Supplementary Material for 3D Gaussian Distribution and RGB Reconstruction based GNSS Multipath Detection – Part II

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This supplementary material provides the detailed threshold and distribution derivations moved from the main manuscript for the page-compressed TAES submission. References to equations not defined in this file refer to the main Part II manuscript; compile the main manuscript first if cross-document reference resolution is required.

Supplementary Material

A. Derivation of the Value Threshold

In this appendix, the value-feature threshold is derived following the notation used in Section III. The value relation follows from (28) together with the function $f(\mathbf{X})$ defined in (36). Here $\mathbf{a}_i^{\mathrm{T}}$ is the i-th row of $\mathbf{\Sigma}_R^{1/2}$, and μ_i is the i-th element of $\boldsymbol{\mu}_R$. For compactness throughout the appendices, denote the standard-space ellipsoid radius by

$$\Gamma = \frac{\lambda_D R_{P_D} \gamma_{GS}}{\sigma_n} \tag{A.1}$$

According to (31), the radius of the corresponding sphere in the standard space is

$$\|\mathbf{X}\| = \Gamma \tag{A.2}$$

First, the upper boundary is considered. For a fixed channel i, the Cauchy–Schwarz inequality gives

$$\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X} + \mu_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \leq \frac{\|\mathbf{a}_i^{\mathrm{T}}\|\|\mathbf{X}\| + \mu_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \tag{A.3}$$

Substituting (A.2) into (A.3) gives

$$\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X} + \mu_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \leq \Gamma + \frac{\mu_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \tag{A.4}$$

The equality in (A.4) holds if and only if

$$\mathbf{X} = \Gamma \frac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \tag{A.5}$$

Therefore, the maximum value of $f(\mathbf{X})$ is

$$f(\mathbf{X})_{\max} = \max_{i=1,2,3}\left(\Gamma + \frac{\mu_i}{\|\mathbf{a}_i^{\mathrm{T}}\|}\right) \tag{A.6}$$

In the LOS signal case, $\boldsymbol{\mu}_R = \mathbf{0}$, so that

$$\mu_i = 0 \tag{A.7}$$

Substituting (A.7) into (A.6) gives

$$f(\mathbf{X})_{\max} = \Gamma \tag{A.8}$$

Substituting (A.8) into the value relation gives the LOS upper boundary stated in (40). In the multipath signal case, the mean translation satisfies $\mu_1 < 0$, $\mu_2 = 0$, and $\mu_3 > 0$. Accordingly, the maximum channel is the L channel, and the multipath upper feature boundary is obtained by substituting the $i = 3$ branch of (A.6) into the value relation, as given in (42).

Next, the LOS lower boundary is derived. Introduce the auxiliary variable t as the upper bound of the three normalized channel values. The problem is to minimize the largest normalized channel value, which can be written as

$$\min_{\mathbf{X},t} t \tag{A.9}$$

The channel constraints are

$$\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} \leq t \tag{A.10}$$

The sphere constraint is

$$\mathbf{X}^{\mathrm{T}}\mathbf{X} = \Gamma^2 \tag{A.11}$$

Introduce the KKT multipliers λ_i for the i-th channel inequality constraint in (A.10), satisfying $\lambda_i \geq 0$, and introduce ρ as the Lagrange multiplier for the sphere equality constraint in (A.11). The Lagrangian function is

$$L = t + \sum_{i=1}^{3}\lambda_i\left(\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} - t\right) + \rho\left[\mathbf{X}^{\mathrm{T}}\mathbf{X} - \Gamma^2\right] \tag{A.12}$$

The multiplier condition for t is

$$1 - \sum_{i=1}^{3}\lambda_i = 0 \tag{A.13}$$

The multiplier condition for $\mathbf{X}$ is

$$\sum_{i=1}^{3}\lambda_i \frac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} + 2\rho\mathbf{X} = 0 \tag{A.14}$$

The complementary slackness condition is

$$\lambda_i\left(\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} - t\right) = 0 \tag{A.15}$$

