State-insensitive optical conveyor belts on integrated nanophotonic waveguides: Supplemental Information

I. LIGHT SHIFT IN A BICHROMATIC FIELD

The method set out here follows closely that set out by Le Kien in Ref. 1 for a monochromatic field. Consider a two-level atom of bare eigenstates $|a\rangle$ and $|b\rangle$ with unperturbed energies $E_a = \hbar\omega_a$ and $E_b = \hbar\omega_b$, irradiated by two monochromatic fields of the form of Eq. (3) with frequencies ω_1 and ω_2. The light shift experienced by state $|a\rangle$ is given by the time-average of the induced dipole moment $\mathbf{p}$ coupled with the bichromatic field $\mathbf{E}$

$$\delta E_a = -\frac{1}{2}\overline{\mathbf{p}\cdot\mathbf{E}} \tag{S1}$$

where the 1/2 indicates that the dipole moment is induced by the field. The induced dipole moment of the atom is given by $\mathbf{p} = \langle\mathbf{d}\rangle = \mathrm{Tr}\{\rho\mathbf{d}\} = \mathbf{d}_{ba}\rho_{ab} + \mathbf{d}_{ab}\rho_{ba}$ where ρ is the density matrix operator and $\mathbf{d}$ is the electric dipole operator $\mathbf{d} = \mathbf{d}_{ba}\,|b\rangle\langle a| + \mathbf{d}_{ab}\,|a\rangle\langle b|$. The time evolution of the atom's density matrix component ρ_{ba} is given by

$$\dot{\rho}_{ba} = -i(\omega_{ba} - i\gamma_{ba}/2)\rho_{ba} + \frac{i}{2\hbar}(\boldsymbol{\mathcal{E}}_1 e^{-i\omega_1 t} + \boldsymbol{\mathcal{E}}_2 e^{-i\omega_2 t} + \text{c.c.})\cdot\mathbf{d}_{ba} \tag{S2}$$

where $\boldsymbol{\mathcal{E}}_\eta$ is the positive-frequency envelope of field η with corresponding amplitude $\mathcal{E}_\eta$ and polarization vector $\hat{\mathbf{u}}_\eta$. In Eq. (S2) we have assumed the populations $\rho_{aa} \simeq 1$ and $\rho_{bb} \simeq 0$ [1]. Assuming a solution of the form $\rho_{ba} = \rho_1^+ e^{-i\omega_1 t} + \rho_1^- e^{i\omega_1 t} + \rho_2^+ e^{-i\omega_2 t} + \rho_2^- e^{i\omega_2 t}$, we find

$$\rho_1^+ = \frac{\boldsymbol{\mathcal{E}}_1\cdot\mathbf{d}_{ba}}{2\hbar}\frac{1}{\omega_{ba} - \omega_1 - i\gamma_{ba}/2}, \tag{S3a}$$

$$\rho_1^- = \frac{\boldsymbol{\mathcal{E}}_1^*\cdot\mathbf{d}_{ba}}{2\hbar}\frac{1}{\omega_{ba} + \omega_1 - i\gamma_{ba}/2}, \tag{S3b}$$

$$\rho_2^+ = \frac{\boldsymbol{\mathcal{E}}_2\cdot\mathbf{d}_{ba}}{2\hbar}\frac{1}{\omega_{ba} - \omega_2 - i\gamma_{ba}/2}, \tag{S3c}$$

$$\rho_2^- = \frac{\boldsymbol{\mathcal{E}}_2^*\cdot\mathbf{d}_{ba}}{2\hbar}\frac{1}{\omega_{ba} + \omega_2 - i\gamma_{ba}/2}, \tag{S3d}$$

where we have assumed steady state solutions $\dot{\rho}_1^\pm = 0$ and $\dot{\rho}_2^\pm = 0$. Using an ansatz of the form $\mathbf{p} = (\mathbf{P}_1 e^{-i\omega_1 t} + \mathbf{P}_1^* e^{+i\omega_1 t} + \mathbf{P}_2 e^{-i\omega_2 t} + \mathbf{P}_2^* e^{i\omega_2 t})/2$ we define the envelope functions

$$\mathbf{P}_1 = 2(\mathbf{d}_{ab}\rho_1^+ + \mathbf{d}_{ba}\rho_1^{-*}), \tag{S4a}$$

