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Semi-Markov Dynamic Stochastic General Equilibrium Models: Theory, Solution, and Estimation

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Abstract

This paper develops a general framework for Dynamic Stochastic General Equilibrium (DSGE) models where structural parameters switch regimes according to a semi-Markov process—the Semi-Markov DSGE (SM-DSGE) model. Unlike standard Markov-switching (MS-DSGE) specifications, which assume constant, memoryless exit probabilities, our framework lets transition probabilities depend on how long the economy has remained in the current regime. This feature captures the duration dependence widely documented in monetary policy and business cycle data.

Our main contributions can be summarized as a complete econometric toolkit—solution, filtering, stability verification, and Bayesian inference—for DSGE models with semi-Markov regime switching. The specific contributions are as follows. First, we embed the semi-Markov process into a DSGE model by constructing an augmented state space that tracks both the current regime and elapsed duration, and we derive first-order perturbation solutions that depend on duration. Second, we develop a Hamilton-Kalman filter that integrates over the augmented discrete state while applying the Kalman filter to the continuous state, producing an exact likelihood evaluator whose cost is linear in the sample size. Third, we establish conditions for mean-square stability of the SM-DSGE solution, extending the determinacy results of Davig and Leeper (2007) and Farmer, Waggoner and Zha (2009) to the semi-Markov setting.

Fourth, we estimate the model using U.S. quarterly data on output growth, inflation, and the federal funds rate (1954 Q3–2007 Q4). The estimation uncovers two monetary policy regimes: a hawkish regime (Regime 1) with an inflation response coefficient $\phi_\pi = 1.50$ [90% CI: 1.38–1.61] and an accommodative regime (Regime 2) with $\phi_\pi = 0.62$ [0.29–1.01]. The Weibull shape parameters for both regimes are estimated at $\hat{\kappa} \approx 1.11$, indicating mildly increasing hazard rates and thus positive duration dependence: the longer a regime has persisted, the more likely it is to end. Duration-dependent impulse responses show that the real effects of a monetary policy shock vary substantially with

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regime tenure. These findings highlight the empirical relevance of moving beyond the geometric-duration assumption embedded in conventional Markov-switching models.
Keywords: DSGE models, semi-Markov processes, regime switching, duration dependence, Bayesian estimation, monetary policy.
JEL Classification: C11, C32, E31, E52.

1 Introduction

Regime-switching DSGE models have become a workhorse for studying changes in monetary policy conduct, fiscal rules, and the volatility of structural shocks.[1] Most of this literature assumes that regime changes follow a Markov chain, which implies a constant exit probability regardless of how long the economy has been in a given regime. The geometric distribution of sojourn times that this assumption generates may be restrictive. Monetary policy regimes may exhibit duration dependence: the probability of a policy shift may rise or fall with the tenure of the current regime.

This paper proposes a generalization—the Semi-Markov DSGE (SM-DSGE) model—in which the process governing regime switches is a semi-Markov process (SMP). In an SMP, the probability of leaving a regime at duration d is described by a duration-dependent hazard function $\lambda_i(d)$ rather than a constant transition probability p_{ij}. This generalization is parsimonious: it requires only one or two additional parameters per regime (for example, the shape parameter of a Weibull sojourn distribution) yet opens a rich space of regime-duration dynamics—including increasing, decreasing, and non-monotonic hazard rates—that a standard Markov chain cannot capture. The economic novelty is straightforward but important: private agents internalize tenure when forming expectations about future policy and macroeconomic conditions. As a result, the economy's law of motion depends on the current regime, the endogenous and exogenous states, and the duration since the last regime change.

Our framework rests on a well-known device from the theory of semi-Markov processes: any SMP on a finite state space can be represented as a Markov chain on an augmented state space that records both the current regime i and the elapsed duration d. By embedding this augmented chain in the DSGE model, we obtain a state-space system whose solution and likelihood evaluation parallel the standard MS-DSGE methodology but with richer dynamics. The main challenge is computational. Once the switching process is no longer memoryless, the state vector must be enlarged to include elapsed duration or a sufficient statistic for duration dependence. We show that this complication can be handled cleanly by embedding the semi-Markov chain in an augmented Markov chain over regime-duration pairs. After truncating duration at a finite upper bound, the model becomes a finite-state regime-switching DSGE system that can be solved with a modification of existing perturbation methods. The same augmented-state representation also makes estimation feasible with a particle-filter-based Bayesian procedure.

Related literature. Hamilton (1989) introduced Markov-switching models in macroeconomics, and subsequent work extended them to DSGE settings (see, e.g., Sims and Zha, 2006; Davig and Leeper, 2007; Farmer, Waggoner and Zha, 2009; Foerster, Waggoner and Zha, 2016). Our work contributes to this literature by relaxing the geometric-duration assumption. There is also a long tradition of duration-dependent regime-switching models in time series econometrics (Durlauf and McCurdy, 1994; Maheu and McCurdy, 2000), but these have remained largely separate from structural DSGE estimation. This paper bridges that gap.Up to our best knowledge, few papers in the literature consider non-constant hazard rates in the regime-switching DSGE models.Bianchi and Melosi (2013, 2017) are among the

[1] See, for example, Sims and Zha (2006), Bianchi (2013), and Bianchi and Melosi (2017).

very few exceptions related to our works.Bianchi and Melosi(2013) model "dormant shocks" that activate with state-dependent probabilities, and Bianchi and Melosi(2017) allow transition probabilities to respond to endogenous outcomes during the Great Recession. Our contribution differs at least in two respects. First, methodologically, we provide a general rational-expectations solution algorithm for duration-dependent regime switching that nests the standard Markov-switching DSGE as a special case and extends naturally to arbitrary sojourn distributions, whereas Bianchi–Melosi's approach augments the state space with regime-specific "dormant" duplicates and solves each as a separate Markov chain. Our semi-Markov framework makes the duration dependence explicit in the transition kernel and proves that the fixed-point iteration converges at a geometric rate, offering a transparent link between the discrete-time Weibull hazard and the model's policy functions. Second, economically, Bianchi–Melosi focus on state-dependent transitions—dormant shocks that "wake up" when the ZLB binds or fiscal stress rises—while we model duration dependence, capturing the institutional fragility of long-lived policy stances independently of current macroeconomic conditions. In short, duration dependence and state dependence are complementary margins—our framework isolates the former and demonstrates its quantitative importance for counterfactual policy analysis, while Bianchi–Melosi enrich the latter. A natural extension would combine both features in a unified framework. Methodologically, our solution approach generalizes the perturbation methods of Foerster, Waggoner and Zha (2016) to the semi-Markov case, and our Hamilton-Kalman filter extends the approximate filter of Kim and Nelson (1999) to the augmented state space. Empirically, our finding of positive duration dependence in U.S. monetary policy regimes complements narrative evidence on the increasing fragility of long-lived policy stances(Clarida, Gali,and Gertler,2000;Gali,2015).

The paper proceeds as follows. Section 2 defines the semi-Markov process and its augmented Markov representation. Section 3 presents the SM-DSGE model. Section 4 develops the perturbation solution and establishes stability conditions. Section 5 describes the Hamilton-Kalman filter and the Bayesian estimation procedure. Section 6 applies the framework to U.S. data. Section 7 discusses extensions, and Section 8 concludes.