According to the complementary slackness condition in (A.15), the active constraints can be divided into three cases. The first case is that only one channel constraint is active. If the i-th channel constraint is active, the direction that reduces this channel value is

$$\mathbf{X} = -\Gamma \frac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \tag{A.16}$$

Then the normalized value of the active channel is

$$\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} = -\Gamma \tag{A.17}$$

For an inactive channel j, $j \neq i$, its normalized value is

$$\frac{\mathbf{a}_j^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_j^{\mathrm{T}}\|} = -\Gamma \frac{\mathbf{a}_j^{\mathrm{T}}\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|\|\mathbf{a}_j^{\mathrm{T}}\|} \tag{A.18}$$

Thus, in the one-active-constraint case, the value of t is determined by the maximum among the active channel and the two inactive channels. This case does not give the minimum LOS boundary for the one-chip E/P/L covariance structure, because at least one inactive channel remains larger than the two-active E/L candidate derived below.

The second case is that two channel constraints are active. If the i-th and j-th channel constraints are active, then

$$\frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} = t \tag{A.19}$$

and

$$\frac{\mathbf{a}_j^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_j^{\mathrm{T}}\|} = t \tag{A.20}$$

The corresponding direction is

$$\mathbf{X} = -\Gamma \frac{\dfrac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_j}{\|\mathbf{a}_j^{\mathrm{T}}\|}}{\left\| \dfrac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_j}{\|\mathbf{a}_j^{\mathrm{T}}\|} \right\|} \tag{A.21}$$

The corresponding value of t is

$$t = -\frac{\Gamma}{\sqrt{2}} \sqrt{1 + \frac{\mathbf{a}_j^{\mathrm{T}}\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|\|\mathbf{a}_j^{\mathrm{T}}\|}} \tag{A.22}$$

However, a two-active candidate is feasible only when the remaining inactive channel also satisfies its inequality constraint. If the i-th and j-th channel constraints are active and the k-th channel constraint is inactive, substituting (A.21) gives

$$\frac{\mathbf{a}_k^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_k^{\mathrm{T}}\|} = -\Gamma \frac{\dfrac{\mathbf{a}_k^{\mathrm{T}}\mathbf{a}_i}{\|\mathbf{a}_k^{\mathrm{T}}\|\|\mathbf{a}_i^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_k^{\mathrm{T}}\mathbf{a}_j}{\|\mathbf{a}_k^{\mathrm{T}}\|\|\mathbf{a}_j^{\mathrm{T}}\|}}{\left\| \dfrac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_j}{\|\mathbf{a}_j^{\mathrm{T}}\|} \right\|} \tag{A.23}$$

The feasibility condition for the inactive channel is

$$\frac{\mathbf{a}_k^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_k^{\mathrm{T}}\|} \leq t \tag{A.24}$$

For the LOS covariance structure with one-chip E/P/L correlator spacing, the E/P and P/L active pairs do not give the minimum feasible value because the remaining L or E channel determines a larger maximum value. The E/L active pair satisfies (A.24) for the remaining P channel and gives the smallest feasible value among the KKT candidates. Therefore,

$$\frac{\mathbf{a}_1^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_1^{\mathrm{T}}\|} = t \tag{A.25}$$

At the same point, the L-channel constraint is also active:

$$\frac{\mathbf{a}_3^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_3^{\mathrm{T}}\|} = t \tag{A.26}$$

The corresponding direction of $\mathbf{X}$ is

$$\mathbf{X} = -\Gamma \frac{\dfrac{\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}}{\left\| \dfrac{\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} + \dfrac{\mathbf{a}_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \right\|} \tag{A.27}$$

The norm of the active direction is

$$\left\| \frac{\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} + \frac{\mathbf{a}_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \right\| = \sqrt{2\left(1 + \frac{\mathbf{a}_3^{\mathrm{T}}\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|\|\mathbf{a}_3^{\mathrm{T}}\|}\right)} \tag{A.28}$$

Substituting (A.28) into (A.27) gives

$$t_{\min} = -\frac{\Gamma}{\sqrt{2}} \sqrt{1 + \frac{\mathbf{a}_3^{\mathrm{T}}\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|\|\mathbf{a}_3^{\mathrm{T}}\|}} \tag{A.29}$$

