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Hierarchical Terminal Sliding Mode Contouring Control for Five-Axis CNC Machine Tools With Combined Contour Error Estimation

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Abstract. Five-axis computer numerical control (CNC) machines achieve superior machining flexibility, but the coupling of three translational and two rotary feed drives makes direct regulation of the tool-tip contour error, the normal-direction deviation from the desired path, substantially harder than in three-axis systems. This paper proposes a *hierarchical terminal sliding mode contouring* (HTSMC) controller for five-axis feed drives. At the lower level, a non-singular terminal sliding mode (NTSM) law with integral action and a continuous boundary-layer/power-rate reaching law drives each of the five drive-axis tracking errors to a finite-time neighbourhood of zero without chattering. At the upper level, a terminal contour sliding surface regulates the tool-tip contour error, whose instantaneous value is estimated on-line by a combined algorithm fusing the delayed Frenet-frame coordinate transformation with iterative curvature-circle refinement; the estimator stays accurate under the tangential lag that a naive instantaneous projection over-reports by more than 80 %. A composite Lyapunov analysis that *retains the inter-level coupling* establishes finite-time convergence of the full hierarchical sliding variable with an explicit settling-time bound, and non-singularity of all terminal surfaces is proven. In simulations on a five-axis model with parametric mismatch and a cutting-force disturbance, and against a conventional linear SMC, a non-singular terminal SMC, and the contour-free ablation, all sharing the *same* continuous reaching law so that the comparison isolates the structural contribution rather than the reaching law, the proposed controller attains the lowest tool-tip contour error. The hierarchical upper level lowers the RMS contour error by 14 % to 19 % over its contour-free ablation, a margin that grows with feed rate, and the closed loop is robust to $\pm 20\%$ variation in every control gain.

Keywords: Five-axis machine tools; Contouring control; Terminal sliding mode control; Contour error estimation.

1 Introduction

Five-axis CNC machine tools combine three translational axes (X, Y, Z) with two rotary axes (A, C), so that the cutting tool can reach a workpiece from almost any orientation. This capability is indispensable in the manufacture of aerospace structural parts and turbine blades, dies and moulds, biomedical implants and other free-form sculptured surfaces, where a fixed tool orientation would gouge the surface or leave large scallop heights. The extra orientational freedom, however, comes at a price: five servo drives with widely different inertias and travel ranges must be coordinated in real time, the rotary axes introduce strong nonlinear coupling between tool orientation and tool-tip position, and cutting forces, friction and parameter variations disturb every axis. All of these effects ultimately surface as *contour error*, the instantaneous normal-direction distance between the actual tool tip and the desired spatial path. Because it is contour error, and not the per-axis tracking error, against which a finished part's dimensional and surface quality is judged, its direct suppression is the central objective of a contouring controller.

Regulating contour error is fundamentally harder than regulating axis tracking error, and the difficulty is amplified in five-axis machining. A set of individually small but correlated axis errors can combine into

a large normal deviation on a curved path, so minimising each axis error independently does not minimise the contour error [12]. Furthermore, the contour error is not measured directly; it must be estimated on-line from the axis positions and the reference geometry. On a curved path the naive estimate, the instantaneous normal projection of the position error at the commanded point, is biased whenever the tool lags tangentially behind the reference, because part of the already-traversed tangential error is then mistaken for normal deviation. This bias grows with feed rate and with path curvature, precisely the conditions under which the contour error is largest, so an accurate on-line estimator is a prerequisite for, not merely an accessory to, high-accuracy contouring.

Contour-error control originates with the variable-gain cross-coupling controller (CCC) of Koren and Lo [12], which feeds a weighted combination of axis errors back to each drive. Subsequent work refined the contour-error model with task-coordinate and global task-frame formulations and extended it to three-dimensional and five-axis geometry. El Khalick and Uchiyama introduced an iterative curvature-circle refinement that removes the finite-radius bias of the linear estimate [4], and later a delayed Frenet-frame transformation together with an orientation contour-error estimator that keeps the tool-tip and tool-orientation motions synchronised [5, 6]; the five-

axis contour model of Sencer and Altintas [23] is widely used for controller design. More recent NC-data, digital-twin and spline-based five-axis schemes achieve high accuracy but generally rely on off-line iteration or identification rather than a real-time finite-time feedback law [10, 14, 24, 25].

The quality of any contouring controller is bounded by the quality of its contour-error estimate, because the controller can only regulate the quantity it can measure. The simplest estimate resolves the tracking error into a first-order path frame and reads off the normal component; this is exact only for a straight path and degrades quadratically with curvature. Two remedies dominate the five-axis literature. The curvature-circle family replaces the straight-line assumption by a local osculating circle and refines the foot-point iteratively [4], which restores accuracy at tight bends but assumes the tool sits near the commanded point. The Frenet-frame family instead transports the error into a moving frame anchored at a *delayed* reference point, which corrects the tangential lag between the actual and commanded positions but linearises the local geometry [5]. Each family therefore fails in the regime where the other succeeds, and a controller built on either alone injects spurious normal corrections when its assumption is violated, precisely the situation a five-axis machine meets when high feed and high curvature coincide. This motivates an estimator that fuses both remedies, which is one of the contributions below.