2 Semi-Markov Processes

2.1 Definitions

Let $\{s_t\}_{t\geq 0}$ be a discrete-time stochastic process taking values in a finite set $\{1, \ldots, M\}$. The process is a semi-Markov process if the next regime and the sojourn time in the current regime are governed by a joint distribution that depends on the current regime. Equivalently, conditionally on the current regime i, the duration d spent in that regime follows a sojourn-time distribution $F_i(d) = \Pr(d_t \leq d \mid s_t = i)$, and the next regime $j \neq i$ is drawn from an embedded transition matrix $\Pi = (\pi_{ij})$ with zero diagonal.

The hazard function for regime i at duration d is

$$\lambda_i(d) = \Pr(\text{exit at } d \mid \text{survived to } d) = \frac{F_i(d) - F_i(d-1)}{1 - F_i(d-1)}.$$

2.2 Contrast with Standard Markov Chains

The key distinction is duration dependence. For a Markov chain the hazard is constant: $\lambda_i(d) = 1 - p_{ii}$ for all d. For a semi-Markov process the hazard may increase, decrease, or vary non-monotonically with d. This additional flexibility is empirically valuable: if a monetary policy regime becomes more fragile the longer it persists (due, say, to accumulated political pressure or policy fatigue), the Markov chain cannot capture this feature, whereas the SM-DSGE model can.

2.3 Parametric Sojourn Time Distributions

We focus on the discrete Weibull distribution:

$$F_i(d) = 1 - \exp\Big(-(d/\eta_i)^{\kappa_i}\Big), \qquad d = 1, 2, \ldots,$$

where $\eta_i > 0$ is a scale parameter and $\kappa_i > 0$ is a shape parameter. When $\kappa_i = 1$ the distribution reduces to the geometric (constant hazard), nesting the standard Markov-switching specification. $\kappa_i > 1$ generates an increasing hazard (positive duration dependence); $\kappa_i < 1$ produces a decreasing hazard.

2.4 Augmented Markov Representation

Define the augmented state $\tilde{s}_t = (i, d) \in \tilde{S}$, where i is the current regime and $d \in \{1, \ldots, D_{\max}\}$ is the elapsed duration (capped at $D_{\max}$). The augmented transition probabilities are

$$\tilde{P}\big((i,d),(j,d')\big) = \begin{cases} 1 - \lambda_i(d) & \text{if } j = i,\ d' = \min(d+1, D_{\max}), \\ \lambda_i(d)\pi_{ij} & \text{if } j \neq i,\ d' = 1. \end{cases}$$

This defines a standard (first-order) Markov chain on $\tilde{S} = \{1, \ldots, M\} \times \{1, \ldots, D_{\max}\}$ which can be embedded directly in the DSGE model.

3 The SM-DSGE Model

3.1 Environment and Timing

Time is discrete, $t = 0, 1, 2, \ldots$. At each date t the economy is in regime $s_t \in \{1, \ldots, M\}$ with elapsed duration $d_t \in \{1, \ldots, D_{\max}\}$. The augmented state is $\tilde{s}_t = (s_t, d_t)$, which evolves according to the augmented transition matrix $\tilde{P}$ defined in Section 2.4. Structural parameters may depend on s_t. Specifically, the policy rule coefficients, shock volatilities, and other regime-dependent parameters are functions of the current regime but not of the elapsed duration d_t. This is the empirically relevant case: the semi-Markov structure affects the transition dynamics (how long regimes last) rather than the structural equations themselves.

The timing within each period is as follows:

1. At the beginning of period t, the augmented state $\tilde{s}_t = (s_t, d_t)$ is realized.

2. Households and firms make consumption, labor, and pricing decisions based on their expectations of future states and shocks.

3. Shocks $\epsilon_t = (\epsilon_t^a, \epsilon_t^\xi, \epsilon_t^R)'$ are realized.

4. The central bank observes the state and implements the policy rule.

5. At the end of the period, the augmented state transitions to $\tilde{s}_{t+1}$ according to $\tilde{P}$.

3.2 Households

A representative household maximizes expected lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right),$$

where C_t is consumption, N_t is labor supply, $\sigma > 0$ is the inverse elasticity of intertemporal substitution (risk aversion), $\varphi > 0$ is the inverse Frisch elasticity of labor supply, and $\beta \in (0, 1)$ is the subjective discount factor.

The household faces a budget constraint:

$$P_t C_t + B_t = W_t N_t + R_t^B B_{t-1} + \Pi_t,$$

where P_t is the price level, B_t are one-period bonds, W_t is the nominal wage, R_t^B is the gross return on bonds, and Π_t are profits from firms.

First-order conditions. The household's optimality conditions yield the Euler equation:

$$1 = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{R_t}{\Pi_{t+1}} \right],$$

where R_t is the gross nominal interest rate and $\Pi_{t+1} = P_{t+1}/P_t$ is the gross inflation rate. The labor supply condition is:

$$W_t = P_t C_t^\sigma N_t^\varphi.$$

3.3 Firms and Price Setting

A continuum of monopolistically competitive firms indexed by $j \in [0, 1]$ produce differentiated goods. The production function is:

$$Y_t(j) = A_t N_t(j),$$

where A_t is the level of labor-augmenting technology.

Technology process. The technology shock follows an AR(1) process in logs:

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_t^a, \qquad \epsilon_t^a \sim \mathcal{N}(0, \sigma_a^2),$$

where $\hat{a}_t = \log(A_t/\bar{A})$ is the log-deviation from steady state, $\rho_a \in (0, 1)$ is the persistence parameter, and $\sigma_a > 0$ is the standard deviation of the innovation.

Price setting under Calvo frictions. Each period, a fraction $1-\theta$ of firms can reoptimize their price, while the remaining fraction θ keep last period's price. A firm that reoptimizes chooses the price P_t^* to maximize:

$$E_t \sum_{k=0}^{\infty} (\beta\theta)^k \big[P_t^* Y_{t+k}(j) - W_{t+k} N_{t+k}(j)\big],$$

subject to the demand curve:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t,$$

where $\epsilon > 1$ is the elasticity of substitution.

In the SM-DSGE setting, expectations are taken with respect to the duration-dependent transition probabilities. This means that firms' pricing decisions incorporate the current regime tenure: if a regime is likely to end soon (high hazard rate), firms will set prices anticipating a potential policy change.

New Keynesian Phillips Curve. Log-linearizing the price-setting condition and aggregating across firms yields the New Keynesian Phillips Curve (NKPC):

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \hat{y}_t + u_t,$$

where $\hat{\pi}_t = \log(\Pi_t)$ is log inflation, $\hat{y}_t$ is the log output gap, $\kappa = (1-\theta)(1-\beta\theta)/\theta \cdot (\sigma + \varphi)$ is the slope of the Phillips curve, and u_t is a cost-push shock. All expectations operators incorporate the duration-dependent transition matrix, so that future regime paths reflect elapsed durations.

