Supplementary Material for 3D Gaussian Distribution and RGB Reconstruction based GNSS Multipath Detection – Part I

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This supplementary material provides the detailed derivations moved from the main manuscript for the page-compressed TAES submission. References to equations not defined in this file refer to the main Part I manuscript; compile the main manuscript first if cross-document reference resolution is required.

Supplementary Material

A. Linearization of the normalized correlation values

In this appendix, the normalized correlation values are linearized, specifically as follows:

Under the assumption of ideal front-end filtering, the normalized variance of the real and imaginary parts of η is given by [1]:

$$\sigma_n^2 = \frac{B_{IF}}{C/N_0 N_s} = \frac{f_s}{2C/N_0 N_s} = \frac{1}{2TC/N_0} \tag{A.1}$$

where B_{IF} is the intermediate frequency bandwidth, N_0 is the noise density, N_s is the sampling quantity, f_s is the sampling frequency.

σ_n^2 does not depend on the correlator. It is noted that all the parameters in (A.1) are known during the simulation process.

Setting the correlator interval to 1 *chip*, using the Cholesky decomposition:

$$\begin{bmatrix}\eta_{EQ}\\ \eta_{PQ}\\ \eta_{LQ}\end{bmatrix} = \sigma_n \begin{bmatrix} 1 & 0 & 0 \\ 1-\dfrac{\Delta}{2T_c} & \sqrt{\dfrac{\Delta}{T_c}\left(1-\dfrac{\Delta}{4T_c}\right)} & 0 \\ 1-\dfrac{\Delta}{T_c} & \dfrac{\sqrt{\Delta}\,(2T_c-\Delta)}{T_c\sqrt{4T_c-\Delta}} & \dfrac{\sqrt{2\Delta T_c\,(2T_c-\Delta)}}{T_c\sqrt{(4T_c-\Delta)}} \end{bmatrix} \begin{bmatrix}\nu_{1Q}\\ \nu_{2Q}\\ \nu_{3Q}\end{bmatrix} \tag{A.2}$$

$$\begin{bmatrix}\eta_{EI}\\ \eta_{PI}\\ \eta_{LI}\end{bmatrix} = \sigma_n \begin{bmatrix} 1 & 0 & 0 \\ 1-\dfrac{\Delta}{2T_c} & \sqrt{\dfrac{\Delta}{T_c}\left(1-\dfrac{\Delta}{4T_c}\right)} & 0 \\ 1-\dfrac{\Delta}{T_c} & \dfrac{\sqrt{\Delta}\,(2T_c-\Delta)}{T_c\sqrt{4T_c-\Delta}} & \dfrac{\sqrt{2\Delta T_c\,(2T_c-\Delta)}}{T_c\sqrt{(4T_c-\Delta)}} \end{bmatrix} \begin{bmatrix}\nu_{1I}\\ \nu_{2I}\\ \nu_{3I}\end{bmatrix} \tag{A.3}$$

where $\nu_1 = \nu_{1I} + j\nu_{1Q}$, $\nu_2 = \nu_{2I} + j\nu_{2Q}$ and $\nu_3 = \nu_{3I} + j\nu_{3Q}$ are three complex Gaussian random variables with independent real and imaginary parts with unit variance. ν_{1Q}, ν_{2Q} and ν_{3Q} are the imaginary parts, ν_{1I}, ν_{2I} and ν_{3I} are the real parts, respectively.

By substituting (A.2)–(A.3) into normalized correlation values and performing a first-order multivariate Taylor expansion, with the expansion point $\mathbf{V} = [\nu_1, \nu_2, \nu_3]^{\mathrm{T}} = \mathbf{0}_{3\times1}$, the linearized correlation values can be obtained.

For LOS signals:

$$\begin{cases} \tilde{EQ}_D \approx \tilde{EQ}_D|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{EQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.4)} \\ \tilde{PQ}_D \approx \tilde{PQ}_D|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{PQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.5)} \\ \tilde{LQ}_D \approx \tilde{LQ}_D|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{LQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.6)} \end{cases}$$

where

$$\begin{cases} \tilde{EQ}_D|_{\mathbf{V}=\mathbf{0}} = \dfrac{R_{E_D}}{R_{P_D}}\sin\delta\varphi_D & \text{(A.7)} \\ \tilde{PQ}_D|_{\mathbf{V}=\mathbf{0}} = \sin\delta\varphi_D & \text{(A.8)} \\ \tilde{LQ}_D|_{\mathbf{V}=\mathbf{0}} = \dfrac{R_{L_D}}{R_{P_D}}\sin\delta\varphi_D & \text{(A.9)} \end{cases}$$

and

$$\begin{cases} \mathbf{V}\dfrac{\partial \tilde{EQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} = \dfrac{\left(R_{P_D}^2 + R_{E_D}^2\sin^2\delta\varphi_D - \left(2-\dfrac{\Delta}{T_c}\right)R_{E_D}R_{P_D}\sin^2\delta\varphi_D\right)}{\lambda_D^2 R_{P_D}^4}\sigma_n^2 & \text{(A.10)} \\ \mathbf{V}\dfrac{\partial \tilde{PQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} = \dfrac{\cos^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^2}\sigma_n^2 & \text{(A.11)} \\ \mathbf{V}\dfrac{\partial \tilde{LQ}_D}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} = \dfrac{\left(R_{P_D}^2 + R_{L_D}^2\sin^2\delta\varphi_D - \left(2-\dfrac{\Delta}{T_c}\right)R_{P_D}R_{L_D}\sin^2\delta\varphi_D\right)}{\lambda_D^2 R_{P_D}^4}\sigma_n^2 & \text{(A.12)} \end{cases}$$

Thus, the specific form of (49)–(51) can be expressed as follows.