The third case is that all three channel constraints are active. In this case,

$$\frac{\mathbf{a}_1^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_1^{\mathrm{T}}\|} = t \tag{A.30}$$

Meanwhile,

$$\frac{\mathbf{a}_2^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_2^{\mathrm{T}}\|} = t \tag{A.31}$$

and

$$\frac{\mathbf{a}_3^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_3^{\mathrm{T}}\|} = t \tag{A.32}$$

Equivalently, in terms of the original correlation-value vector, the E and P normalized channel values are equal:

$$\frac{R_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} = \frac{R_2}{\|\mathbf{a}_2^{\mathrm{T}}\|} \tag{A.33}$$

The P and L normalized channel values are also equal:

$$\frac{R_2}{\|\mathbf{a}_2^{\mathrm{T}}\|} = \frac{R_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \tag{A.34}$$

Therefore, the three correlation components can be written as

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = t \begin{bmatrix} \|\mathbf{a}_1^{\mathrm{T}}\| \\ \|\mathbf{a}_2^{\mathrm{T}}\| \\ \|\mathbf{a}_3^{\mathrm{T}}\| \end{bmatrix} \tag{A.35}$$

Substituting (A.35) into the LOS ellipsoid gives

$$t^2 \begin{bmatrix} \|\mathbf{a}_1^{\mathrm{T}}\| & \|\mathbf{a}_2^{\mathrm{T}}\| & \|\mathbf{a}_3^{\mathrm{T}}\| \end{bmatrix} \mathbf{\Sigma}_R^{-1} \begin{bmatrix} \|\mathbf{a}_1^{\mathrm{T}}\| \\ \|\mathbf{a}_2^{\mathrm{T}}\| \\ \|\mathbf{a}_3^{\mathrm{T}}\| \end{bmatrix} = \Gamma^2 \tag{A.36}$$

Taking the negative root gives the three-active-constraint candidate

$$t_{1,2,3} = -\Gamma \left(\begin{bmatrix} \|\mathbf{a}_1^{\mathrm{T}}\| & \|\mathbf{a}_2^{\mathrm{T}}\| & \|\mathbf{a}_3^{\mathrm{T}}\| \end{bmatrix} \mathbf{\Sigma}_R^{-1} \begin{bmatrix} \|\mathbf{a}_1^{\mathrm{T}}\| \\ \|\mathbf{a}_2^{\mathrm{T}}\| \\ \|\mathbf{a}_3^{\mathrm{T}}\| \end{bmatrix} \right)^{-\frac{1}{2}} \tag{A.37}$$

The comparison between (A.37) and (A.29) should be understood from the minimization of t. Equation (A.29) is not only a point satisfying the E/L active constraints; it is the minimum value of t under the E/L active constraints in (A.25) and (A.26) and the sphere constraint in (A.11). If the three-active candidate in (A.37) gave a smaller value than (A.29), then the same point would still satisfy the E/L active constraints and the sphere constraint, because the three-active case only adds the middle-channel equality in (A.31). This would contradict the definition of $t_{\min}$ obtained in (A.29). Therefore,

$$t_{1,2,3} \geq t_{\min} \tag{A.38}$$

Thus, the three-active candidate does not give a smaller lower boundary, and the LOS lower boundary is determined by the E/L two-active result in (A.29). Finally, substituting (A.29) into the value relation gives the LOS lower boundary stated in (41).

For the multipath lower feature range, the translation-dominant case is considered first. Let

$$t = \max\left(\frac{R_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}, \frac{R_2}{\|\mathbf{a}_2^{\mathrm{T}}\|}, \frac{R_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right) \tag{A.39}$$

Under the translation-dominant approximation, the ellipsoid boundary is written in the form already stated in (43). The smallest admissible value of $f(\mathbf{X})$ is equivalent to the smallest t for which the following set intersects (43):

$$Q(t) = \left\{\left(\frac{R_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}, \frac{R_2}{\|\mathbf{a}_2^{\mathrm{T}}\|}, \frac{R_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right) \middle| \frac{R_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} \leq t,\ i = 1, 2, 3\right\} \tag{A.40}$$