$$\mathbf{P}_1^* = 2(\mathbf{d}_{ab}\rho_1^- + \mathbf{d}_{ba}\rho_1^{+*}), \tag{S4b}$$

$$\mathbf{P}_2 = 2(\mathbf{d}_{ab}\rho_2^+ + \mathbf{d}_{ba}\rho_2^{-*}), \tag{S4c}$$

$$\mathbf{P}_2^* = 2(\mathbf{d}_{ab}\rho_2^- + \mathbf{d}_{ba}\rho_2^{+*}), \tag{S4d}$$

such that the inner product $\mathbf{p}\cdot\mathbf{E}$ is

$$\mathbf{p}\cdot\mathbf{E} = \frac{1}{4}\Big(\mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_1 e^{-2i\omega_1 t} + \mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_2 e^{-i(\omega_1+\omega_2)t} + \mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_1^* + \mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_2^* e^{-i(\omega_1-\omega_2)t} + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_1 e^{-i(\omega_1+\omega_2)t} + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_2 e^{-2i\omega_2 t} + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_1^* e^{i(\omega_1-\omega_2)t} + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_2^* + \text{c.c.}\Big). \tag{S5}$$

We assume that ω_1 and ω_2 are symmetrically detuned about a central frequency ω such that $\omega_1 = \omega + \delta\omega/2$, $\omega_2 = \omega - \delta\omega/2$ and $\omega_1 - \omega_2 = \delta\omega$. The form of Eq. (S5) upon time-averaging is dependent on the magnitude of this mutual detuning, for which we consider two cases. The first is when $\delta\omega \sim \omega$, for which we arrive to the standard summation over light shifts for two far-mutually-detuned fields

$$\delta E_a = \delta E_a^1 + \delta E_a^2 \tag{S6}$$

where δE_a^1 and δE_a^2 are the light shifts due to the fields of ω_1 and ω_2, respectively, and can be written as expectation values of the V^{ls} operator of Eq. (6) in the main text. The form of solution (S6) applies for red- and blue-detuned two-color potentials and pairs of red- or blue-detuned modes in a compensated scheme (that is, counter-propagating red- or blue-detuned modes mutually detuned by $\sim$GHz).

The second case we consider is that in which $\delta\omega \ll \omega$. Upon time-averaging, the light shift takes the form

$$\delta E_a = -\frac{1}{4}\,\text{Re}\left(\mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_1^* + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_2^* + \mathbf{P}_1\cdot\boldsymbol{\mathcal{E}}_2^* e^{-i\delta\omega t} + \mathbf{P}_2\cdot\boldsymbol{\mathcal{E}}_1^* e^{i\delta\omega t}\right). \tag{S7}$$

By approximating $\mathbf{E}_1$ and $\mathbf{E}_2$ as forwards and backwards traveling modes respectively with $\omega_1 \approx \omega_2$, and collecting terms, the energy shift Eq. (S7) can be written

$$\delta E_a = -\frac{1}{4}\,\text{Re}\left(\mathbf{P}(t)\cdot\boldsymbol{\mathcal{E}}^{\text{SW}}(t)\right) \tag{S8}$$

where

$$\mathbf{P}(t) = \mathbf{P}_1 e^{-\delta\omega t/2} + \mathbf{P}_2 e^{i\delta\omega t/2} \tag{S9}$$

and

$$\boldsymbol{\mathcal{E}}^{\text{SW}}(t) = \begin{pmatrix} 2\mathcal{E}_x\cos\left(\beta z - \delta\omega t/2\right) \\ 2\mathcal{E}_y\cos\left(\beta z - \delta\omega t/2\right) \\ -2\mathcal{E}_z\sin\left(\beta z - \delta\omega t/2\right) \end{pmatrix}. \tag{S10}$$

Equation (S8) corresponds to the light shift experienced by an atom in state $|a\rangle$ in a moving optical lattice. It corresponds to an expectation value of the light shift operator V^{ls} of Eq. (6) in the main text with a time-dependent field, but time-independent dynamic polarizabilities.