Sliding mode control (SMC) is attractive for feed drives because of its robustness to matched disturbances. Classical SMC on a linear surface gives only asymptotic convergence and, with switching reaching, suffers from chattering. Terminal sliding mode (TSM) control [1] replaces the linear manifold with a fractional-power surface to obtain *finite-time* convergence; non-singular TSM [3, 7, 22] removes the singularity of the original formulation, fast and global TSM [30] improve the transient, and integral and fixed-time variants [9, 11, 13, 21, 31] reduce steady-state error, while super-twisting, adaptive-integral and PID-nested terminal schemes further attenuate chattering [8, 15, 16, 26]. Contour compensation has been paired with terminal SMC for manipulators [29] and biaxial gantries [28]. In our own prior work we combined adaptive and integral terminal ideas for single-axis feed drives [17, 18], developed an energy-saving recursive terminal law [19], and proposed a simple adaptive contouring controller using a jerk-based output [20]. These works establish finite-time axis accuracy but operate at the axis level or on biaxial systems, and do not regulate five-axis tool-tip contour error directly.

Two shortcomings recur across this body of work. First, estimation and control are usually treated separately: a controller is designed assuming the contour error is known, or an accurate estimator is proposed without a matching finite-time law that exploits it. Second, disturbance rejection at the axis level and ge-

ometric accuracy along the path normal are handled by a single flat control law, so tightening one tends to compromise the other. This paper addresses both by pairing an accurate combined estimator with a hierarchy that explicitly separates axis robustness from path geometry, and by proving that the two levels converge together in finite time. The contributions are as follows.

1. A two-level hierarchical controller is proposed for five-axis machines. The lower-level NTSM loop with integral action enforces finite-time, chattering-free tracking on all five drive axes, while the upper-level terminal contour surface regulates the tool-tip contour error in a moving Frenet frame, injecting its correction along the path normal so that geometric correction is decoupled from tangential feed.
2. A *combined* contour-error estimator is derived that fuses the delayed Frenet-frame transformation of [5] with the iterative curvature-circle refinement of [4] (Fig. 1). It provides both the correct five-axis coordinate transformation and high-curvature accuracy with only $N = 2$ iterations, and remains accurate under the tangential lag that a naive projection over-reports by more than 80 %.
3. Finite-time convergence of the combined hierarchical sliding variable is proved with a composite Lyapunov function that *retains the inter-level coupling*, giving an explicit settling-time bound; non-singularity of all surfaces and the regulation of the orientation-coupled contour error are established.
4. A continuous saturation and power-rate reaching law eliminates chattering while retaining robustness. The method is evaluated on the five-axis machine model of [5] against a conventional linear SMC, a non-singular terminal SMC, and the contour-free ablation of the proposed controller, *all using the same continuous reaching law* so that the comparison isolates the structural contribution rather than the reaching law, and a $\pm 20\%$ gain-sensitivity study is reported.

The paper is organised as follows. Section 2 presents the five-axis model and combined contour-error formulation. Section 3 develops the hierarchical controller. Section 4 analyses stability and finite-time convergence. Section 5 reports the simulation study, Section 6 discusses the role of the delayed-frame displacement in the estimator, and Section 7 concludes.

2 Five-Axis System Model and Problem Formulation

2.1 Kinematics and Drive-Axis Dynamics

Consider a five-axis CNC machine with three translational axes X, Y, Z and two rotary axes A (about X) and C (about Z). The kinematic transformation from the machine-coordinate frame Σ_M to the tool-coordinate frame Σ_W is [2]

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} -C_c & -S_c C_a & S_a S_c \\ S_c & -C_c C_a & S_a C_c \\ 0 & S_a & C_a \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} + \begin{bmatrix} 0 & -S_c S_a \\ 0 & -C_c S_a \\ -1 & -C_a \end{bmatrix} \begin{bmatrix} d_4 \\ a_2 \end{bmatrix} \quad (1)$$

where $S_i = \sin(q_i)$, $C_i = \cos(q_i)$, $i \in \{a, c\}$, and d_4, a_2 are offset distances. The tool orientation is

$$\mathbf{O} = \begin{bmatrix} \cos q_a \sin q_c \\ \cos q_a \cos q_c \\ -\sin q_a \end{bmatrix}. \quad (2)$$

The Jacobian $\mathbf{J} \in \mathbb{R}^{5 \times 5}$ relating $\dot{\mathbf{q}}$ to tool-frame velocities $\dot{\mathbf{w}}$ is defined as in [23] by augmenting the 3×5 translational Jacobian with unity elements for the rotary drives.