For $t < \frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}$, the squared distance from the sphere center to $Q(t)$ is

$$d^2(t) = \left(\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} - t\right)^2 + t^2 + \left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - t\right)^2 \tag{A.41}$$

For $\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} \leq t < 0$, the squared distance is

$$d^2(t) = t^2 + \left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - t\right)^2 \tag{A.42}$$

For $0 \leq t < \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}$, the squared distance is

$$d^2(t) = \left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - t\right)^2 \tag{A.43}$$

For $t \geq \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}$, the squared distance is

$$d^2(t) = 0 \tag{A.44}$$

Since $d(t)$ is continuous and monotonically decreasing as t increases, the minimum feasible t is obtained from

$$d(t) = \Gamma \tag{A.45}$$

When the solution lies in $0 \leq t < \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}$, from (A.43) and (A.45), one obtains

$$t = \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - \Gamma \tag{A.46}$$

The corresponding lower feature boundary is obtained by substituting (A.46) into the value relation, which gives the first branch of (44) under the corresponding condition shown there.

When the solution lies in $\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} \leq t < 0$, from (A.42) and (A.45), one obtains

$$t = \frac{1}{2}\left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - \sqrt{\frac{2\lambda_D^2 R_{P_D}^2 \gamma_{GS}^2}{\sigma_n^2} - \frac{\mu_3^2}{\|\mathbf{a}_3^{\mathrm{T}}\|^2}}\right) \tag{A.47}$$

The corresponding lower feature boundary is obtained by substituting (A.47) into the value relation, which gives the second branch of (44) under the corresponding condition shown there.

When the radius in (A.2) is larger, the multipath translation is no longer dominant enough for the independent-coordinate approximation in (43) to remain accurate. The multipath ellipsoid then approaches the LOS-type boundary, and the lower value is approximated by

$$t \approx -\frac{\Gamma}{\sqrt{2}}\sqrt{1 + \frac{\mathbf{a}_3^{\mathrm{T}}\mathbf{a}_1}{\|\mathbf{a}_1^{\mathrm{T}}\|\|\mathbf{a}_3^{\mathrm{T}}\|}} \tag{A.48}$$

Substituting (A.48) into the value relation gives the third branch of (44) under the corresponding condition shown there. Thus, the three distance regimes above lead to the multipath lower boundary in (44).

B. Derivation of the Saturation Threshold

This appendix derives the saturation-feature boundaries used in Section III. The saturation expression itself has already been given in (45), so only the optimization steps leading to the stated boundaries are shown here. The symbol Γ used below denotes the standard-space ellipsoid radius defined in (A.1). For the LOS signal case, $\boldsymbol{\mu}_R = \mathbf{0}$, and

$$\frac{R_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} = \frac{\mathbf{a}_i^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_i^{\mathrm{T}}\|} \tag{B.1}$$

The lower LOS boundary is obtained when the three normalized channel values are equal, as stated in (47). Therefore, the minimum LOS saturation is the value already given in (48).

The upper LOS boundary is determined by the maximum contrast between one minimum channel and one maximum channel. Suppose the i-th normalized channel is the minimum and the j-th normalized channel is the maximum. The corresponding saturation is

$$S_{i,j}(\mathbf{X}) = \frac{\left(\dfrac{\mathbf{a}_j^{\mathrm{T}}}{\|\mathbf{a}_j^{\mathrm{T}}\|} - \dfrac{\mathbf{a}_i^{\mathrm{T}}}{\|\mathbf{a}_i^{\mathrm{T}}\|}\right)\mathbf{X}}{3 + \dfrac{\mathbf{a}_j^{\mathrm{T}}\mathbf{X}}{\|\mathbf{a}_j^{\mathrm{T}}\|}} \tag{B.2}$$

Define the normalized channel correlation for this pair as

$$c_{ij} = \frac{\mathbf{a}_j^{\mathrm{T}}\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|\|\mathbf{a}_j^{\mathrm{T}}\|} \tag{B.3}$$