II. DYNAMIC POLARIZABILITIES

In the main text, we wrote the light shift operator of Eq. (6) in terms of the conventional scalar, vector and tensor polarizabilities

$$\alpha_F^s = \frac{1}{\sqrt{3(2F+1)}}\alpha_F^{(0)} \tag{S11a}$$

$$\alpha_F^v = -\sqrt{\frac{2F}{(F+1)(2F+1)}}\alpha_F^{(1)} \tag{S11b}$$

$$\alpha_F^T = \sqrt{\frac{2F(2F-1)}{3(F+1)(2F+1)(2F+3)}}\alpha_F^{(2)} \tag{S11c}$$

where $\alpha_F^{(K)}$ is the $K = 0, 1, 2$ reduced polarizability of the hyperfine level $|nJF\rangle$. As the dipole matrix elements do not couple to the nuclear degrees of freedom, the reduced polarizabilities $\alpha_F^{(K)}$ can be written[2]

$$\alpha_{nF}^{(K)} = (-1)^{J+I+F+K}(2F+1)\begin{Bmatrix} F & K & F \\ J & I & J \end{Bmatrix}\alpha_{nJ}^{(K)} \tag{S12}$$

where the quantum numbers I and F have been factored out, providing us with reduced polarizabilities for the fine structure level $|nJ\rangle$

$$\begin{aligned}\alpha_{nJ}^{(K)} =& (-1)^{J+K+1}\sqrt{2K+1}\sum_{n''J''}(-1)^{J''}\begin{Bmatrix} 1 & K & 1 \\ J & J'' & J \end{Bmatrix} \times |\langle n''J''\,\|\mathbf{d}\|\,nJ\rangle|^2 \\ &\times \frac{1}{\hbar}\,\mathrm{Re}\left(\frac{1}{\omega_{n''J''nJ} - \omega - i\gamma_{n''J''nJ}/2} + \frac{(-1)^K}{\omega_{n''J''nJ} + \omega + i\gamma_{n''J''nJ}/2}\right)\end{aligned} \tag{S13}$$

where we have assumed that $\omega_{nJF} \approx \omega_{nJ}$ and $\gamma_{nJF} \approx \gamma_{nJ}$. Here $\langle n''J''\,\|\mathbf{d}\|\,nJ\rangle$ denotes the reduced matrix element of the electric dipole operator in the fine structure basis, and can be obtained from the oscillator strength $f_{nJn''J''}$. See references 2 and 1 for detailed derivations of the above quantities. We determine the reduced polarizabilities of Eq. (S13) with the Alkali Rydberg Calculator [3], which computes the *conventional* polarizabilities for the fine structure level $|nJ\rangle$

$$\alpha_{nJ}^s = \frac{1}{\sqrt{3(2J+1)}}\alpha_{nJ}^{(0)} \tag{S14a}$$

$$\alpha_{nJ}^v = -\sqrt{\frac{2J}{(J+1)(2J+1)}}\alpha_{nJ}^{(1)} \tag{S14b}$$

$$\alpha_{nJ}^T = \sqrt{\frac{2J(2J-1)}{3(J+1)(2J+1)(2J+3)}}\alpha_{nJ}^{(2)}. \tag{S14c}$$