$$\begin{cases} \tilde{EQ}_D \sim \mathcal{N}\left(\dfrac{R_{E_D}}{R_{P_D}}\sin\delta\varphi_D, \left(\dfrac{R_{P_D}^2 + R_{E_D}^2\sin^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^4} - \dfrac{\left(2-\dfrac{\Delta}{T_c}\right)R_{E_D}R_{P_D}\sin^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^4}\right)\sigma_n^2\right) & \text{(A.13)} \\ \tilde{PQ}_D \sim \mathcal{N}\left(\sin\delta\varphi_D, \dfrac{\cos^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^2}\sigma_n^2\right) & \text{(A.14)} \\ \tilde{LQ}_D \sim \mathcal{N}\left(\dfrac{R_{L_D}}{R_{P_D}}\sin\delta\varphi_D, \left(\dfrac{R_{P_D}^2 + R_{L_D}^2\sin^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^4} - \dfrac{\left(2-\dfrac{\Delta}{T_c}\right)R_{P_D}R_{L_D}\sin^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^4}\right)\sigma_n^2\right) & \text{(A.15)} \end{cases}$$

Based on formulas (3)–(14) and (26)–(38), it can be observed that the NLOS signal is formally consistent with the LOS signal. So, similar to LOS signals, normalized correlation values in NLOS signal case can be represented as follows.

$$\begin{cases} \tilde{EQ}_N \approx \tilde{EQ}_N|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{EQ}_N}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.16)} \\ \tilde{PQ}_N \approx \tilde{PQ}_N|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{PQ}_N}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.17)} \\ \tilde{LQ}_N \approx \tilde{LQ}_N|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\dfrac{\partial \tilde{LQ}_N}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.18)} \end{cases}$$

Thus, the specific form of (52)–(54) can be expressed as:

$$\begin{cases} \tilde{EQ}_N \sim \mathcal{N}\left(\frac{R_{E_N}}{R_{P_N}}\sin\delta\varphi_N, \begin{pmatrix} \frac{R_{P_N}^2 + R_{E_N}^2\sin^2\delta\varphi_N}{\lambda_N^2 R_{P_N}^4} \\ -\frac{\left(2-\frac{\Delta}{T_c}\right)R_{E_N}R_{P_N}\sin^2\delta\varphi_N}{\lambda_N^2 R_{P_N}^4} \end{pmatrix}\sigma_n^2\right) & \text{(A.19)} \\ \tilde{PQ}_N \sim \mathcal{N}\left(\sin\delta\varphi_N, \frac{\cos^2\delta\varphi_N}{\lambda_N^2 R_{P_N}^2}\sigma_n^2\right) & \text{(A.20)} \\ \tilde{LQ}_N \sim \mathcal{N}\left(\frac{R_{L_N}}{R_{P_N}}\sin\delta\varphi_N, \begin{pmatrix} \frac{R_{P_N}^2 + R_{L_N}^2\sin^2\delta\varphi_N}{\lambda_N^2 R_{P_N}^4} \\ -\frac{\left(2-\frac{\Delta}{T_c}\right)R_{P_N}R_{L_N}\sin^2\delta\varphi_N}{\lambda_N^2 R_{P_N}^4} \end{pmatrix}\sigma_n^2\right) & \text{(A.21)} \end{cases}$$

For multipath signals, it is necessary to consider the impact of both LOS and NLOS signals simultaneously, the normalized correlation values can be expressed as follows.

$$\begin{cases} \tilde{EQ}_M \approx \tilde{EQ}_M|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\frac{\partial \tilde{EQ}_M}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.22)} \\ \tilde{PQ}_M \approx \tilde{PQ}_M|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\frac{\partial \tilde{PQ}_M}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.23)} \\ \tilde{LQ}_M \approx \tilde{LQ}_M|_{\mathbf{V}=\mathbf{0}} + \mathbf{V}\frac{\partial \tilde{LQ}_M}{\partial \mathbf{V}^{\mathrm{T}}}|_{\mathbf{V}=\mathbf{0}} & \text{(A.24)} \end{cases}$$

Thus, the specific form of (55)–(57) can be represented as:

$$\begin{cases} \tilde{EQ}_M \sim \mathcal{N}\left(\frac{\kappa_{EQ}}{\sqrt{\lambda_M}}, \begin{pmatrix} \lambda_M^2 + \kappa_{EQ}^2\kappa_{PQ}^2 + \kappa_{EQ}^2\kappa_{PI}^2 \\ -\left(2-\frac{\Delta}{T_c}\right)\lambda_M\kappa_{EQ}\kappa_{PQ} \end{pmatrix}\frac{\sigma_n^2}{\lambda_M^3}\right) & \text{(A.25)} \\ \tilde{PQ}_M \sim \mathcal{N}\left(\frac{\kappa_{PQ}}{\sqrt{\lambda_M}}, \frac{\left(\lambda_M - \kappa_{PQ}^2\right)^2 + \kappa_{PQ}^2\kappa_{PI}^2}{\lambda_M^3}\sigma_n^2\right) & \text{(A.26)} \\ \tilde{LQ}_M \sim \mathcal{N}\left(\frac{\kappa_{LQ}}{\sqrt{\lambda_M}}, \begin{pmatrix} \lambda_M^2 + \kappa_{LQ}^2\kappa_{PQ}^2 + \kappa_{LQ}^2\kappa_{PI}^2 \\ -\left(2-\frac{\Delta}{T_c}\right)\lambda_M\kappa_{LQ}\kappa_{PQ} \end{pmatrix}\frac{\sigma_n^2}{\lambda_M^3}\right) & \text{(A.27)} \end{cases}$$

where

$$\begin{cases} \kappa_{EQ} = \lambda_D R_{E_D}\sin\delta\varphi_D + \lambda_N R_{E_N}\sin\delta\varphi_N & \text{(A.28)} \\ \kappa_{PQ} = \lambda_D R_{P_D}\sin\delta\varphi_D + \lambda_N R_{P_N}\sin\delta\varphi_N & \text{(A.29)} \\ \kappa_{LQ} = \lambda_D R_{L_D}\sin\delta\varphi_D + \lambda_N R_{L_N}\sin\delta\varphi_N & \text{(A.30)} \\ \kappa_{PI} = \lambda_D R_{P_D}\cos\delta\varphi_D + \lambda_N R_{P_N}\cos\delta\varphi_N & \text{(A.31)} \end{cases}$$