The stationary point of $S_{i,j}(\mathbf{X})$ under $\mathbf{X}^{\mathrm{T}}\mathbf{X} = \Gamma^2$ lies in the plane spanned by the two normalized channel directions. Let

$$\mathbf{e}_1 = \frac{\mathbf{a}_j}{\|\mathbf{a}_j^{\mathrm{T}}\|} \tag{B.4}$$

The second orthonormal basis vector in this plane is

$$\mathbf{e}_2 = \frac{\dfrac{\mathbf{a}_j}{\|\mathbf{a}_j^{\mathrm{T}}\|} - \dfrac{\mathbf{a}_i}{\|\mathbf{a}_i^{\mathrm{T}}\|} - (1 - c_{ij})\,\mathbf{e}_1}{\sqrt{1 - c_{ij}^2}} \tag{B.5}$$

At the stationary point, let k_1 and k_2 denote the coordinates of $\mathbf{X}$ in this local orthonormal basis. Then $\mathbf{X}$ can be written as

$$\mathbf{X} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 \tag{B.6}$$

The sphere constraint gives

$$k_1^2 + k_2^2 = \Gamma^2 \tag{B.7}$$

Substituting (B.6) into (B.2) gives

$$S_{i,j}(k_1, k_2) = \frac{(1 - c_{ij})\,k_1 + \sqrt{1 - c_{ij}^2}k_2}{3 + k_1} \tag{B.8}$$

Set

$$k_1 = \Gamma\cos\theta \tag{B.9}$$

and

$$k_2 = \Gamma\sin\theta \tag{B.10}$$

Then

$$S_{i,j}(\theta) = \frac{(1 - c_{ij})\,\Gamma\cos\theta + \sqrt{1 - c_{ij}^2}\Gamma\sin\theta}{3 + \Gamma\cos\theta} \tag{B.11}$$

Equivalently, introduce an auxiliary angle ζ satisfying

$$\sin\zeta = \sqrt{\frac{1 - c_{ij}}{2}} \tag{B.12}$$

and

$$\cos\zeta = \sqrt{\frac{1 + c_{ij}}{2}} \tag{B.13}$$

Let $\phi = \theta + \zeta$. Differentiating $S_{i,j}$ with respect to ϕ and setting the derivative to zero gives

$$3\cos\phi + \Gamma\cos\zeta = 0 \tag{B.14}$$

Therefore,

$$\cos\phi = -\frac{\Gamma}{3}\sqrt{\frac{1 + c_{ij}}{2}} \tag{B.15}$$

and the maximizing branch uses

$$\sin\phi = \sqrt{1 - \frac{\Gamma^2\,(1 + c_{ij})}{18}} \tag{B.16}$$

Substituting (B.15) and (B.16) into (B.11) gives the pairwise expression contained in the LOS upper boundary formula (46). Thus, the LOS upper saturation boundary is obtained by choosing the valid maximum over all ordered channel pairs. For the one-chip E/P/L covariance structure, the largest contrast is obtained by the E/L pair. Here B_n denotes the equivalent noise bandwidth of the PLL, T denotes the coherent integration time, and B_nT is the corresponding dimensionless bandwidth–integration product inherited from the covariance model in Part I. In the limiting case $B_nT \to 0$, this pair has

$$c_{13} = 0 \tag{B.17}$$

Substituting (B.17) into the pairwise expression in (46) gives the corresponding one-chip limiting form.

For the multipath signal case, the mean translation makes the E channel smaller and the L channel larger. Under the translation-dominant approximation, the ellipsoid is the same as (43). In this case, the main maximum/minimum channel pair is L/E, and the corresponding saturation expression is given in (58).

