X -sector coherence $\rho_{00,11}^g = icsS_\lambda e^{-i\phi}$ and populations $(c^2, 0, s^2C_\lambda^2, s^2S_\lambda^2)$ it carries $\rho_{00,10}^g = csC_\lambda e^{-i\phi}$ (sector 3) and $\rho_{10,11}^g = is^2C_\lambda S_\lambda$ (sector 4). The sector rules of Sec. III give the exact final state

$$\begin{aligned} \rho_{00,11}^f &= \Lambda_{AB} icsS_\lambda e^{-i\phi}, \\ \rho_{00,10}^f &= k_A(1-u_B)csC_\lambda e^{-i\phi}, \quad \rho_{01,11}^f = k_A u_B csC_\lambda e^{-i\phi}, \\ \rho_{00,01}^f &= k_B d_A is^2C_\lambda S_\lambda, \quad \rho_{10,11}^f = k_B(1-d_A)is^2C_\lambda S_\lambda, \end{aligned} \quad (\text{D1})$$

with populations from Eq. (16). Under partial transposition the X -sector element moves to $(01, 10)$ while the sector-3 elements stay in place and the sector-4 elements are conjugated in place, so $\rho^{f\Gamma}$ is a full Hermitian 4×4 matrix whose spectrum is the root set of a quartic that factorizes only when $C_\lambda S_\lambda = 0$; this single case is therefore evaluated numerically in the figures. Writing $\lambda = 1 + \epsilon$, all discarded coherences are $O(C_\lambda) = O(\epsilon)$, and in $\rho^{f\Gamma}$ they occupy positions coupling the inner block to the outer block; since the relevant negative eigenvalue of the truncated matrix is nondegenerate with an $O(1)$ gap to the outer-block spectrum, its shift is second order in the coupling. Hence the X -truncated master formula—Eq. (29) with $|c_g| = cs|S_\lambda|$, $w_g = 0$, and the exact R_x populations—satisfies $N_{R_x}^f = N_{R_x}^{\text{tr}} + O(\epsilon^2)$. Numerically $|N_{R_x}^f - N_{R_x}^{\text{tr}}|/\epsilon^2 \rightarrow 0.556$ for $\epsilon \rightarrow 0$ under the aggressive parameter set at $\theta = 0.3\pi$ (Table VI), confirming the error is exactly of order ϵ^2 .

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