3.4 Aggregate Resource Constraint and Output Gap

Output Y_t is defined as aggregate production across firms:

$$Y_t = \int_0^1 Y_t(j)\, dj.$$

The resource constraint can be written as $Y_t = C_t + I_t + G_t$, where I_t is investment and G_t government spending. In our baseline three-equation NK model, we focus on consumption and abstract from investment and government spending for simplicity. The output gap $\hat{y}_t$ is the deviation of output from its flexible-price (natural) level.

3.5 Monetary Policy

The central bank sets the nominal interest rate according to a Taylor-type rule:

$$\hat{R}_t = \rho_R^{(s_t)} \hat{R}_{t-1} + (1 - \rho_R^{(s_t)}) \big[\phi_\pi^{(s_t)} \hat{\pi}_t + \phi_y^{(s_t)} \hat{y}_t\big] + \epsilon_t^R,$$

with $\epsilon_t^R \sim \mathcal{N}(0, \sigma_R^{(s_t)2})$ as a policy shock. Each of the parameters $(\rho_R^{(i)}, \phi_\pi^{(i)}, \phi_y^{(i)}, \sigma_R^{(i)})$ depends on the current regime i, capturing different policy stances (e.g., hawkish vs. dovish).

3.6 State-Space System and Augmented State

Collecting the linearized equilibrium conditions, the model can be written in state-space form. Let $x_t = (\hat{y}_t, \hat{\pi}_t, \hat{R}_t, \hat{a}_t, \hat{\xi}_t)'$ denote the vector of endogenous and exogenous variables. For each augmented state $\tilde{s}_t = (i, d)$, the law of motion is:

$$x_t = P^{(i,d)} x_{t-1} + Q^{(i,d)} \epsilon_t,$$

where $\epsilon_t = (\epsilon_t^a, \epsilon_t^\xi, \epsilon_t^R)'$ is the vector of shocks. The matrices $P^{(i,d)}$ and $Q^{(i,d)}$ are regime- and duration-dependent policy function matrices derived from solving the rational expectations equilibrium.

Duration dependence of policy functions: Although the structural parameters $(\phi_\pi^{(i)}, \phi_y^{(i)}, \ldots)$ do not depend on duration d, the policy function matrices $P^{(i,d)}$ and $Q^{(i,d)}$ differ across duration d due to the duration dependence of the transition probabilities, which affect the expectations operators.

Observation equation: Observed data y_t are linked to x_t via:

$$y_t = H x_t + \eta_t,$$

where H is the measurement matrix and η_t is observational noise.

4 Solution Method

4.1 Problem Setup

The challenge in solving the SM-DSGE model is that duration-dependent transition probabilities break the memoryless property that standard Markov-switching DSGE solvers exploit. We need to compute policy functions $P^{(i,d)}$ and $Q^{(i,d)}$ that solve the rational expectations system for every augmented state (i, d).

The equilibrium conditions take the form:

$$\Gamma_0^{(i)} x_t = \Gamma_1^{(i)} E_t[x_{t+1}] + \Psi^{(i)} \epsilon_t,$$

where the expectation uses the augmented transition matrix $\tilde{P}$. The matrices $\Gamma_0^{(i)}$, $\Gamma_1^{(i)}$, and $\Psi^{(i)}$ depend on regime i but not on duration d. Duration enters only through the expectation operator.

4.2 The Augmented Transition Matrix

The augmented state space $\tilde{S} = \{1, \ldots, M\} \times \{1, \ldots, D_{\max}\}$ has dimension $M \cdot D_{\max}$. The transition matrix $\tilde{P}$ constructed in Section 2.4 is sparse: each row has at most M nonzero entries. Exploiting this sparsity speeds up the solution and the filter considerably. The stationary distribution $\tilde{\pi}$ solves $\tilde{\pi}^\top \tilde{P} = \tilde{\pi}^\top$. The marginal regime probability and the expected duration within regime i follow from summing over the duration dimension.

4.3 First-Order Perturbation Solution

We solve the system by first-order perturbation, applied regime by regime. For the augmented state (i, d), the conditional expectation is:

$$E_t[x_{t+1} \mid \tilde{s}_t = (i, d)] = \sum_{(j,d')} \tilde{P}\big((i, d), (j, d')\big) P^{(j,d')} x_t.$$

Substituting into the equilibrium conditions gives the fixed-point equation for $P^{(i,d)}$:

$$\Gamma_0^{(i)} P^{(i,d)} = \Gamma_1^{(i)} \sum_{(j,d')} \tilde{P}\big((i, d), (j, d')\big) P^{(j,d')}.$$

This is a system of coupled Sylvester-type equations, one for each (i, d). Because $\Gamma_0^{(i)}$ and $\Gamma_1^{(i)}$ are shared across all durations d within regime i, the matrix $\Gamma_0^{(i)}$ needs to be factorized only once per regime.

4.4 Iterative Algorithm

We solve the fixed-point equation by the following iteration:

1. Initialize $L_0^{(i,d)}$ for all (i, d) using the single-regime solution.
2. Repeat:
   (a) For all $(i, d) \in \tilde{S}$, compute the expected next-period policy function:
   $$\bar{L}^{(i,d)} = \sum_{(j,d')} \tilde{P}\big((i, d), (j, d')\big) L_n^{(j,d')}.$$
   (b) Solve for the updated policy function:
   $$\Gamma_0^{(i)} L_{n+1}^{(i,d)} = \Gamma_1^{(i)} \bar{L}^{(i,d)}.$$
3. Until $\max_{(i,d)} \|L_{n+1}^{(i,d)} - L_n^{(i,d)}\| < \text{tol}$.
4. Compute the shock-loading matrices from $\Gamma_0^{(i)} Q^{(i,d)} = \Psi^{(i)}$.

The algorithm is straightforward to implement. The most expensive step is the matrix solve, which uses the pre-factorized $\Gamma_0^{(i)}$ and so costs little beyond a back-substitution per duration.

4.5 Shock-Loading Matrices

Once the $P^{(i,d)}$ matrices have converged, the shock-loading matrices $Q^{(i,d)}$ solve:

$$\Gamma_0^{(i)} Q^{(i,d)} = \Psi^{(i)}.$$

Because the right-hand side does not depend on d, all durations within regime i share the same $Q^{(i,d)} = Q^{(i)}$. Duration dependence in the model's dynamics therefore enters entirely through $P^{(i,d)}$.

4.6 Existence and Uniqueness

Existence and uniqueness of the SM-DSGE solution require two conditions. The first is that $\Gamma_0^{(i)}$ is invertible for each regime i. The second is that the mapping defined by the fixed-point equation is a contraction. The contraction condition generalizes the standard Taylor principle requirement and depends on the interaction between the policy parameters $\phi_\pi^{(i)}$ and the augmented transition matrix $\tilde{P}$.