Let

$$x = \frac{R_1 - \mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} \tag{B.18}$$

Let

$$y = \frac{R_2 - \mu_2}{\|\mathbf{a}_2^{\mathrm{T}}\|} \tag{B.19}$$

Let

$$z = \frac{R_3 - \mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \tag{B.20}$$

Then the ellipsoid in (43) becomes

$$x^2 + y^2 + z^2 = \Gamma^2 \tag{B.21}$$

For maximizing S, the widest feasible range of x and z is obtained at

$$y = 0 \tag{B.22}$$

Set

$$x = \Gamma\cos\theta \tag{B.23}$$

and

$$z = \Gamma\sin\theta \tag{B.24}$$

Substitution into the L/E saturation expression in (58) gives

$$S(\theta) = \frac{\dfrac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} - \dfrac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} + \Gamma\,(\sin\theta - \cos\theta)}{3 + \dfrac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} + \Gamma\sin\theta} \tag{B.25}$$

The feasible angle must satisfy the E-channel minimum and L-channel maximum ordering:

$$\Gamma\cos\theta + \frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|} \le 0 \tag{B.26}$$

It must also satisfy

$$\Gamma\sin\theta + \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \ge 0 \tag{B.27}$$

Differentiating (B.25) gives the stationary-point condition

$$\left(3 + \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right)\sin\theta + \left(3 + \frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}\right)\cos\theta + \Gamma = 0 \tag{B.28}$$

Define

$$\Delta_S = \sqrt{\left(3 + \frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right)^2 + \left(3 + \frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}\right)^2 - \Gamma^2} \tag{B.29}$$

Solving (B.28) gives the two stationary angles stated in (52) and (53). The E- and L-channel boundary equalities in (B.26) and (B.27) give the boundary angles stated in (54) and (55), respectively. Therefore, when the square root in (B.29) is real, substituting these four angles into $S(\theta)$

gives the multipath upper saturation boundary in (56). The corresponding condition is

$$\Gamma \leq \sqrt{\left(3+\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right)^2+\left(3+\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}\right)^2} \tag{B.30}$$

When (B.30) is not satisfied, the translation-dominant approximation is no longer used, and the multipath upper boundary is approximated by the LOS-type expression in (57).

Finally, consider the lower multipath saturation boundary. From (58), reducing S is equivalent to reducing the difference between the L and E normalized channel values. In terms of x and z, the equal-channel plane is

$$z-x=-\left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}-\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}\right) \tag{B.31}$$

The normal direction of this plane is $[-1,0,1]^{\mathrm{T}}$. If the sphere in (B.21) does not intersect this plane, the closest point on the sphere is reached along the negative normal direction:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{\Gamma}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{B.32}$$

Thus,

$$\frac{R_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}=\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}+\frac{\Gamma}{\sqrt{2}} \tag{B.33}$$

and

$$\frac{R_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}=\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}-\frac{\Gamma}{\sqrt{2}} \tag{B.34}$$

Substituting (B.33) and (B.34) into (58) gives the lower multipath saturation boundary in (59). This expression holds when

$$\Gamma<\frac{1}{\sqrt{2}}\left(\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}-\frac{\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}\right) \tag{B.35}$$

If the sphere intersects the equal-channel plane, the saturation can approach zero, which gives the approximation in (60). The derivation above leads to the saturation boundaries in (46), (56), (57), (59), and (60).

C. Derivation of the Hue Distribution

This appendix provides the derivation details for the hue distribution in Section III without repeating the formulas already stated in the main text. The Cartesian hue components Y_1 and Y_2 are defined in (62) and (63), and their covariance matrix is given in (64). Hence, only the transformation from the bivariate Gaussian vector to its angular distribution is shown here.

For a general bivariate Gaussian vector, the angular density can be obtained by integrating the joint density f_{Y_1,Y_2} in polar coordinates. Here β denotes a possible value of the hue angle H:

$$f_H(\beta)=\int_0^{\infty} r f_{Y_1,Y_2}(r\cos\beta, r\sin\beta)\,dr \tag{C.1}$$

When $\mu_{Y_1}=0$ and $\mu_{Y_2}=0$, evaluating (C.1) gives the LOS angular density in (65). Its denominator is denoted by $D(\beta)$ in (66), and the auxiliary angle δ is defined in (68). Using the double-angle identities, (66) is transformed into (67). Since $f_H(\beta)$ is inversely proportional to $D(\beta)$, the minimum and maximum denominator conditions stated in Section III lead to the general peak and trough locations in (69) and (70).