B. Derivation of steady-state multipath error

In this appendix, the focus is on deriving the steady-state multipath errors processed by the code loop discriminator and the carrier loop discriminator. The Early-Minus-Late Power (EMLP) discriminator, which is widely used with a chip spacing of 1 *chip*, is employed as the discriminator in the code tracking loop. The specific phase discrimination formula for the EMLP discriminator is given by:

$$\delta\hat{\tau} = \frac{1}{2}\frac{EI^2 + EQ^2 - LI^2 - LQ^2}{EI^2 + EQ^2 + LI^2 + LQ^2} \tag{B.1}$$

Let $\delta\hat{\tau} = 0$ and ignore the influence of noise, that is:

$$EI^2 + EQ^2 = LI^2 + LQ^2 \tag{B.2}$$

The relationship between code loop multipath steady-state error and multipath parameters can be obtained by (B.2).

For LOS signals, substituting (3) and (5) into (B.2), it can be obtained as follows.

$$\begin{aligned} &(\lambda_D R_{E_D}\cos\delta\varphi_D)^2 + (\lambda_D R_{E_D}\sin\delta\varphi_D)^2 \\ &= (\lambda_D R_{L_D}\cos\delta\varphi_D)^2 + (\lambda_D R_{L_D}\sin\delta\varphi_D)^2 \end{aligned} \tag{B.3}$$

Thus,

$$\delta\tau_D = 0 \tag{B.4}$$

For NLOS signals, substituting (9) and (11) into (B.2), it can be obtained as follows.

$$\begin{aligned} &(\lambda_N R_{E_N}\cos\delta\varphi_N)^2 + (\lambda_N R_{E_N}\sin\delta\varphi_N)^2 \\ &= (\lambda_N R_{L_N}\cos\delta\varphi_N)^2 + (\lambda_N R_{L_N}\sin\delta\varphi_N)^2 \end{aligned} \tag{B.5}$$

Thus,

$$\delta\tau_N = 0 \tag{B.6}$$

For multipath signals, substituting (15) and (17) into (B.2), it can be obtained as follows.

$$\begin{aligned} &(\lambda_D R_{E_D})^2 + (\lambda_N R_{E_N})^2 + 2\lambda_D R_{E_D}\lambda_N R_{E_N}\cos(\delta\varphi_N - \delta\varphi_D) \\ &= (\lambda_D R_{L_D})^2 + (\lambda_N R_{L_N})^2 + 2\lambda_D R_{L_D}\lambda_N R_{L_N}\cos(\delta\varphi_N - \delta\varphi_D) \end{aligned} \tag{B.7}$$

when $|\delta\tau_N| \le \frac{1}{2}$, i.e. short multipath:

$$\delta\tau_D = \frac{\lambda_N^2 + \lambda_D\lambda_N\cos(\delta\varphi_N - \delta\varphi_D)}{\lambda_D^2 + \lambda_D\lambda_N\cos(\delta\varphi_N - \delta\varphi_D)}\delta\tau_N \tag{B.8}$$

The Costas loop is employed as the phase-locking loop. The phase discrimination function of the Costas loop is given by:

$$\delta\hat{\varphi} = \arctan\frac{PQ}{PI} \tag{B.9}$$

Let $\delta\hat{\varphi} = 0$ and ignore the influence of noise, that is:

$$PQ = 0 \tag{B.10}$$

The relationship between phase loop multipath steady-state error and multipath parameters can be obtained by (B.10).

For LOS signals, substituting (4) into (B.10), it can be obtained as follows.

$$\lambda_D R_{P_D}\sin\delta\varphi_D = 0 \tag{B.11}$$

Thus,

$$\delta\varphi_D = 0 \tag{B.12}$$

For NLOS signals, substituting (10) into (B.10), it can be obtained as follows.

$$\lambda_N R_{P_N}\sin\delta\varphi_N = 0 \tag{B.13}$$

Thus,

$$\delta\varphi_N = 0 \tag{B.14}$$

For multipath signals, substituting (16) into (B.10), it can be obtained as follows.

$$\lambda_D R_{P_D}\sin\delta\varphi_D + \lambda_N R_{P_N}\sin\delta\varphi_N = 0 \tag{B.15}$$

Thus,

$$\delta\varphi_D = \arctan\left(-\frac{\lambda_N R_{P_N}\sin(\delta\varphi_N - \delta\varphi_D)}{\lambda_N R_{P_N}\cos(\delta\varphi_N - \delta\varphi_D) + \lambda_D R_{P_D}}\right) \tag{B.16}$$

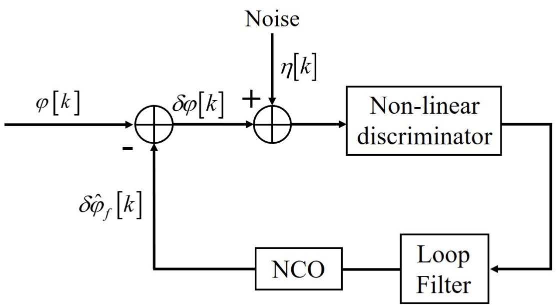

Fig. C.1. Simulation model of the phase tracking loop

C. Derivation of correlation value distribution

In this appendix, the focus is on deriving the distribution of correlation values following processing through the code loop discriminator and the carrier loop discriminator.

Under idealized noiseless conditions, phase discriminator processing could yield a perfect phase difference estimation ($\delta\hat{\varphi} = 0$). However, in practical PLL operation, noise introduces non-negligible fluctuations even under phase lock, as shown in Fig. C.1. These fluctuations affect correlation value distributions, especially the variance, thereby necessitating the inclusion of post-PLL estimated phase error $\delta\hat{\varphi}_f$ in the distribution derivation.