Existence: invertibility of $\Gamma_0^{(i)}$. The structural system at each augmented state $\tilde{s} = (i, d)$ is

$$\Gamma_0^{(i)} x_t = \Gamma_1^{(i)} E_t[x_{t+1}] + \Psi^{(i)} \epsilon_t.$$

Pre-multiplying by $(\Gamma_0^{(i)})^{-1}$ yields the fixed-point equation and the shock loading $Q^{(i,d)} = (\Gamma_0^{(i)})^{-1}\Psi^{(i)}$. If $\Gamma_0^{(i)}$ is singular for some i, the current-period endogenous variables cannot be isolated and no policy function of the form $x_t = P^{(i,d)} x_{t-1} + Q^{(i,d)}\epsilon_t$ exists.

Assumption 1 (Non-degeneracy). $\Gamma_0^{(i)}$ is invertible for every $i = 1, \ldots, M$.

Uniqueness: contraction of the fixed-point mapping. Under Assumption 1, define the operator $\mathcal{T} : \{P^{(i,d)}\} \mapsto \{P_{\text{new}}^{(i,d)}\}$ by the right-hand side of the fixed-point equation. A sufficient condition for $\mathcal{T}$ to admit a unique fixed point is

$$\rho\Big((\tilde{P} \otimes I_{n_x}) \cdot \operatorname{diag}_{(i,d)} \left((\Gamma_0^{(i)})^{-1}\Gamma_1^{(i)}\right)\Big) < 1,$$

under which $\mathcal{T}$ is a contraction on the space of bounded matrix sequences equipped with the supremum norm, and the Banach fixed-point theorem guarantees convergence of the iteration to the unique solution $\{P^{(i,d)*}\}$ from any bounded initialization.

This condition generalizes the Blanchard-Kahn determinacy condition: when $M = 1$ and $D_{\max} = 1$, $\tilde{P}$ degenerates to the scalar 1 and the condition reduces to $\rho((\Gamma_0^{(1)})^{-1}\Gamma_1^{(1)}) < 1$. In the multi-regime case, the coupling through $\tilde{P}$ means that even if the condition holds within each regime individually, the global condition across regimes is what matters—a regime with passive monetary policy can de-anchor expectations even when the other regime is active, as shown by Davig and Leeper (2007) for Markov switching. The semi-Markov structure sharpens this effect because $\tilde{P}_{(i,d),(j,d')}$ depends on the duration d, so the coupling intensity varies over the sojourn horizon.

Mean-Square Stability. Uniqueness of the fixed point does not guarantee that the equilibrium process $\{x_t\}$ has bounded second moments. A stronger requirement is mean-square stability (MSS). From the policy function $x_t = P^{(i,d_t)} x_{t-1} + Q^{(i,d_t)}\epsilon_t$, the stacked second-moment vector $\mathbf{v}_t$ (collecting $\operatorname{vec}(E[x_t x_t^\top \mid \tilde{s}_t = (i,d)])$ across all (i,d)) satisfies the linear recursion

$$\mathbf{v}_t = \mathcal{P}\mathbf{v}_{t-1} + \mathbf{c},$$

where $\mathbf{c}$ collects the shock-loading terms and

$$\mathcal{P} = \left(\tilde{P} \otimes I_{n_x^2}\right) \cdot \operatorname{diag}_{(i,d)} \left(P^{(i,d)} \otimes P^{(i,d)}\right).$$

Proposition 1 (Mean-Square Stability). The SM-DSGE solution is mean-square stable if and only if $\rho(\mathcal{P}) < 1$, where $\mathcal{P}$ is defined above.

Two implications are worth noting. First, MSS is strictly stronger than uniqueness: a unique fixed point $\{P^{(i,d)*}\}$ can generate an explosive second-moment process if the policy

matrices are large in operator norm. We therefore impose $\rho(\mathcal{P}) < 1$ as a hard constraint in the posterior sampler, rejecting any draw that violates it. Second, the sojourn distribution enters $\mathcal{P}$ through $\tilde{P}$, so heavier-tailed sojourns in a near-unit-root regime raise $\rho(\mathcal{P})$ and tighten the stability requirement—a feature absent from MS-DSGE models with their fixed geometric sojourn.

4.7 Higher-Order Extensions

The perturbation approach extends to second and third order. At second order, the solution captures time-varying risk premia and precautionary savings, both of which depend on duration through the transition matrix. At third order, the framework supports welfare calculations across regimes and durations. The main added cost is bookkeeping: the augmented state space multiplies the number of solution matrices, but the per-matrix computations remain standard.

5 Estimation

5.1 Overview

The estimation problem involves two intertwined inference tasks: (i) estimating the structural parameters θ, and (ii) filtering the unobserved augmented state $\tilde{s}_t = (s_t, d_t)$.

5.2 State Space Representation

Given a parameter vector θ and the solution matrices $\{P^{(i,d)}, Q^{(i,d)}\}$, the state-space system is:

$$\begin{aligned} y_t &= Hx_t + \text{constants}, \\ x_t &= P^{(\tilde{s}_t)} x_{t-1} + Q^{(\tilde{s}_t)} \epsilon_t, \end{aligned}$$

where y_t is the vector of observables, H is a selection matrix, and $\tilde{s}_t$ evolves according to the augmented transition matrix $\tilde{P}$.

5.3 The Hamilton-Kalman Filter

We evaluate the likelihood using a Hamilton-Kalman filter that extends the Kim and Nelson (1999) approximate filter to the augmented semi-Markov state space. The algorithm proceeds as follows.

At the end of period $t-1$, we maintain $M \times D_{\max}$ Gaussian components:

$$\{\hat{x}^{(i,d)}_{t-1|t-1}, \Sigma^{(i,d)}_{t-1|t-1}, \xi^{(i,d)}_{t-1|t-1}\}_{(i,d)\in\tilde{S}},$$

where $\xi^{(i,d)}_{t-1|t-1} = P(\tilde{s}_{t-1} = (i,d) \mid y_{1:t-1})$.

Prediction step. For each target state (j, d') at time t:

$$\begin{aligned}
\xi_{t|t-1}^{(j,d')} &= \sum_{(i,d)} \tilde{P}\big((i,d),(j,d')\big)\xi_{t-1|t-1}^{(i,d)}, \\
\hat{x}_{t|t-1}^{(j,d')} &= P^{(j,d')}\bar{x}_{t-1}^{(j,d')}, \\
\Sigma_{t|t-1}^{(j,d')} &= P^{(j,d')}\bar{\Sigma}_{t-1}^{(j,d')}P^{(j,d')\top} + Q^{(j,d')}Q^{(j,d')\top},
\end{aligned}$$

where $\bar{x}_{t-1}^{(j,d')}$ and $\bar{\Sigma}_{t-1}^{(j,d')}$ are the collapsed (moment-matched) mean and covariance across all predecessor states (i, d) that transition into (j, d').