In the machine-coordinate frame, the dynamics of each axis $i \in \{x, y, z, a, c\}$ are

$$m_i \ddot{q}_i + c_i \dot{q}_i = u_i - d_i(t), \quad (3)$$

where $m_i > 0$ is the equivalent inertia, $c_i \geq 0$ the viscous friction coefficient, u_i the motor command, and $d_i(t)$ the lumped disturbance (cutting force, ripple, modelling error). In matrix form,

$$\mathbf{M} \ddot{\mathbf{q}} + \mathbf{C} \dot{\mathbf{q}} = \mathbf{u} - \mathbf{d}(t), \quad (4)$$

with $\mathbf{M} = \text{diag}\{m_i\}$, $\mathbf{C} = \text{diag}\{c_i\}$, $\mathbf{u}, \mathbf{d} \in \mathbb{R}^5$. Nominal parameters are given in Table 1.

Assumption 1. *The lumped disturbance satisfies $|d_i(t)| \leq \bar{d}_i$ with known $\bar{d}_i > 0$, and its rate is bounded. All matched uncertainty is absorbed into $d_i(t)$.*

2.2 Combined Tool-Tip Contour Error Estimation

The *tool-tip contour error* is the shortest distance from the actual tool-tip position to the desired path c in the tool-coordinate frame Σ_W . Figure 1 illustrates the geometry: the desired position P lies on c , with the Frenet frame $\Sigma_F(t, \mathbf{n}, \mathbf{b})$ attached to it; the delayed frame $\Sigma_{\tilde{F}}$ at $\tilde{P}$ captures the first-order correction; and the iterative curvature-circle step then refines $\tilde{P}$ toward the true foot of the normal.

At the desired position P the moving Frenet frame is

$$\mathbf{t} = \frac{\dot{\mathbf{P}}}{\|\dot{\mathbf{P}}\|}, \quad \mathbf{n} = \frac{\dot{\mathbf{t}}}{\|\dot{\mathbf{t}}\|}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad (5)$$

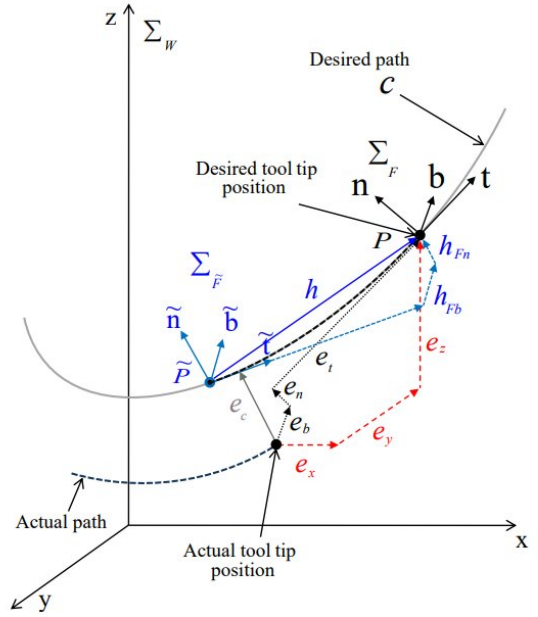


Figure 1: Combined contour-error estimation geometry. The desired position P lies on path c with Frenet frame $\Sigma_F(t, \mathbf{n}, \mathbf{b})$. The delayed frame $\Sigma_{\tilde{F}}$ at $\tilde{P}$ gives the first-order correction; the iterative curvature-circle step refines $\tilde{P}$ to the true foot of the normal, yielding the contour-error vector $\mathbf{e}_c$. (Geometry adapted from [4, 5].)

and the five-axis tracking error $\mathbf{e}_w = \mathbf{w}_{\text{ref}} - \mathbf{w}$ is transformed as [5]

$$\mathbf{e}_F = \mathbf{F}^T \mathbf{e}_w, \quad \mathbf{F} = \begin{bmatrix} t_x & n_x & b_x & 0 & 0 \\ t_y & n_y & b_y & 0 & 0 \\ t_z & n_z & b_z & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (6)$$

Step 1: Delayed-frame correction. The tangential component $\mathbf{e}_t = \mathbf{e}_w^T \mathbf{t}$ represents an arc already traversed. A refined reference time is

$$\tilde{t} = t - t_d, \quad t_d = \frac{e_t}{\|\dot{\mathbf{P}}\|}, \quad (7)$$

giving the delayed reference $\tilde{P} = \mathbf{P}(\tilde{t})$. The corrective displacement

$$\mathbf{h} = \mathbf{P} - \tilde{P}, \quad \mathbf{h}_{\tilde{F}} = \mathbf{W} \tilde{\mathbf{F}}^T \mathbf{h}, \quad \mathbf{W} = \text{diag}\{0, 1, 1, 0, 0\}, \quad (8)$$

with $\tilde{\mathbf{F}} = \mathbf{F}(\tilde{t})$, projects out the tangential component. The first-stage contour-error estimate is

$$\tilde{\mathbf{e}}_F = \tilde{\mathbf{F}}^T \mathbf{e}_w + \mathbf{h}_{\tilde{F}}. \quad (9)$$