For LOS signals, under small error conditions, Phase error output $\delta\hat{\varphi}$ of phase discriminator can be expressed as follows.

$$\delta\hat{\varphi} = \arctan\frac{PQ}{PI} \approx \frac{PQ}{PI} = \frac{\lambda_D R_{P_D}\sin\delta\varphi_D + \eta_{PQ}}{\lambda_D R_{P_D}\cos\delta\varphi_D + \eta_{PI}} \tag{C.1}$$

In the case of steady-state error ($\delta\varphi_D \approx 0$, $\delta\tau_D \approx 0$). $\delta\hat{\varphi}$ can be written as:

$$\delta\hat{\varphi} \approx \delta\varphi_D + \frac{\eta_{PQ}}{\lambda_D R_{P_D}} \tag{C.2}$$

Thus, $\delta\hat{\varphi}$ can be approximated by following a Gaussian distribution.

As $\delta\hat{\varphi}$ undergoes further processing through linear loop filters (standard in tracking loops) and numerically controlled oscillator (NCO), the phase error estimation of PLL output $\delta\hat{\varphi}_f$ retains Gaussian distribution:

$$\delta\hat{\varphi}_f \sim \mathcal{N}\left(0, \sigma_f^2\right) \tag{C.3}$$

where σ_f^2 is the variance of the phase error estimation.

For the phase-locked loop of the classical filter (e.g., PID controller), under steady-state conditions, σ_f^2 can be approximated as [2]–[6]:

$$\sigma_f^2 = 2B_n T\mathcal{D}\left(\delta\hat{\varphi}\right) \tag{C.4}$$

where B_n denotes the noise bandwidth.

Notably, following processing by the loop filter and the NCO, the variance $\mathcal{D}(\delta\hat{\varphi}_f)$ is determined by the filter parameters and exhibits a mathematically linear relationship with $\mathcal{D}(\delta\hat{\varphi})$ under typical conditions, while the noise-induced correlation between $\delta\hat{\varphi}_f$ and $\delta\hat{\varphi}$ is ignored.

Based on (A.13)–(A.15), it can be seen that normalized and linearized correlation value distribution parameters critically depend on the true phase difference $\delta\varphi_D$, yet practical implementations face fundamental constraints: true $\delta\varphi_D$ remains inaccessible due to measurement noise, necessitating substitution with estimated phase difference $\delta\hat{\varphi}_f$ for the distribution derivation. Thus, correlation value can be written as:

$$\begin{cases}
\tilde{EQ}_D \sim \mathcal{N}\left(\dfrac{R_{E_D}}{R_{P_D}}\sin\delta\hat{\varphi}_f, \left(\dfrac{R_{P_D}^2 + R_{E_D}^2\sin^2\delta\hat{\varphi}_f}{\lambda_D^2 R_{P_D}^4} - \dfrac{\left(2-\dfrac{\Delta}{T_c}\right)R_{E_D}R_{P_D}\sin^2\delta\hat{\varphi}_f}{\lambda_D^2 R_{P_D}^4}\right)\sigma_n^2\right) & \text{(C.5)}\\
\tilde{PQ}_D \sim \mathcal{N}\left(\sin^2\delta\hat{\varphi}_f, \dfrac{\cos^2\delta\varphi_D}{\lambda_D^2 R_{P_D}^2}\sigma_n^2\right) & \text{(C.6)}\\
\tilde{LQ}_D \sim \mathcal{N}\left(\dfrac{R_{L_D}}{R_{P_D}}\sin^2\delta\hat{\varphi}_f, \left(\dfrac{R_{P_D}^2 + R_{L_D}^2\sin^2\delta\hat{\varphi}_f}{\lambda_D^2 R_{P_D}^4} - \dfrac{\left(2-\dfrac{\Delta}{T_c}\right)R_{P_D}R_{L_D}\sin^2\delta\hat{\varphi}_f}{\lambda_D^2 R_{P_D}^4}\right)\sigma_n^2\right) & \text{(C.7)}
\end{cases}$$

However, the stochastic nature of $\delta\hat{\varphi}_f$ requires probabilistic treatment. Under typical urban canyon environmental conditions ($C/N_0 \geq 25$ dB-Hz) where $\sigma_f^2 \ll 1$, small-perturbation approximations ($\sin\delta\hat{\varphi} \approx \delta\hat{\varphi}$, $\sin^2\delta\hat{\varphi} \approx 0$) become applicable. Thus, the correlation values can be expressed as follows.

$$\begin{cases}
\tilde{EQ}_D \approx \dfrac{R_{E_D}}{R_{P_D}}\delta\hat{\varphi}_f + \dfrac{1}{\lambda_D R_{P_D}}\eta_{EQ} & \text{(C.8)}\\
\tilde{PQ}_D \approx \delta\hat{\varphi}_f + \dfrac{1}{\lambda_D R_{P_D}}\eta_{PQ} & \text{(C.9)}\\
\tilde{LQ}_D \approx \dfrac{R_{L_D}}{R_{P_D}}\delta\hat{\varphi}_f + \dfrac{1}{\lambda_D R_{P_D}}\eta_{LQ} & \text{(C.10)}
\end{cases}$$

Substituting (C.3) into (C.8)–(C.10), the distributions of the LOS correlation values (59)–(61) can be obtained.

Similarly, for NLOS signals, the distributions of the NLOS correlation values (64)–(66) can be obtained.