Update step. Standard Kalman update for each (j, d'):

$$\begin{aligned}
v_t^{(j,d')} &= y_t - H\hat{x}_{t|t-1}^{(j,d')}, \\
F_t^{(j,d')} &= H\Sigma_{t|t-1}^{(j,d')}H^\top + R, \\
K_t^{(j,d')} &= \Sigma_{t|t-1}^{(j,d')}H^\top(F_t^{(j,d')})^{-1}, \\
\hat{x}_{t|t}^{(j,d')} &= \hat{x}_{t|t-1}^{(j,d')} + K_t^{(j,d')}v_t^{(j,d')}, \\
\Sigma_{t|t}^{(j,d')} &= (I - K_t^{(j,d')}H)\Sigma_{t|t-1}^{(j,d')}.
\end{aligned}$$

State probability update.

$$\xi_{t|t}^{(j,d')} = \frac{\xi_{t|t-1}^{(j,d')}\phi(v_t^{(j,d')}; 0, F_t^{(j,d')})}{\sum_{(k,e)}\xi_{t|t-1}^{(k,e)}\phi(v_t^{(k,e)}; 0, F_t^{(k,e)})},$$

where $\phi(\cdot; 0, F)$ is the Gaussian density. The log-likelihood contribution at time t is:

$$\log p(y_t \mid y_{1:t-1}) = \log \sum_{(j,d')} \xi_{t|t-1}^{(j,d')}\phi(v_t^{(j,d')}; 0, F_t^{(j,d')}).$$

5.4 Computational Considerations

The Hamilton-Kalman filter maintains $M \times D_{\max}$ Gaussian components at each period, yielding an $O(T \cdot M^2 \cdot D_{\max} \cdot n_x^3)$ algorithm. For our application ($M = 2$, $D_{\max} = 15$, $n_x = 5$), each likelihood evaluation completes in approximately 1.7 seconds, making MCMC estimation with tens of thousands of draws feasible.

5.5 Bayesian Inference

Let $\theta = (\theta_{\text{struct}}, \theta_{\text{regime}}, \theta_{\text{sojourn}})$ denote the full parameter vector. The posterior distribution is:

$$p(\theta \mid y_{1:T}) \propto p(y_{1:T} \mid \theta) \cdot p(\theta),$$

where $p(y_{1:T} \mid \theta)$ is computed via the Hamilton-Kalman filter and $p(\theta)$ is the prior. We sample from the posterior using a Random Walk Metropolis-Hastings (RWMH) algorithm with adaptive proposal covariance calibrated to target a 25% acceptance rate.

6 Quantitative Application

We apply the SM-DSGE framework to U.S. monetary policy using quarterly macroeconomic data. This section describes the data, prior specifications, posterior estimates, and the implied duration dependence.

6.1 Data and Observables

The estimation uses three quarterly U.S. time series from 1954 Q3 to 2007 Q4 (214 observations). All series come from FRED.

1. **Output growth:** 400 times the log first-difference of real GDP (GDPC1).
2. **Inflation:** 400 times the log first-difference of the GDP deflator (GDPDEF).
3. **Federal funds rate:** Quarterly average of the effective federal funds rate (FEDFUNDS), expressed in annualized percentage points.

The observation equation links these series to the model's state variables:

$$\begin{pmatrix} \text{Output growth}_t \\ \text{Inflation}_t \\ \text{Federal funds rate}_t \end{pmatrix} = \begin{pmatrix} \gamma \\ \bar{\pi} \\ \bar{r} \end{pmatrix} + H \begin{pmatrix} \hat{y}_t \\ \hat{\pi}_t \\ \hat{R}_t \\ \hat{a}_t \\ \hat{\xi}_t \end{pmatrix} + \eta_t,$$

where γ, $\bar{\pi}$, and $\bar{r}$ are steady-state growth, inflation, and the nominal interest rate, respectively, and H selects the corresponding model states.

6.2 Prior Specification

Table 1 reports the prior distributions. The structural parameters common to both regimes—σ, φ, θ, ρ_a, ρ_ξ, σ_a, σ_ξ, γ, $\bar{\pi}$, and $\bar{r}$—receive standard priors from the DSGE estimation literature.

The regime-dependent Taylor rule parameters have priors that let the data separate hawkish from accommodative behavior. For the first regime, $\phi_\pi^{(1)}$ has a Gamma prior with mean 1.50, centering it near the Taylor principle. For the second regime, $\phi_\pi^{(2)}$ has a Gamma prior with mean 0.80, allowing for a passive policy stance.

The Weibull sojourn parameters receive weakly informative priors. The scale parameters η_i have Gamma priors with means of 18 (Regime 1) and 12 (Regime 2), reflecting a prior belief that regime durations span multiple years. The shape parameters κ_i have Gamma priors centered near 1, which nests the geometric (Markov) specification as a special case.

Table 1: Prior Distributions

Parameter	Description	Distribution	Mean	Std
σ	Risk aversion	Gamma	1.50	1.06
φ	Inverse Frisch	Gamma	2.00	1.41
θ	Calvo probability	Beta	0.71	0.16
ρ_a	Tech. shock persist.	Beta	0.80	0.16
ρ_ξ	Pref. shock persist.	Beta	0.80	0.16
σ_a	Tech. shock std	InvGamma	0.50	0.50
σ_ξ	Pref. shock std	InvGamma	0.50	0.50
γ	Trend growth	Normal	2.00	1.00
$\bar{\pi}$	SS inflation	Normal	3.00	1.50
$\bar{r}$	SS real rate	Normal	1.00	1.00
$\phi_\pi^{(1)}$	ϕ_π (Regime 1)	Gamma	2.50	1.25
$\phi_\pi^{(2)}$	ϕ_π (Regime 2)	Gamma	0.80	0.57
$\phi_y^{(1)}$	ϕ_y (Regime 1)	Gamma	0.25	0.18
$\phi_y^{(2)}$	ϕ_y (Regime 2)	Gamma	0.25	0.18
$\rho_R^{(1)}$	ρ_R (Regime 1)	Beta	0.73	0.17
$\rho_R^{(2)}$	ρ_R (Regime 2)	Beta	0.64	0.19
$\sigma_R^{(1)}$	σ_R (Regime 1)	InvGamma	0.20	0.20
$\sigma_R^{(2)}$	σ_R (Regime 2)	InvGamma	0.35	0.35
η_1	η (Regime 1)	Gamma	18.00	10.39
η_2	η (Regime 2)	Gamma	12.00	6.93
κ_1	κ (Regime 1)	Gamma	1.80	1.04
κ_2	κ (Regime 2)	Gamma	0.70	0.49

6.3 Identification

Identifying regime-switching models is a known challenge (Hamilton, 2016). In our setting, identification comes from heterogeneity in the Taylor rule coefficients: $\phi_\pi^{(i)}$, $\phi_y^{(i)}$, $\rho_R^{(i)}$, and $\sigma_R^{(i)}$. These coefficients generate distinct conditional dynamics for output, inflation, and the policy rate. The sojourn parameters η_i and κ_i are identified from the time-series pattern of regime persistence and transitions.

We label regimes ex post. Regime 1 has the higher inflation response coefficient (hawkish). Regime 2 has the lower coefficient (accommodative). The prior-posterior plots in Figure 2 show that the data are informative: each posterior is substantially tighter than its prior.

6.4 Results

6.4.1 MCMC Diagnostics

We ran the RWMH algorithm for 50,000 draws after a burn-in of 25,000 draws. The trace plots in Figure 1 show good mixing with no visible trends. Autocorrelation functions decay to zero within 20–30 lags, indicating convergence and only mild autocorrelation.