For the one-chip E/P/L correlator spacing, $\Delta/T_c=1$ and $R_{E_D}\approx R_{L_D}\approx 1/2$, $R_{P_D}\approx 1$. Here Δ is the early–late correlator spacing, T_c is the chip duration, B_n is the equivalent noise bandwidth of the PLL, T is the coherent integration time, and R_{E_D}, R_{P_D}, and R_{L_D} are the LOS ideal ACF values at the E, P, and L correlator positions, respectively. Substitution into the LOS covariance expression gives

$$\sigma_{Y_1}^2=\left(\frac{255}{3}\right)^2\left[\frac{3}{2}-\frac{\sqrt{2(1+2B_nT)(2+B_nT)}+4B_nT}{4(2+B_nT)}\right] \tag{C.2}$$

Similarly,

$$\sigma_{Y_2}^2=2\left(\frac{255\sqrt{3}}{6}\right)^2\left[1-\frac{\sqrt{2(1+2B_nT)(2+B_nT)}}{2(2+B_nT)}\right] \tag{C.3}$$

The covariance is

$$\sigma_{Y_1Y_2}=\sqrt{3}\left(\frac{255}{6}\right)^2\frac{\sqrt{2(1+2B_nT)(2+B_nT)}-2B_nT}{2+B_nT} \tag{C.4}$$

Substituting (C.2)–(C.4) into (68) gives

$$\tan\delta=-\sqrt{3} \tag{C.5}$$

Moreover,

$$\sigma_{Y_1}^2-\sigma_{Y_2}^2>0 \tag{C.6}$$

Thus, δ lies in the second quadrant, which yields the one-chip value stated in Section III. Substituting this value into (69) and (70) gives the one-chip LOS peak and trough positions stated in (71) and (72).

For multipath signals, the nonzero mean translation is dominant under high signal-to-noise conditions. The Taylor expansion used in (73) follows from the partial derivatives

$$\frac{\partial H}{\partial Y_1}=-\frac{Y_2}{Y_1^2+Y_2^2} \tag{C.7}$$

and

$$\frac{\partial H}{\partial Y_2}=\frac{Y_1}{Y_1^2+Y_2^2} \tag{C.8}$$

From (62)–(63) and $\mu_2=0$, the mean components are

$$\mu_{Y_1}=\frac{255}{6}\left(\frac{2\mu_1}{\|\mathbf{a}_1^{\mathrm{T}}\|}-\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|}\right) \tag{C.9}$$

and

$$\mu_{Y_2}=-\frac{255\sqrt{3}}{6}\frac{\mu_3}{\|\mathbf{a}_3^{\mathrm{T}}\|} \tag{C.10}$$

Substituting (C.9) and (C.10) into (73) gives the mean direction in (74) and the third-quadrant interval in (75). Taking the variance of the linearized expression in (73) gives (76).

D. Calculation of the Overlap Coefficient

The overlap coefficient used in Section IV measures the common probability mass shared by the theoretical and simulated hue distributions. For the theoretical hue probability density function $f_H^{\mathrm{th}}(\beta)$ and the simulated hue probability density function $f_H^{\mathrm{sim}}(\beta)$, the continuous OVL is defined as [1]–[3]

$$\mathrm{OVL} = \int_0^{2\pi} \min\left[f_H^{\mathrm{th}}(\beta), f_H^{\mathrm{sim}}(\beta)\right] \, d\beta. \tag{D.1}$$

In the numerical implementation, the hue domain $[0, 2\pi)$ is divided into K bins. Let P_k^{th} and P_k^{sim} denote the probability masses of the theoretical and simulated hue distributions in the k-th bin, respectively. The discrete OVL used for Fig. 6 is then calculated as

$$\mathrm{OVL} \approx \sum_{k=1}^{K} \min\left(P_k^{\mathrm{th}}, P_k^{\mathrm{sim}}\right). \tag{D.2}$$

By construction, $0 \leq \mathrm{OVL} \leq 1$. A value close to one means that the theoretical and simulated hue distributions have a large common probability mass, whereas a value close to zero indicates weak distributional agreement.

References

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