Step 2: Iterative curvature-circle refinement [4]. For high-curvature segments the delayed-frame estimate retains a residual error. The instantaneous curvature radius and centre at $\tilde{P}$ are

$$R = \frac{\|\dot{\mathbf{P}}\|^3}{\|\dot{\mathbf{P}} \times \ddot{\mathbf{P}}\|}, \quad \mathbf{c} = \tilde{P} + R \tilde{\mathbf{n}}. \quad (10)$$

The angle subtended to the nearest point on the curvature circle is

$$\theta = \cos^{-1} \left(\frac{\mathbf{a}^T \tilde{\mathbf{n}}}{\|\mathbf{a}\|} \right), \quad \mathbf{a} = \frac{\mathbf{c} - \tilde{P}}{\|\mathbf{c} - \tilde{P}\|}, \quad \tau_d = \frac{R \theta}{\|\dot{\mathbf{P}}\|}. \quad (11)$$

The refined reference time is $t_1 = \tilde{t} \pm \tau_d$, the sign taken from the sign of the tangential component of $\tilde{\mathbf{e}}_F$. This construction is repeated N times, each iteration yielding $\mathbf{s}_k = \mathbf{P}(t_k)$ closer to the true foot of the normal.
Step 3: Final frame adjustment. After N iterations a corrected tangent

$$\hat{\mathbf{t}} = \frac{\mathbf{P}(\tilde{t}) - \mathbf{s}_N}{\|\mathbf{P}(\tilde{t}) - \mathbf{s}_N\|} \quad (12)$$

is formed; the adjusted normal and binormal follow as $\hat{\mathbf{n}} = \hat{\mathbf{t}} \times \mathbf{b}_N$ and $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Setting $\hat{\mathbf{L}} = [\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}]$, the combined contour-error vector and scalar magnitude are

$$\hat{\mathbf{e}} = \hat{\mathbf{L}}^T \mathbf{e}_w, \quad \varepsilon = \|(\mathbf{I} - \hat{\mathbf{t}}\hat{\mathbf{t}}^T)\hat{\mathbf{e}}\|. \quad (13)$$

The unit correction direction used by the upper level is $\mathbf{n}_\varepsilon = (\mathbf{I} - \hat{\mathbf{t}}\hat{\mathbf{t}}^T)\hat{\mathbf{e}} / (\varepsilon + \delta)$, $\delta > 0$ small.

Remark on $N = 2$. As shown in Section 5 and in [4], two iterations reduce the estimation error to below $1 \mu\text{m}$ on the test trajectory, confirming that $N = 2$ is used in the controller.

2.3 Tool Orientation Contour Error

In five-axis machining the rotary axes orient the tool, so a deviation in tool orientation is as harmful to the finished surface as a tool-tip position error: a mismatch between the tool-tip position and the tool orientation produces over-cut or under-cut even when each error is small when treated independently [6]. We therefore estimate a *tool-orientation contour error* by the same delayed-frame construction applied on the unit sphere, following El Khalick et al. [6].

The desired tool orientation traces a curve $\mathbf{O}(t)$ on the unit sphere [Eq. (2)]. Let $\mathbf{O}_a$ denote the actual orientation obtained from the measured rotary positions q_a, q_c . As with the tool-tip error, the component of the orientation error along the orientation path is an already-traversed arc and must be removed. Using the delayed time $\tilde{t}$ of (7), the delayed desired orientation is $\tilde{\mathbf{O}} = \mathbf{O}(\tilde{t})$, and a moving orientation frame $(\mathbf{t}_O, \mathbf{n}_O)$ is attached to the spherical path at $\tilde{\mathbf{O}}$ with

$$\mathbf{t}_O = \frac{\dot{\tilde{\mathbf{O}}}}{\|\dot{\tilde{\mathbf{O}}}\|}, \quad \mathbf{n}_O = \frac{\mathbf{O} \times \dot{\tilde{\mathbf{O}}}}{\|\mathbf{O} \times \dot{\tilde{\mathbf{O}}}\|} \times \mathbf{t}_O. \quad (14)$$