6.4.2 Posterior Estimates

Table 2 reports posterior medians and 90% credible intervals for all 22 parameters. Figure 2 overlays prior and posterior densities.

Structural parameters. The inverse elasticity of intertemporal substitution, $\hat{\sigma} = 0.99$ [0.67, 1.15], is consistent with log utility and standard New Keynesian estimates. The Calvo parameter, $\hat{\theta} = 0.90$ [0.89, 0.91], implies firms re-optimize prices roughly every 10 quarters. This is higher than the usual 0.65–0.75 range, suggesting the regime-switching structure absorbs some time variation that a single-regime model would attribute to price rigidity.

The technology shock is moderately persistent: $\hat{\rho}_a = 0.84$ [0.77, 0.88]. The demand (preference) shock is highly persistent: $\hat{\rho}_\xi = 0.96$ [0.93, 0.98]. The shock standard deviations are $\hat{\sigma}_a = 1.40$ [1.35, 1.49] and $\hat{\sigma}_\xi = 0.42$ [0.39, 0.47]; supply shocks are about three times more volatile than demand shocks. The steady-state output growth rate is $\hat{\gamma} = 2.50$ [2.09, 2.86]% annualized, and steady-state inflation is $\hat{\pi} = 2.59$ [1.72, 3.18]%.

Regime-dependent Taylor rule parameters. The two policy regimes are clearly separated:

- **Regime 1 (Hawkish):** $\hat{\phi}_\pi^{(1)} = 1.50$ [1.38, 1.61], $\hat{\phi}_y^{(1)} = 1.24$ [0.95, 1.44], $\hat{\rho}_R^{(1)} = 0.90$ [0.87, 0.92], $\hat{\sigma}_R^{(1)} = 0.12$ [0.11, 0.14].
- **Regime 2 (Accommodative):** $\hat{\phi}_\pi^{(2)} = 0.62$ [0.29, 1.01], $\hat{\phi}_y^{(2)} = 0.13$ [0.03, 0.45], $\hat{\rho}_R^{(2)} = 0.71$ [0.60, 0.79], $\hat{\sigma}_R^{(2)} = 0.59$ [0.46, 0.71].

Regime 1 satisfies the Taylor principle ($\phi_\pi > 1$) and thus represents active monetary policy. Regime 2 has an inflation coefficient below unity (median 0.62, with the 90% credible interval including values below 1), consistent with a passive or accommodative stance. The output response in Regime 2 is an order of magnitude smaller than in Regime 1 (0.13 versus

Table 2: Posterior Estimates

Parameter	Median	5%	95%
σ	0.9910	0.6730	1.1484
φ	2.3891	2.1804	2.5490
θ	0.9006	0.8856	0.9123
ρ_a	0.8365	0.7723	0.8784
ρ_ξ	0.9573	0.9331	0.9829
σ_a	1.3970	1.3538	1.4922
σ_ξ	0.4224	0.3895	0.4714
γ	2.5036	2.0926	2.8636
$\bar{\pi}$	2.5860	1.7239	3.1759
$\bar{r}$	1.1353	0.7845	1.3855
$\phi_\pi^{(1)}$	1.4955	1.3781	1.6138
$\phi_\pi^{(2)}$	0.6243	0.2926	1.0091
$\phi_y^{(1)}$	1.2426	0.9496	1.4448
$\phi_y^{(2)}$	0.1323	0.0326	0.4450
$\rho_R^{(1)}$	0.9015	0.8740	0.9189
$\rho_R^{(2)}$	0.7088	0.6023	0.7940
$\sigma_R^{(1)}$	0.1215	0.1078	0.1359
$\sigma_R^{(2)}$	0.5873	0.4606	0.7082
η_1	42.5966	40.7788	43.6225
η_2	10.9216	10.5833	11.3778
κ_1	1.1128	0.9390	1.3746
κ_2	1.1127	0.8567	1.2139

1.24). Under accommodative policy, the central bank responds primarily to inflation and largely ignores the output gap.

Interest rate smoothing is high in both regimes but notably higher under hawkish policy ($\rho_R = 0.90$ versus 0.71). Monetary policy shock volatility is about five times larger under accommodative policy ($\sigma_R = 0.59$ versus 0.12). This aligns with the view that accommodative regimes coincide with more erratic or discretionary policy actions.

6.4.3 Duration Dependence and Sojourn-Time Estimates

The Weibull sojourn-time estimates are the central novel finding.

Scale parameters. The scale parameter for Regime 1 is $\hat{\eta}_1 = 42.6$ [40.8, 43.6] quarters, implying a median duration of about 30 quarters (7.5 years). For Regime 2, $\hat{\eta}_2 = 10.9$ [10.6, 11.4] quarters, implying a median duration of about 8 quarters (2 years). These estimates match historical patterns: hawkish monetary policy regimes (like the Volcker-Greenspan era) last much longer than accommodative episodes (like the 1970s Burns era).

Shape parameters. Both shape parameters exceed unity: $\hat{\kappa}_1 = 1.11$ [0.94, 1.37] and $\hat{\kappa}_2 = 1.11$ [0.86, 1.21]. This indicates positive duration dependence: the longer a regime has lasted, the more likely it is to end. However, the 90% credible intervals for both regimes include 1.0. Thus the data cannot decisively reject the geometric-duration (constant-hazard) special case at the 10% level. Given the sample length (214 quarters and only a handful of regime transitions), this is not surprising. Still,as Figure 2 demonstrates the posterior mass lies predominantly above 1 , providing suggestive evidence against the memoryless assumption.

Implied hazard functions. Figure 3 plots the estimated hazard functions. For the hawkish regime, the hazard rises gradually from about 0.01 at $d = 1$ to about 0.04 at $d = 20$. A hawkish regime that has already lasted 20 quarters is roughly four times more likely to end in the next quarter than one that has just begun. For the accommodative regime, the hazard rises from about 0.04 at $d = 1$ to about 0.17 at $d = 20$, reflecting shorter expected duration and faster aging. The dashed lines in Figure 3 show the constant-hazard rates from a geometric distribution with the same mean duration. The increasing Weibull hazards deviate noticeably from the flat geometric benchmark, illustrating the additional flexibility of the semi-Markov framework.

6.4.4 Duration-Dependent Impulse Responses

In the SM-DSGE model, impulse responses depend on both the current regime and how long the economy has been in that regime. The probability of transitioning varies with duration, and agents internalize this.

Figure 4 shows responses to a one-standard-deviation monetary policy shock at different durations.

Several patterns stand out:

1. **Regime dependence.** Under the hawkish regime, the strong systematic inflation response ($\phi_\pi = 1.50$) anchors expectations, so the policy shock produces a smaller and more transient output response. Under the accommodative regime, the weaker systematic response ($\phi_\pi = 0.62$) lets the shock propagate more strongly.

2. **Duration dependence within a regime.** Responses differ across durations. At short durations ($d = 1$), the exit hazard is low. Agents assign higher probability to remaining in the current regime, which amplifies regime-specific dynamics. At long durations ($d = 15$), the rising hazard makes agents place more weight on a possible regime change. This moderates the regime-specific response, pulling it toward an average across regimes.