For multipath signals, under small error conditions, (B.9) can be expressed as follows.

$$\begin{aligned}
\delta\hat{\varphi} &= \arctan\frac{PQ}{PI} \approx \frac{PQ}{PI}\\
&= \frac{\lambda_D R_{P_D}\sin\delta\varphi_D + \lambda_N R_{P_N}\sin\delta\varphi_N + \eta_{PQ}}{\lambda_D R_{P_D}\cos\delta\varphi_D + \lambda_N R_{P_N}\cos\delta\varphi_N + \eta_{PI}}
\end{aligned} \tag{C.11}$$

In the case of steady-state error (at this point, the discriminator incorrectly locked the code delay and carrier phase to the Multipath signal parameters, that is, $\kappa_{PQ} \approx 0$, $\left|\tilde{E}_M\right| \approx \left|\tilde{L}_M\right|$). $\delta\hat{\varphi}$ becomes:

$$\delta\hat{\varphi} \approx \frac{\kappa_{PQ}}{\sqrt{\lambda_M}} + \frac{\eta_{PQ}}{\sqrt{\lambda_M}} \tag{C.12}$$

Similarly, $\delta\hat{\varphi}_f$ can be expressed as follows.

$$\delta\hat{\varphi}_f \sim \mathcal{N}\left(0, \sigma_f^2\right) \tag{C.13}$$

From (B.15), it can be obtained:

$$\begin{cases}
\sin\delta\varphi_N = \dfrac{-\lambda_D R_{P_D}}{\lambda_N R_{P_N}}\sin\delta\varphi_D & \text{(C.14)}\\
\sin\delta\varphi_D = \dfrac{-\lambda_N R_{P_N}\sin\left(\delta\varphi_N - \delta\varphi_D\right)}{\sqrt{\lambda_M}} & \text{(C.15)}
\end{cases}$$

Note that, assuming $\kappa_{PQ}=\varepsilon,\varepsilon\to 0$, (C.14 and C.15) can be given by

$$\begin{cases}\sin\delta\varphi_N=\dfrac{-\lambda_D R_{P_D}}{\lambda_N R_{P_N}}\sin\delta\varphi_D+\delta_N & \text{(C.16)}\\[2ex] \sin\delta\varphi_D=\dfrac{-\lambda_N R_{P_N}\sin\left(\delta\varphi_N-\delta\varphi_D\right)}{\sqrt{\lambda_M}}+\delta_D & \text{(C.17)}\end{cases}$$

where δ_N and δ_D are small deviations introduced by ε.

Substituting (C.14) into (A.28) and (A.29) yields:

$$\delta_N=\frac{\kappa_{PQ}}{\lambda_N R_{P_N}} \tag{C.18}$$

$$\kappa_{EQ}=\lambda_D\sin\delta\varphi_D\left(R_{E_D}-\frac{R_{E_N}R_{P_D}}{R_{P_N}}\right)+\lambda_N R_{E_N}\delta_N \tag{C.19}$$

By combining the above two equations, we can obtain:

$$\kappa_{EQ}=\lambda_D\sin\delta\varphi_D\left(R_{E_D}-\frac{R_{E_N}R_{P_D}}{R_{P_N}}\right)+\frac{R_{E_N}}{R_{P_N}}\kappa_{PQ} \tag{C.20}$$

Substituting (C.17) into (C.20) and only retaining the dominant item yields:

$$\kappa_{EQ}=\frac{R_{E_N}}{R_{P_N}}\kappa_{PQ}+\frac{\lambda_D\lambda_N\left(R_{P_D}R_{E_N}-R_{E_D}R_{P_N}\right)\sin\left(\delta\varphi_N-\delta\varphi_D\right)}{\sqrt{\lambda_M}} \tag{C.21}$$

Similarly, the relationship between κ_{LQ} and κ_{PQ} can be given by:

$$\kappa_{LQ}=\frac{R_{L_N}}{R_{P_N}}\kappa_{PQ}+\frac{\lambda_D\lambda_N\left(R_{P_D}R_{L_N}-R_{L_D}R_{P_N}\right)\sin\left(\delta\varphi_N-\delta\varphi_D\right)}{\sqrt{\lambda_M}} \tag{C.22}$$

Using $\delta\hat{\varphi}_f$ and substitute (C.21) and (C.22) into (A.25), (A.26) and (A.27), the correlation values can be expressed as follows.

$$\begin{cases}\tilde{EQ}_M\approx\dfrac{\lambda_D\lambda_N}{\lambda_M}\left(R_{P_D}R_{E_N}-R_{E_D}R_{P_N}\right)\\ \qquad\times\sin\left(\delta\varphi_N-\delta\varphi_D\right) & \text{(C.23)}\\ \qquad+\dfrac{R_{E_M}}{R_{P_M}}\delta\hat{\varphi}_f+\dfrac{1}{\sqrt{\lambda_M}}\eta_{EQ}\\ \tilde{PQ}_M\approx\delta\hat{\varphi}+\dfrac{1}{\sqrt{\lambda_M}}\eta_{PQ} & \text{(C.24)}\\ \tilde{LQ}_M\approx\dfrac{\lambda_D\lambda_N}{\lambda_M}\left(R_{P_D}R_{L_N}-R_{L_D}R_{P_N}\right)\\ \qquad\times\sin\left(\delta\varphi_N-\delta\varphi_D\right) & \text{(C.25)}\\ \qquad+\dfrac{R_{L_M}}{R_{P_M}}\delta\hat{\varphi}_f+\dfrac{1}{\sqrt{\lambda_M}}\eta_{LQ}\end{cases}$$

Based on (C.13), the distributions of the multipath correlation values (68)–(70) can be obtained.

D. Derivation of the covariance of the joint distribution

In this appendix, the derivation of the covariance for the 3D Gaussian of correlation values is presented.