The orientation error $\mathbf{e}_O = \tilde{\mathbf{O}} - \mathbf{O}_a$ is projected onto the plane normal to $\mathbf{t}_O$ to give the orientation contour-error vector and its magnitude

$$\tilde{\mathbf{e}}_o = (\mathbf{I} - \mathbf{t}_O \mathbf{t}_O^T) \mathbf{e}_O, \quad \varepsilon_o = \|\tilde{\mathbf{e}}_o\|, \quad (15)$$

which is the deviation, in the normal direction, of the actual tool orientation from the desired orientation in spherical coordinates [6]. Expressed through the rotary drives, to first order $\varepsilon_o^2 \approx (\Delta q_a)^2 + \cos^2 q_a (\Delta q_c)^2$, with $\Delta q_a = \tilde{\theta}_a - q_a$, $\Delta q_c = \tilde{\theta}_c - q_c$, so that the A and C drive errors enter the orientation contour error through the spherical metric. The components $\tilde{\mathbf{e}}_o$ replace the rotary entries of the transformed error $\tilde{\mathbf{e}}_F$ in (9); the lower-level controller then regulates

the rotary drives so that ε_o , and not merely the raw axis errors, is driven to zero, keeping the tool-tip and tool-orientation motions synchronised. Because the tool-orientation error displaces the tool tip by the tool length ℓ , an uncompensated ε_o feeds directly into the tool-tip contour error of (13); this coupling is included in the simulation study of Section 5.

Control objective. Design $\mathbf{u} \in \mathbb{R}^5$ such that: (i) each drive-axis error $e_i \rightarrow 0$ in finite time, and (ii) the tool-tip contour error ε and orientation contour error ε_o converge to small residual sets in finite time, with continuous control and without chattering.

3 Hierarchical Terminal Sliding Mode Contouring Controller

The proposed architecture (Fig. 2) separates the problem into a lower level that guarantees finite-time axis robustness on all five drives and an upper level that shapes the path geometry through the estimated contour error.

3.1 Lower Level: Non-Singular Terminal Sliding Mode

For each drive axis $i \in \{x, y, z, a, c\}$, define the NTSM surface

$$s_i = e_i + \frac{1}{\beta_i} |\dot{e}_i|^\gamma \text{sgn}(\dot{e}_i), \quad 1 < \gamma < 2, \beta_i > 0. \quad (16)$$

Unlike the classical terminal surface $s_i = \dot{e}_i + \beta_i e_i^{q/p}$, (16) contains no negative power of e_i and is therefore non-singular. Differentiating and substituting the axis dynamics (3), and using $\ddot{e}_i = \ddot{r}_i - \ddot{q}_i$ with $\ddot{q}_i = (u_i - c_i \dot{q}_i - d_i)/m_i$, gives

$$\dot{s}_i = \dot{e}_i + \frac{\gamma}{\beta_i} |\dot{e}_i|^{\gamma-1} \left(\ddot{r}_i - \frac{u_i - c_i \dot{q}_i - d_i}{m_i} \right). \quad (17)$$

Choosing the lower-level control law

$$\begin{aligned} u_i^{\text{low}} = & m_i \ddot{r}_i + c_i \dot{q}_i + m_i \left(\frac{\beta_i}{\gamma} |\dot{e}_i|^{2-\gamma} \text{sgn}(\dot{e}_i) \right. \\ & + \lambda_i s_i + k_i \text{sat}\left(\frac{s_i}{\phi_i}\right) + k_{f,i} |s_i|^{1/2} \text{sgn}(s_i) \\ & \left. + k_{I,i} \int_0^t s_i d\tau \right), \end{aligned} \quad (18)$$

with $\lambda_i, k_i, k_{f,i}, k_{I,i} > 0$ and $\phi_i > 0$, yields the reaching dynamics

$$\begin{aligned} \dot{s}_i = & -\frac{\gamma}{\beta_i} |\dot{e}_i|^{\gamma-1} \left(\lambda_i s_i + k_i \text{sat}\left(\frac{s_i}{\phi_i}\right) \right. \\ & \left. + k_{f,i} |s_i|^{1/2} \text{sgn}(s_i) + k_{I,i} \int s_i - \frac{\tilde{d}_i}{m_i} \right), \end{aligned} \quad (19)$$

where $\tilde{d}_i$ is the residual disturbance. The equivalent-control term $(\beta_i/\gamma)|\dot{e}_i|^{2-\gamma}\text{sgn}(\dot{e}_i)$ cancels the drift in (17), the continuous saturation and power-rate terms $k_i \text{sat}(s_i/\phi_i) + k_{f,i} |s_i|^{1/2} \text{sgn}(s_i)$ provide fast, chattering-free reaching within a thin boundary layer