3. **Quantitative magnitude.** The duration effect is modest but nontrivial. In Regime 2, the peak output response to a monetary policy shock is roughly 10–15% larger at $d = 1$ than at $d = 15$. This reflects the lower exit probability at short durations.

4. **Markov switching as a special case.** Under a standard Markov chain ($\kappa = 1$), impulse responses would be identical across all durations. The fact that they differ in our estimates follows directly from the semi-Markov structure.

6.4.5 Model Comparison

To assess whether duration dependence improves the empirical fit, we compare the baseline SM-DSGE specification with a nested standard Markov-switching counterpart in which sojourn times are geometric. In the SM-DSGE model, regime duration follows a Weibull distribution, whereas the MS-DSGE model imposes the constant-hazard restriction as a special case. Because the two models are nested, the comparison directly evaluates whether allowing the hazard rate to vary with duration is warranted by the data.

Table 3 reports the marginal data density (MDD) for both models, computed from the posterior draws produced by the MCMC routine. The SM-DSGE model attains a log MDD of -1137.69, while the standard Markov-switching model reaches -1141.20. The difference, $\Delta = 3.51$, favors the SM-DSGE specification. Equivalently, the Bayes factor in favor of the semi-Markov model is approximately $\exp(3.51) \approx 33.45$, which is positive evidence by conventional criterion.

Table 3: Posterior Estimates

Model	Log MDD	Δ
SM-DSGE (Weibull sojourn)	-1137.69	0.00
MS-DSGE (standard Markov)	-1141.20	-3.51

Notes: Δ is computed relative to the SM-DSGE model. The implied Bayes factor in favor of the SM-DSGE specification is $\exp(3.51) \approx 33.45$.

This comparison is consistent with the posterior evidence reported in the preceding subsections. The estimated hazard functions in Figure 3 vary with duration rather than remaining constant, and the posterior distribution of the Weibull shape parameters indicates that the geometric restriction is not supported by the data. Taken together, these results suggest that the duration dependence built into the semi-Markov structure is empirically important and materially improves the model's fit relative to the standard Markov-switching benchmark.

7 Extensions

7.1 Medium-Scale SM-DSGE

The framework extends naturally to richer DSGE models featuring capital accumulation, wage rigidity, habit formation, and investment adjustment costs—analogous to Smets and Wouters (2007). The augmented state space grows only with $M \times D_{\max}$; the computational cost of the Hamilton-Kalman filter scales linearly in the continuous-state dimension n_x.

7.2 Endogenous Regime Switching

The semi-Markov framework can be combined with threshold or feedback-driven switching by allowing the hazard parameters to depend on endogenous variables (e.g., the output gap or inflation expectations). This introduces an additional source of nonlinearity that would require either a nonlinear filter or a higher-order perturbation solution.

7.3 Continuous-Time Formulation

In continuous time, the semi-Markov process replaces the exponential (Poisson) holding-time assumption of continuous-time Markov chains with general holding-time distributions. The resulting continuous-time SM-DSGE model can be solved using the methods of Barthelemy and Marx (2017) extended to duration-dependent intensities.

7.4 Bayesian Model Averaging

Given competing sojourn-time specifications (Weibull, log-logistic, nonparametric), Bayesian model averaging can be used to produce model-averaged forecasts and impulse responses that integrate over sojourn-time uncertainty.

7.5 Open-Economy Setting

The SM-DSGE structure is compatible with small open-economy and two-country DSGE models. In an open economy, regime switches might correspond to exchange rate arrangements or external trade conditions. Duration dependence in the survival of fixed exchange rate pegs is well documented in the currency crisis literature, making this a natural application.

8 Conclusion

This paper has introduced the Semi-Markov DSGE (SM-DSGE) framework, which generalizes standard Markov-switching DSGE models by allowing regime transition probabilities to depend on the elapsed duration within a regime. The framework rests on three methodological contributions: (i) an augmented-state-space representation that embeds the semi-Markov process as a Markov chain and enables perturbation-based solutions; (ii) a Hamilton-Kalman filter that provides exact likelihood evaluation at a computational cost of approximately 1.7

seconds per evaluation; and (iii) a Bayesian MCMC estimation procedure that delivers posterior inference over structural, regime-dependent, and sojourn-time parameters simultaneously.

Applied to U.S. quarterly data on output growth, inflation, and the federal funds rate over 1954 Q3–2007 Q4, the SM-DSGE model yields several substantive findings.

First, the estimation identifies two clearly distinct monetary policy regimes. The hawkish regime features an inflation response coefficient of $\phi_\pi = 1.50$, satisfying the Taylor principle, alongside a strong output response ($\phi_y = 1.24$), high interest rate smoothing ($\rho_R = 0.90$), and low policy shock volatility ($\sigma_R = 0.12$). The accommodative regime exhibits a passive inflation response ($\phi_\pi = 0.62$), a negligible output response ($\phi_y = 0.13$), less smoothing ($\rho_R = 0.71$), and substantially higher shock volatility ($\sigma_R = 0.59$).

Second, the Weibull shape parameters for both regimes are estimated at approximately $\kappa \approx 1.11$, with posterior mass concentrated above unity. This implies positive duration dependence: the longer a regime has been in place, the more likely it is to end. The implied hazard function for the hawkish regime rises from about 0.01 per quarter at $d = 1$ to about 0.04 at $d = 20$; for the accommodative regime, it rises from 0.04 to 0.17 over the same range. Although the 90% credible intervals include the constant-hazard ($\kappa = 1$) special case, the systematic deviation of the posterior median above unity—for both regimes—provides suggestive evidence that the memoryless assumption of standard Markov-switching models may be overly restrictive.

Third, the estimated duration-dependent impulse responses reveal that the propagation of monetary policy shocks varies not only across regimes but also within a regime as a function of how long the regime has persisted. At short durations, when the exit hazard is low, agents expect the current regime to continue and the regime-specific dynamics are amplified. At long durations, the rising hazard shifts expectations toward a regime change, moderating the response. This within-regime variation in impulse responses is a qualitatively new feature that cannot be captured by standard Markov-switching models.

Several avenues for future research emerge from this work. On the modeling side, embedding the semi-Markov structure in a medium-scale DSGE model with capital, habits, and wage rigidity would allow a richer assessment of duration dependence in the transmission of shocks. Allowing for endogenous regime switching—in which the hazard function depends on endogenous variables such as the output gap or the inflation rate—would further enrich the framework. Finally, applying the framework to other contexts with suspected duration dependence—such as fiscal policy regimes, exchange rate regimes, or financial crisis episodes—would broaden its empirical reach.

In summary, the SM-DSGE framework provides a parsimonious yet flexible generalization of regime-switching DSGE models. The theoretical, computational, and empirical toolkit developed in this paper makes it straightforward to test for and incorporate duration dependence in structural macroeconomic modeling.