Based on the definition of covariance:

$$\begin{cases}\sigma_{12}^2=\sigma_{21}^2=\operatorname{cov}\left(\tilde{EQ},\tilde{PQ}\right) & \text{(D.1)}\\ \sigma_{13}^2=\sigma_{31}^2=\operatorname{cov}\left(\tilde{EQ},\tilde{LQ}\right) & \text{(D.2)}\\ \sigma_{23}^2=\sigma_{32}^2=\operatorname{cov}\left(\tilde{PQ},\tilde{LQ}\right) & \text{(D.3)}\end{cases}$$

According to (C.8)–(C.10), it can be seen that the distribution of the output of the LOS/NLOS/multipath signal after normalization and linearization can be expressed as a linear combination of ν_1, ν_2 and ν_3. After being processed by the code and carrier discriminator, we can obtain:

For LOS signals:

$$\begin{cases}\tilde{EQ}_D\approx\dfrac{R_{E_D}}{R_{P_D}}\delta\hat{\varphi}+\dfrac{\sigma_n}{\lambda_D R_{P_D}}\nu_{1Q} & \text{(D.4)}\\ \tilde{PQ}_D\approx\delta\hat{\varphi}_f+\left(1-\dfrac{\Delta}{2T_c}\right)\dfrac{\sigma_n\nu_{1Q}}{\lambda_D R_{P_D}}\\ \qquad+\sqrt{\dfrac{\Delta}{T_c}\left(1-\dfrac{\Delta}{4T_c}\right)}\dfrac{\sigma_n\nu_{2Q}}{\lambda_D R_{P_D}} & \text{(D.5)}\\ \tilde{LQ}_D\approx\dfrac{R_{L_D}}{R_{P_D}}\delta\hat{\varphi}_f\\ \qquad+\dfrac{\sqrt{2\Delta T_c(2T_c-\Delta)}}{T_c\sqrt{4T_c-\Delta}}\dfrac{\sigma_n\nu_{3Q}}{\lambda_D R_{P_D}}\\ \qquad+\dfrac{\sqrt{\Delta}(2T_c-\Delta)}{T_c\sqrt{4T_c-\Delta}}\dfrac{\sigma_n\nu_{2Q}}{\lambda_D R_{P_D}} & \text{(D.6)}\\ \qquad+\left(1-\dfrac{\Delta}{T_c}\right)\dfrac{\sigma_n\nu_{1Q}}{\lambda_D R_{P_D}}\end{cases}$$

For NLOS signals:

$$\begin{cases}\tilde{EQ}_N\approx\dfrac{R_{E_N}}{R_{P_N}}\delta\hat{\varphi}+\dfrac{\sigma_n}{\lambda_N R_{P_N}}\nu_{1Q} & \text{(D.7)}\\ \tilde{PQ}_N\approx\delta\hat{\varphi}_f+\left(1-\dfrac{\Delta}{2T_c}\right)\dfrac{\sigma_n\nu_{1Q}}{\lambda_N R_{P_N}}\\ \qquad+\sqrt{\dfrac{\Delta}{T_c}\left(1-\dfrac{\Delta}{4T_c}\right)}\dfrac{\sigma_n\nu_{2Q}}{\lambda_N R_{P_N}} & \text{(D.8)}\\ \tilde{LQ}_N\approx\dfrac{R_{L_N}}{R_{P_N}}\delta\hat{\varphi}_f\\ \qquad+\dfrac{\sqrt{2\Delta T_c(2T_c-\Delta)}}{T_c\sqrt{4T_c-\Delta}}\dfrac{\sigma_n\nu_{3Q}}{\lambda_D R_{P_N}}\\ \qquad+\dfrac{\sqrt{\Delta}(2T_c-\Delta)}{T_c\sqrt{4T_c-\Delta}}\dfrac{\sigma_n\nu_{2Q}}{\lambda_N R_{P_N}} & \text{(D.9)}\\ \qquad+\left(1-\dfrac{\Delta}{T_c}\right)\dfrac{\sigma_n\nu_{1Q}}{\lambda_N R_{P_N}}\end{cases}$$

For multipath signals:

$$\begin{cases} \tilde{EQ}_M \approx \frac{\lambda_D \lambda_N}{\lambda_M} \left(R_{P_D} R_{E_N} - R_{E_D} R_{P_N}\right) \\ \qquad \times \sin\left(\delta\varphi_N - \delta\varphi_D\right) + \frac{R_{E_M}}{R_{P_M}} \delta\hat{\varphi} + \frac{\sigma_n \nu_{1Q}}{\sqrt{\lambda_M}} & \text{(D.10)} \\ \tilde{PQ}_M = \delta\hat{\varphi} + \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n \nu_{1Q}}{\sqrt{\lambda_M}} \\ \qquad + \sqrt{\frac{\Delta}{T_c}\left(1 - \frac{\Delta}{4T_c}\right)} \frac{\sigma_n \nu_{2Q}}{\sqrt{\lambda_M}} & \text{(D.11)} \\ \tilde{LQ}_M \approx \frac{\lambda_D \lambda_N}{\lambda_M} \left(R_{P_D} R_{L_N} - R_{L_D} R_{P_N}\right) \\ \qquad \times \sin\left(\delta\varphi_N - \delta\varphi_D\right) \\ \qquad + \left(1 - \frac{\Delta}{T_c}\right) \frac{\sigma_n \nu_{1Q}}{\sqrt{\lambda_M}} \\ \qquad + \frac{\sqrt{\Delta}\left(2T_c - \Delta\right)}{T_c \sqrt{4T_c - \Delta}} \frac{\sigma_n \nu_{2Q}}{\sqrt{\lambda_M}} & \text{(D.12)} \\ \qquad + \frac{\sqrt{2\Delta T_c \left(2T_c - \Delta\right)}}{T_c \sqrt{\left(4T_c - \Delta\right)}} \frac{\sigma_n \nu_{3Q}}{\sqrt{\lambda_M}} \\ \qquad + \frac{R_{L_M}}{R_{P_M}} \delta\hat{\varphi}_f \end{cases}$$

By substituting (D.4)–(D.12) into (D.1)–(D.3), based on the linear and translation invariant properties of covariance, we can obtain:

For LOS signals:

$$\begin{cases} \operatorname{cov}\left(\tilde{EQ}_D, \tilde{PQ}_D\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_D^2 R_{P_D}^2} + \frac{R_{E_D}}{R_{P_D}} \sigma_f^2 & \text{(D.13)} \\ \operatorname{cov}\left(\tilde{EQ}_D, \tilde{LQ}_D\right) = \left(1 - \frac{\Delta}{T_c}\right) \frac{\sigma_n^2}{\lambda_D^2 R_{P_D}^2} + \frac{R_{E_D} R_{L_D}}{R_{P_D}^2} \sigma_f^2 & \text{(D.14)} \\ \operatorname{cov}\left(\tilde{PQ}_D, \tilde{LQ}_D\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_D^2 R_{P_D}^2} + \frac{R_{L_D}}{R_{P_D}} \sigma_f^2 & \text{(D.15)} \end{cases}$$

The covariance form of the NLOS/multipath signal is consistent with that of the LOS signal.