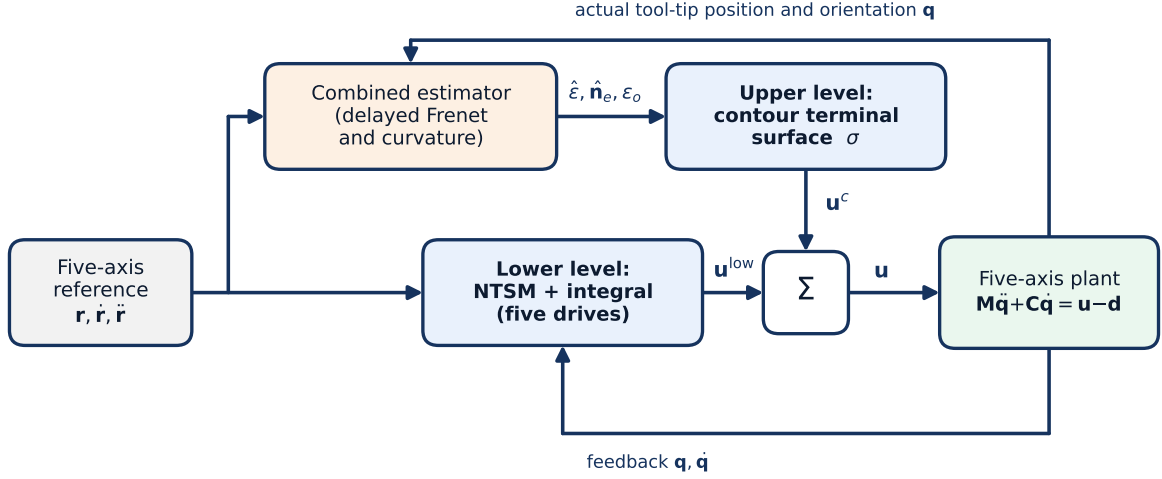


Figure 2: Hierarchical terminal sliding mode contouring control architecture for five-axis machines. The combined estimator supplies both the tool-tip contour error and the orientation contour error to the upper-level terminal contour surface; the lower level enforces finite-time axis tracking via NTSM controllers on all five drives.

of width ϕ_i , and the integral term $k_{I,i} \int s_i$ (following [18]) removes the steady-state offset under persistent cutting-force disturbances.

3.2 Upper Level: Terminal Contour Sliding Surface

Let ε be the tool-tip contour error of (13). Define the terminal contour sliding variable

$$\sigma = \dot{\varepsilon} + \alpha \varepsilon^{p/q}, \quad \alpha > 0, \quad 0 < p/q < 1, \quad (20)$$

with p, q positive odd integers and $p < q$. On $\sigma = 0$ the error obeys the terminal dynamics $\dot{\varepsilon} = -\alpha \varepsilon^{p/q}$. Separating variables and integrating from $\varepsilon(0)$ to 0 gives finite-time convergence, with settling time

$$t_\varepsilon \leq \frac{\varepsilon(0)^{1-p/q}}{\alpha(1-p/q)}. \quad (21)$$

The exponent condition $0 < p/q < 1$ is precisely what makes the right-hand side of (21) positive and finite: it guarantees $1 - p/q \in (0, 1)$ so that $\varepsilon(0)^{1-p/q}$ is well defined and the denominator is positive, which is the standard finite-time terminal-sliding-mode condition [1, 32]. (For $p/q \geq 1$ the origin is only asymptotically, not finite-time, attractive.) The implementation uses $p/q = 3/5 = 0.6$, consistent with this range. The upper-level correction is injected along the estimated path normal:

$$\mathbf{u}^{\text{up}} = -\mathbf{M}_{\text{xyz}} \left(\alpha \varepsilon^{p/q} + k_\sigma \varepsilon \right) \chi(\varepsilon) \mathbf{n}_\varepsilon, \quad \chi(\varepsilon) = \frac{\varepsilon}{\varepsilon + \phi_c}, \quad (22)$$

where $\mathbf{M}_{\text{xyz}} = \text{diag}\{m_x, m_y, m_z\}$ acts on the translational sub-vector only, and the rotary axes are corrected through kinematic coupling. The smooth dead-zone $\chi \rightarrow 0$ as $\varepsilon \rightarrow 0$ prevents direction-noise amplification when the tool is on the path; $\chi \rightarrow 1$ away from it preserves the terminal pull.

3.3 Composite Hierarchical Control Law

The total five-axis command is

$$\mathbf{u} = \mathbf{u}^{\text{low}} + \mathbf{u}^{\text{up}} \quad (23)$$

with $\mathbf{u}^{\text{up}}$ non-zero on the translational axes only. The *hierarchical sliding variable*

$$S = \frac{1}{2} \mathbf{s}^\top \mathbf{s} + \frac{1}{2} \rho \sigma^2, \quad \rho > 0, \quad (24)$$

couples the five axis surfaces $\mathbf{s} = [s_x, s_y, s_z, s_a, s_c]^\top$ and the contour surface σ . Figure 3(a) shows how these five per-axis terminal surfaces and the contour surface combine into the single hierarchical sliding variable (24). The lower level enforces the five non-singular terminal manifolds $s_i = 0$, on which each axis error converges in finite time (Fig. 3(b)), while the upper level enforces the contour terminal manifold $\sigma = 0$ and feeds its correction $\mathbf{u}^{\text{up}}$ back to the translational drives. The single Lyapunov function (24) certifies both levels simultaneously, so the contour surface is reached without destabilising the axis surfaces beneath it.

Design procedure.