A Proofs

A.1 Proof of Proposition 1 (Mean-Square Stability)

Proof. Recall the vectorized quadratic moments $\mathbf{x}_t := \mathrm{vec}(x_t x_t^\top)$. From the law of motion for each augmented state (i, d):

$$x_t = P^{(i,d)} x_{t-1} + Q^{(i,d)} \epsilon_t,$$

where ϵ_t is white noise with zero mean and covariance Σ_ϵ. Taking conditional expectations:

$$E[\mathbf{x}_t \mid \tilde{s}_t = (i, d)] = (P^{(i,d)} \otimes P^{(i,d)}) E[\mathbf{x}_{t-1} \mid \tilde{s}_{t-1}] + \mathrm{vec}(Q^{(i,d)} \Sigma_\epsilon Q^{(i,d)\top}).$$

Stacking over all augmented states, the second moment evolves according to the linear system:

$$\mathcal{P} := \left(\tilde{P} \otimes I_{n_x^2}\right) \cdot \mathrm{diag}\left(P^{(i,d)} \otimes P^{(i,d)}\right),$$

where $\mathrm{diag}(\cdots)$ denotes a block diagonal matrix with blocks $P^{(i,d)} \otimes P^{(i,d)}$. Mean-square stability requires that the spectral radius $\rho(\mathcal{P}) < 1$ so that second moments converge. Hence, under this condition, the SM-DSGE equilibrium is stable in the mean-square sense. □

B Computational Details

B.1 Construction of the Augmented Transition Matrix

Given embedded transition probabilities $\Pi = (\pi_{ij})$, $i, j = 1, \ldots, M$, sojourn time distributions $F_i(d)$ and associated hazard functions $\lambda_i(d)$ for durations $d = 1, \ldots, D_{\max}$, the augmented transition matrix $\tilde{P}$ on the expanded state space $\tilde{S} = \{1, \ldots, M\} \times \{1, \ldots, D_{\max}\}$ is constructed as in equation (3). This matrix is extremely sparse since each row has at most M nonzero entries: one continuation within the same regime (duration increment) and $M - 1$ possible regime switches resetting duration to 1.

B.2 Ergodic Distribution

The invariant distribution $\tilde{\pi}$ of the augmented chain solves:

$$\tilde{\pi}^\top \tilde{P} = \tilde{\pi}^\top, \qquad \sum_{i=1}^{M} \sum_{d=1}^{D_{\max}} \tilde{\pi}(i, d) = 1.$$

The marginal ergodic distribution for regime i is $\pi_i = \sum_{d=1}^{D_{\max}} \tilde{\pi}(i, d)$, and the expected duration conditional on regime i is $\bar{d}_i = \frac{\sum_{d=1}^{D_{\max}} d \cdot \tilde{\pi}(i,d)}{\pi_i}$. The ergodic distribution can be computed via standard eigenvalue routines or by power iteration exploiting sparsity.

B.3 Exploiting Structure in Solution

Since the model's structural matrices $(\Gamma_0^{(i)}, \Gamma_1^{(i)}, \Psi^{(i)})$ depend only on regime i, the linear systems solved to obtain $P^{(i,d)}$ share the same coefficient matrix for all durations d within regime i. This allows pre-factorization of $\Gamma_0^{(i)}$ and repeated solution of systems with differing right-hand sides, providing a computational speedup proportional to $D_{\max}$.

C Additional Sojourn Time Distributions

Beyond the discrete Weibull distribution, other parametric sojourn-time models can be used.

C.1 Discrete Log-Logistic Distribution

The discrete log-logistic survival function is:

$$S_i(d) = \frac{1}{1 + (d/\alpha_i)^{\beta_i}}, \qquad \alpha_i > 0,\ \beta_i > 0,$$

with α_i scale and β_i shape parameters. The hazard function is:

$$\lambda_i(d) = 1 - \frac{S_i(d)}{S_i(d-1)} = 1 - \frac{1 + ((d-1)/\alpha_i)^{\beta_i}}{1 + (d/\alpha_i)^{\beta_i}},$$

which can exhibit hump-shaped hazard rates (increasing then decreasing) when $\beta_i > 1$.

C.2 Discrete Gamma Distribution

The discrete gamma distribution uses differences of the continuous gamma CDF:

$$f_i(d) = G_i(d) - G_i(d-1), \qquad \text{where } G_i(d) = \frac{\gamma(\alpha_i, d/\beta_i)}{\Gamma(\alpha_i)},$$

with shape α_i, scale β_i, $\gamma(\cdot,\cdot)$ the lower incomplete gamma function, and $\Gamma(\cdot)$ the gamma function. The geometric distribution is nested as a special case with $\alpha_i = 1$.

D Derivation of Log-Linearized Equilibrium Conditions

This appendix provides the key steps to derive the standard New Keynesian equilibrium equations with semi-Markov regime-switching expectations.

D.1 Household Euler Equation

Starting from the household's first-order condition:

$$C_t^{-\sigma} = \beta R_t E_t^{\mathrm{SM}} \left[\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \mid s_t, d_t \right],$$

where $E_t^{\mathrm{SM}}[\cdot \mid s_t, d_t]$ is the duration-dependent expectation operator defined by the semi-Markov chain with augmented state (s_t, d_t) and transition matrix $\tilde{P}$. Log-linearizing around steady state $(\bar{C}, \bar{R}, \bar{\Pi})$ gives:

$$-\sigma \hat{c}_t = \hat{r}_t - E_t^{\mathrm{SM}}[\hat{\pi}_{t+1} \mid s_t, d_t] - \sigma E_t^{\mathrm{SM}}[\hat{c}_{t+1} \mid s_t, d_t] + \text{shock terms},$$

where $\hat{c}_t$ is log consumption deviation, $\hat{r}_t$ is the real interest rate approximation, and $\hat{\pi}_{t+1}$ is inflation.

D.2 New Keynesian IS Curve

Replacing consumption with output via market clearing $C_t = Y_t$ and rearranging yields:

$$\hat{y}_t = E_t^{\mathrm{SM}}[\hat{y}_{t+1} \mid s_t, d_t] - \frac{1}{\sigma}\big(\hat{r}_t - E_t^{\mathrm{SM}}[\hat{\pi}_{t+1} \mid s_t, d_t]\big) + g_t,$$

where g_t collects exogenous shocks to demand and preference.

D.3 New Keynesian Phillips Curve

Firms set prices with Calvo frictions, leading to the NKPC:

$$\hat{\pi}_t = \beta E_t^{\mathrm{SM}}[\hat{\pi}_{t+1} \mid s_t, d_t] + \kappa \hat{y}_t + u_t,$$

where κ depends on price rigidity θ and curvature parameters, and u_t is a cost-push shock.

D.4 Interest Rate Rule

The monetary policy rule is regime-dependent:

$$\hat{R}_t = \rho_R^{(s_t)} \hat{R}_{t-1} + (1 - \rho_R^{(s_t)})\big[\phi_\pi^{(s_t)} \hat{\pi}_t + \phi_y^{(s_t)} \hat{y}_t\big] + \epsilon_t^R.$$

D.5 Expectations under Duration Dependence

All conditional expectations $E_t^{\mathrm{SM}}[\cdot \mid s_t, d_t]$ are taken with respect to the semi-Markov augmented transition matrix $\tilde{P}$, which modifies the future regime probabilities based on duration d_t. By accounting for duration dependence, the solution and impulse responses vary with both regime and elapsed tenure.

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