Similarly, for NLOS signals:

$$\begin{cases} \operatorname{cov}\left(\tilde{EQ}_N, \tilde{PQ}_N\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_N^2 R_{P_N}^2} + \frac{R_{E_N}}{R_{P_N}} \sigma_f^2 & \text{(D.16)} \\ \operatorname{cov}\left(\tilde{EQ}_N, \tilde{LQ}_N\right) = \left(1 - \frac{\Delta}{T_c}\right) \frac{\sigma_n^2}{\lambda_N^2 R_{P_N}^2} + \frac{R_{E_N} R_{L_N}}{R_{P_N}^2} \sigma_f^2 & \text{(D.17)} \\ \operatorname{cov}\left(\tilde{PQ}_N, \tilde{LQ}_N\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_N^2 R_{P_N}^2} + \frac{R_{L_N}}{R_{P_N}} \sigma_f^2 & \text{(D.18)} \end{cases}$$

For multipath signals:

$$\begin{cases} \operatorname{cov}\left(\tilde{EQ}_M, \tilde{PQ}_M\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_M} + \frac{R_{E_M}}{R_{P_M}} \sigma_f^2 & \text{(D.19)} \\ \operatorname{cov}\left(\tilde{EQ}_M, \tilde{LQ}_M\right) = \left(1 - \frac{\Delta}{T_c}\right) \frac{\sigma_n^2}{\lambda_M} + \frac{R_{E_M} R_{L_M}}{R_{P_M}^2} \sigma_f^2 & \text{(D.20)} \\ \operatorname{cov}\left(\tilde{PQ}_M, \tilde{LQ}_M\right) = \left(1 - \frac{\Delta}{2T_c}\right) \frac{\sigma_n^2}{\lambda_M} + \frac{R_{L_M}}{R_{P_M}} \sigma_f^2 & \text{(D.21)} \end{cases}$$

E. Homogeneous coordinates and matrices

In this appendix, it is explained in Cartesian coordinate system, how to use homogeneous coordinates to uniformly represent scaling, rotation, and translation transformations through homogeneous matrices.

Considering the translation transformation: Move the point along the X, Y and Z axes by t_x, t_y, and t_z, respectively. The corresponding homogeneous transformation matrix is as follows:

$$\tilde{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.1}$$

Coordinate transformation formula can be expressed as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \tilde{\mathbf{T}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix} \tag{E.2}$$

Considering the scaling transformation: scale the points in the X-, Y- and Z- axis directions by s_x, s_y and s_z times respectively. The corresponding homogeneous transformation matrix is:

$$\tilde{\mathbf{S}} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.3}$$

Coordinate transformation formula can be expressed as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \tilde{\mathbf{S}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \\ 1 \end{bmatrix} \tag{E.4}$$

Considering the rotation transformation: rotation transformation requires specifying the rotation axis and angle θ (counterclockwise is positive). The following is the standard rotation matrix around each coordinate axis:

$$\tilde{\mathbf{Q}}_x\left(\theta\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.5}$$

$$\tilde{\mathbf{Q}}_y\left(\theta\right) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.6}$$

$$\tilde{\mathbf{Q}}_z\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{E.7}$$

Coordinate transformation formula can be expressed as (Taking the Z-axis as an example):

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \tilde{\mathbf{Q}}_z\left(\theta\right) \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \\ 1 \end{bmatrix} \tag{E.8}$$

F. Spectral decomposition and perturbation theory of matrices

In this appendix, the specific steps and forms of spectral decomposition of the noise covariance matrix $\mathbf{C}$ and based on perturbation theory, how $\mathbf{\Sigma}_{R,f}$ as a perturbation term affects $\mathbf{\Sigma}_{R,\eta}$ are explained.

Based on the relationship between correlators, $\mathbf{C}$ can be written as:

$$\mathbf{C} = \begin{bmatrix} 1 & 1-\dfrac{\Delta}{2T_c} & 1-\dfrac{\Delta}{T_c} \\ 1-\dfrac{\Delta}{2T_c} & 1 & 1-\dfrac{\Delta}{2T_c} \\ 1-\dfrac{\Delta}{T_c} & 1-\dfrac{\Delta}{2T_c} & 1 \end{bmatrix} \tag{F.1}$$

$\mathbf{C}$ is a real symmetric matrix, so spectral decomposition can be performed. The form of spectral decomposition is as follows.

$$\mathbf{C} = \mathbf{Q}\mathbf{S}^2\mathbf{Q}^{-1} \tag{F.2}$$

where $\mathbf{S}^2$ is the eigenvalue matrix, $\mathbf{Q}$ is the normalized eigenvector matrix corresponding to $\mathbf{S}^2$.

Step 1: solving the characteristic equation to obtain the eigenvalues:

$$\det(\mathbf{C} - \lambda\mathbf{I}) = 0 \tag{F.3}$$

where λ is the eigenvalue.