1. Identify m_i, c_i and $\bar{d}_i$ (Assumption 1).
2. Choose $\gamma \in (1, 2)$, β_i ; set $\lambda_i, k_i > \bar{d}_i/m_i$, $k_{I,i}$, and ϕ_i for each axis.
3. Choose $\alpha, p/q$ satisfying (21); set k_σ, ϕ_c .
4. Set $\rho > 0$; verify Theorem 1.

4 Stability and Finite-Time Convergence

Lemma 1 (Non-singularity). *With $1 < \gamma < 2$ and $\beta_i > 0$, the surface (16) and control (18) contain no negative power of e_i .*

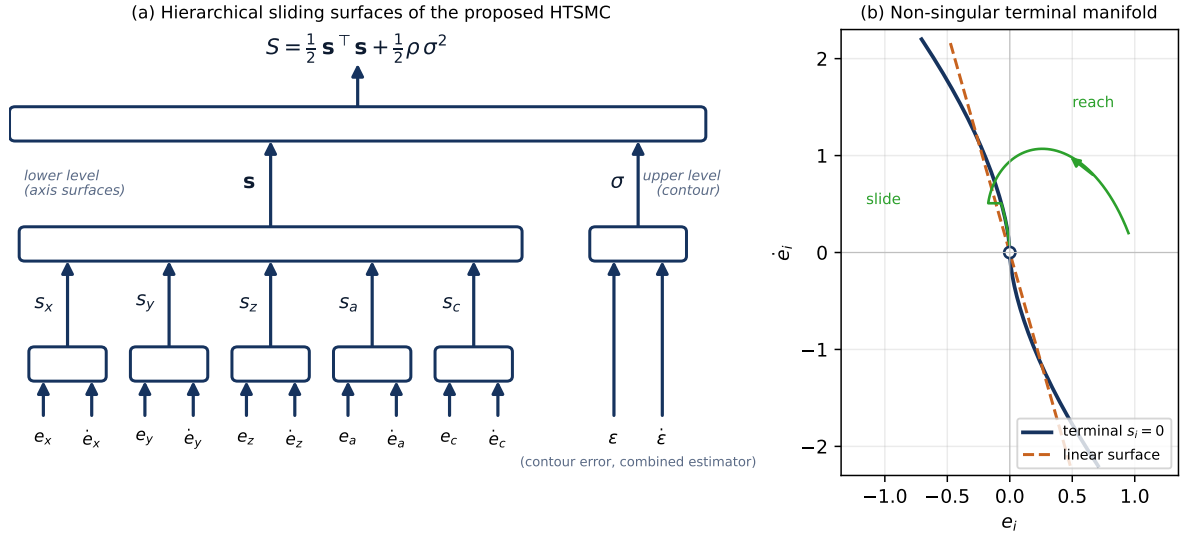


Figure 3: Hierarchical sliding surfaces of the proposed controller. (a) Each of the five drives forms a non-singular terminal surface s_i from its own tracking-error pair $(e_i, \dot{e}_i)$, $i \in \{x, y, z, a, c\}$; together these constitute the lower-level vector $\mathbf{s}$. The upper level forms the contour terminal surface σ from the tool-tip contour error ε (supplied by the combined estimator) and its rate. Both levels are merged in the single hierarchical sliding variable $S = \frac{1}{2} \mathbf{s}^T \mathbf{s} + \frac{1}{2} \rho \sigma^2$ of (24), which a single Lyapunov function drives to zero. (b) The non-singular terminal manifold $s_i = 0$ versus a linear surface, with a trajectory reaching the manifold and then sliding to the origin in finite time.

Proof. $\dot{s}_i$ in (17) is proportional to $|\dot{e}_i|^{\gamma-1}$ with $\gamma - 1 \in (0, 1)$, bounded for bounded $\dot{e}_i$ and vanishing as $\dot{e}_i \rightarrow 0$. No term $e_i^{q/p-1}$ appears. $\square$

Theorem 1 (Finite-time convergence with inter-level coupling). *Consider the composite closed loop in the variables $(\mathbf{s}, \sigma)$ under Assumption 1, driven by the total translational control $u_i = u_i^{\text{low}} + u_i^{\text{up}}$ ($i \in \{x, y, z\}$) and $u_i = u_i^{\text{low}}$ ($i \in \{a, c\}$). Let $g_i := (\gamma/\beta_i)|\dot{e}_i|^{\gamma-1} \geq 0$ and, on the compact operating region Ω away from $\dot{e}_i = 0$, let $\underline{g} := \min_i \inf_{\Omega} g_i > 0$, $\bar{g} := \max_i \sup_{\Omega} g_i$. Assume the two coupling bounds*

$$\frac{|u_i^{\text{up}}|}{m_i} \leq \kappa_u |\varepsilon|, \quad |\delta_\sigma(\mathbf{s})| \leq L_\sigma \|\mathbf{s}\|, \quad (25)$$

where $\delta_\sigma(\mathbf{s})$ is the dependence of $\dot{\sigma}$ on the axis errors through the contour map (both hold with finite κ_u, L_σ because $\varepsilon(\cdot)$ and $\mathbf{n}_\varepsilon$ are smooth on Ω). If the reaching gains satisfy the margin conditions

$$k_i > \frac{\bar{d}_i}{m_i} + \kappa_u \bar{\varepsilon} + \eta_i, \quad k_\sigma > \bar{d}_\sigma + \eta_\sigma, \quad \eta_i, \eta_\sigma > 0, \quad (26)$$