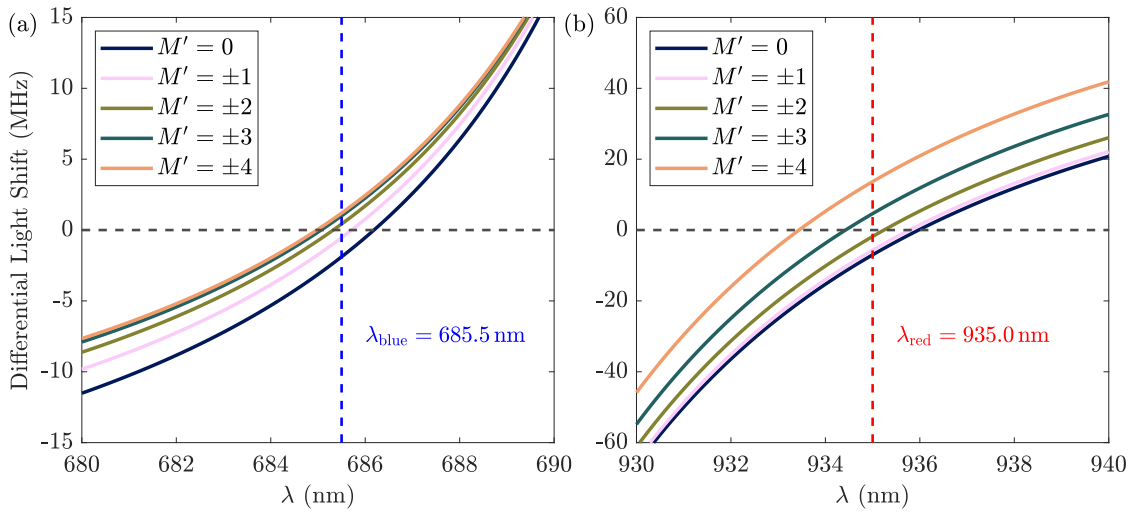


FIG. 1. Differential light shifts between the $6P_{3/2}\,|F'=4,M'\rangle$ and $6S_{1/2}\,|F=4,M\rangle$ magnetic sublevels as a function of wavelength. Light was simulated under linear polarization with an optical intensity $I = 1$ MW/cm^2. The blue-detuned magic wavelengths in (a) have values $\lambda_{\rm blue}^{M'=0} = 686.2$, $\lambda_{\rm blue}^{M'=\pm1} = 685.7$, $\lambda_{\rm blue}^{M'=\pm2} = 685.3$, $\lambda_{\rm blue}^{M'=\pm3} = 685.1$ and $\lambda_{\rm blue}^{M'=\pm4} = 685.0$ nm. Likewise, (b) features red-detuned magic wavelengths $\lambda_{\rm red}^{M'=0} = 936.0$, $\lambda_{\rm red}^{M'=\pm1} = 935.8$, $\lambda_{\rm red}^{M'=\pm2} = 935.2$, $\lambda_{\rm red}^{M'=\pm3} = 934.4$ and $\lambda_{\rm red}^{M'=\pm4} = 933.4$ nm. The averages $\lambda_{\rm blue} = 685.5$ nm and $\lambda_{\rm red} = 935.0$ nm are labeled in (a) and (b), respectively.

These are converted to their reduced forms by cancellation of the prefactors.

III. MAGIC WAVELENGTHS

The red- and blue-detuned magic wavelengths $\lambda_{\rm red}$ and $\lambda_{\rm blue}$ were calculated following the method outlined in Refs. 4 and 5. Note however that we make use only of the light shift operator and neglect the hfs interaction. Assuming linear polarization and an intensity $I = 1$ MW/cm^2, we diagonalize the light shift operator of Eq. (6) to retrieve the energy shifts of the $6S_{1/2}\,|F=4,M\rangle$ and $6P_{3/2}\,|F'=4,M'\rangle$ states as a function of wavelength. The differential (transition [4, 5]) light shift is defined as the difference between the light shifts of the upper and lower levels; magic wavelengths correspond to points where this differential light shift vanishes. Note that due to the assumed linear polarization of light and vanishing tensor polarizability $\alpha_F^T(6S_{1/2}) = 0$, all magnetic sublevels of the $6S_{1/2}$ state have identical light shift. The magnetic sublevels of the $6P_{3/2}$ state experience quadratic splitting due to

the non-zero tensor polarizability $\alpha_F^T(6P_{3/2})$.

In Fig. 1 we plot the differential light shift as a function of wavelength for the $6S_{1/2}\,|F=4,M\rangle \rightarrow 6P_{3/2}\,|F'=4,M'\rangle$ transitions in two wavelength regimes of interest: blue-detuned from the D_2 line (Fig. 1(a)) and red-detuned from the D_2 line (Fig. 1(b)). Exact magic wavelengths for each M' sublevel are quoted in the Fig. caption. To ensure a trapping potential for all the $6S_{1/2}\,|FM\rangle$ and $6P_{3/2}\,|F'M'\rangle$ sublevels, we select our magic wavelengths as the average of the blue- and red-detuned values, with $\lambda_{\rm blue} = 685.5$ nm and $\lambda_{\rm red} = 935.0$ nm, respectively. These differ by less than 1 nm from those reported in Refs. 5 and 6. At this optical intensity, the magnitude of the differential shift at $\lambda_{\rm blue}$ is less than 2 MHz for all M', and likewise at $\lambda_{\rm red}$ less than 15 MHz for all for all M'. This is sufficiently small to ensure trapping of atoms in all ground and excited state sublevels.

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