Thus, $\mathbf{S}^2$ is given by:

$$\mathbf{S}^2 = \begin{bmatrix} \dfrac{3-\dfrac{\Delta}{T_c}+\sqrt{9-\dfrac{10\Delta}{T_c}+\dfrac{3\Delta^2}{T_c^2}}}{2} & 0 & 0 \\ 0 & \dfrac{\Delta}{T_c} & 0 \\ 0 & 0 & \dfrac{3-\dfrac{\Delta}{T_c}-\sqrt{9-\dfrac{10\Delta}{T_c}+\dfrac{3\Delta^2}{T_c^2}}}{2} \end{bmatrix} \tag{F.4}$$

Step 2: Calculate the eigenvector for each eigenvalue using the following equation:

$$(\mathbf{C} - \lambda_i\mathbf{I})\,\mathbf{q}_i = 0 \tag{F.5}$$

Thus, $\mathbf{q}_i$ can be obtained as follows.

$$\begin{cases} \mathbf{q}_1 = \left[\dfrac{1}{2}, \dfrac{\dfrac{\Delta}{T_c}-1+\sqrt{9-\dfrac{10\Delta}{T_c}+\dfrac{3\Delta^2}{T_c^2}}}{2\left(2-\dfrac{\Delta}{T_c}\right)}, \dfrac{1}{2}\right]^{\mathrm{T}} & \text{(F.6)} \\ \mathbf{q}_2 = \left[-\dfrac{1}{2}, 0, \dfrac{1}{2}\right]^{\mathrm{T}} & \text{(F.7)} \\ \mathbf{q}_3 = \left[\dfrac{1}{2}, \dfrac{\dfrac{\Delta}{T_c}-1-\sqrt{9-\dfrac{10\Delta}{T_c}+\dfrac{3\Delta^2}{T_c^2}}}{2\left(2-\dfrac{\Delta}{T_c}\right)}, \dfrac{1}{2}\right]^{\mathrm{T}} & \text{(F.8)} \end{cases}$$

Step 3: Normalize the eigenvector, and (114) can be obtained.

Based on the above steps, $\mathbf{\Sigma}_R$ can be decomposed into:

$$\begin{aligned} \mathbf{\Sigma}_R &= \mathbf{Q}_R\mathbf{S}_R^2\mathbf{Q}_R = \mathbf{\Sigma}_{R,\eta} + \mathbf{\Sigma}_{R,f} \\ &= \mathbf{Q}_{R,\eta}\mathbf{S}_{R,\eta}^2\mathbf{Q}_{R,\eta} + \mathbf{Q}_{R,f}\mathbf{S}_{R,f}^2\mathbf{Q}_{R,f} \end{aligned} \tag{F.9}$$

In general, $\mathbf{Q}_{R,\eta} \neq \mathbf{Q}_{R,f}$, so it cannot be expressed as a simple sum of diagonal matrices. To quantitatively analyze the impact of $\mathbf{\Sigma}_{R,f}$ on $\mathbf{\Sigma}_{R,\eta}$, matrix perturbation theory is employed.

Based on matrix perturbation theory, when $\|\mathbf{\Sigma}_{R,f}\| \ll \|\mathbf{\Sigma}_{R,\eta}\|$, the changes in eigenvalues $\mathbf{S}_R^2$ and eigenvectors $\mathbf{Q}_R$ of $\mathbf{\Sigma}_R$ can be approximated as follows [7].

$$\lambda_R(i) \approx \lambda_{R,\eta}(i) + \mathbf{Q}_{R,\eta}^{\mathrm{T}}(i)\,\mathbf{\Sigma}_{R,f}\mathbf{Q}_{R,\eta}(i) \tag{F.10}$$

$$\mathbf{Q}_R(i) \approx \mathbf{Q}_{R,\eta}(i) + \sum_{j\neq i}\frac{\mathbf{Q}_{R,\eta}^{\mathrm{T}}(j)\,\mathbf{\Sigma}_{R,f}\mathbf{Q}_{R,\eta}(i)}{\lambda_{R,\eta}(i)-\lambda_{R,\eta}(j)}\mathbf{Q}_{R,\eta}(j) \tag{F.11}$$

Each eigenvalue increases $\mathbf{Q}_{R,\eta}^{\mathrm{T}}(i)\,\mathbf{\Sigma}_{R,f}\mathbf{Q}_{R,\eta}(i)$, which is the projection of $\mathbf{\Sigma}_{R,f}$ in the $i\text{-}th$ eigenvector direction of $\mathbf{\Sigma}_{R,\eta}$. Each eigenvector has increased by $\mathbf{\Sigma}_{R,f}$, which means that the direction of the eigenvector has rotated, and the amount of rotation depends on the projection of $\mathbf{\Sigma}_{R,f}$ in other eigenvector directions and the difference in eigenvalues.

Taking LOS signal case as an example, $\mathbf{\Sigma}_{R,f}$ can be expressed as:

$$\mathbf{\Sigma}_{R,f} = 2TB_n\mathbf{\Sigma}_f = 2TB_n\begin{bmatrix}\dfrac{R_{E_D}}{R_{P_D}} \\ 1 \\ \dfrac{R_{L_D}}{R_{P_D}}\end{bmatrix}\begin{bmatrix}\dfrac{R_{E_D}}{R_{P_D}} \\ 1 \\ \dfrac{R_{L_D}}{R_{P_D}}\end{bmatrix}^{\mathrm{T}} \tag{F.12}$$

Thus, $\mathbf{Q}_{R,\eta}^{\mathrm{T}}(i)\,\mathbf{\Sigma}_{R,f}\mathbf{Q}_{R,\eta}(i)$ can be written as:

$$\mathbf{Q}_{R,\eta}^{\mathrm{T}}(i)\,\mathbf{\Sigma}_{R,f}\mathbf{Q}_{R,\eta}(i) = \left(\begin{bmatrix}\dfrac{R_{E_D}}{R_{P_D}} \\ 1 \\ \dfrac{R_{L_D}}{R_{P_D}}\end{bmatrix}^{\mathrm{T}}\mathbf{Q}_{R,\eta}(i)\right)^2 \tag{F.13}$$

Thus, $\mathbf{S}_f$ can be written as:

$$\mathbf{S}_f(i,i) = \begin{bmatrix}\dfrac{R_{E_D}}{R_{P_D}} \\ 1 \\ \dfrac{R_{L_D}}{R_{P_D}}\end{bmatrix}^{\mathrm{T}}\mathbf{Q}_{R,\eta}(i) \tag{F.14}$$

Substituting (F.4), (114), (F.4) and (F.14) into (F.11), (116)–(118) can be obtained.

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