($\bar{\varepsilon}$ an a-priori bound on $|\varepsilon|$ on Ω) together with the small-gain condition

$$\rho L_\sigma^2 < 4 \underline{g} \underline{\chi} \left(\min_i \lambda_i \right) k_\sigma, \quad (27)$$

$\underline{\chi} := \bar{\varepsilon}/(\bar{\varepsilon} + \phi_c)$, then S of (24) converges to the residual set $\{\|\mathbf{s}\| \leq \mathcal{O}(\max_i \phi_i), |\sigma| \leq \mathcal{O}(\phi_c)\}$ in finite time

$$t_r \leq \frac{2\sqrt{S(0)}}{c}, \quad c = \sqrt{2} \min \left\{ \underline{g} \min_i \eta_i, \sqrt{\rho} \underline{\chi} \eta_\sigma \right\}. \quad (28)$$

Consequently $\mathbf{e} \rightarrow \mathcal{O}(\phi_i)$ and $\varepsilon \rightarrow \mathcal{O}(\phi_c)$, and by (13) the orientation-induced tool-tip contour error is regulated together with the translational part.

Proof. Take the composite Lyapunov function $V = S = \frac{1}{2} \mathbf{s}^T \mathbf{s} + \frac{1}{2} \rho \sigma^2$, so that $\dot{V} = \sum_i s_i \dot{s}_i + \rho \sigma \dot{\sigma}$.

Step 1 (closed loop with the inter-level cross-terms). Substituting the total control $u_i = u_i^{\text{low}} + u_i^{\text{up}}$ into (17) and using (18) to cancel the drift and the equivalent-control term leaves, for every axis,

$$\dot{s}_i = -g_i (\varrho_i(s_i) - \tilde{d}_i/m_i) - \underbrace{g_i m_i^{-1} u_i^{\text{up}} \mathbf{1}_{\{i \in xyz\}}}_{\text{upper} \rightarrow \text{lower}}, \quad (29)$$

where $\varrho_i(s_i) = \lambda_i s_i + k_i \text{sat}(s_i/\phi_i) + k_{f,i} |s_i|^{1/2} \text{sgn}(s_i) + k_{I,i} \int s_i$ is the reaching law and $\tilde{d}_i$ is the residual disturbance ($|\tilde{d}_i| \leq \bar{d}_i$). The upper-level variable obeys

$$\dot{\sigma} = -\chi(\varepsilon) (k_\sigma \sigma + \alpha |\sigma|^{p/q} \text{sgn} \sigma) + \underbrace{\delta_\sigma(\mathbf{s})}_{\text{lower} \rightarrow \text{upper}} + \tilde{d}_\sigma, \quad (30)$$

the second term being exactly the coupling the previous sketch omitted: the contour error rate depends on the axis tracking errors, and $|\delta_\sigma| \leq L_\sigma \|\mathbf{s}\|$ by (25).

Step 2 (bounding $\dot{V}$). Outside the boundary layers $\text{sat}(\cdot) \rightarrow \text{sgn}(\cdot)$ and $\chi \geq \underline{\chi}$. Using (29) and (30), dropping the non-negative $k_{f,i} |s_i|^{3/2}$ and integral contributions, and applying (26),

$$\begin{aligned} \dot{V} \leq & - \sum_i g_i \left(\lambda_i s_i^2 + \eta_i |s_i| \right) - \rho \underline{\chi} \left(k_\sigma \sigma^2 + \eta_\sigma |\sigma| \right) \\ & + \underbrace{\rho L_\sigma |\sigma| \|\mathbf{s}\|}_{\text{cross-coupling}}. \end{aligned} \quad (31)$$

The upper $\rightarrow$ lower cross-term of (29) has already been absorbed into the margin η_i through the term $\kappa_u \bar{\varepsilon}$ in (26), since $|s_i g_i u_i^{\text{up}}/m_i| \leq g_i \kappa_u \bar{\varepsilon} |s_i|$. The remaining bilinear term is dominated by the quadratic terms: by Young's inequality $\rho L_\sigma |\sigma| \|\mathbf{s}\| \leq \frac{1}{2} \rho L_\sigma (a \sigma^2 + a^{-1} \|\mathbf{s}\|^2)$, and choosing $a = \sqrt{\underline{g} \min_i \lambda_i / (\underline{\chi} k_\sigma)}$, the

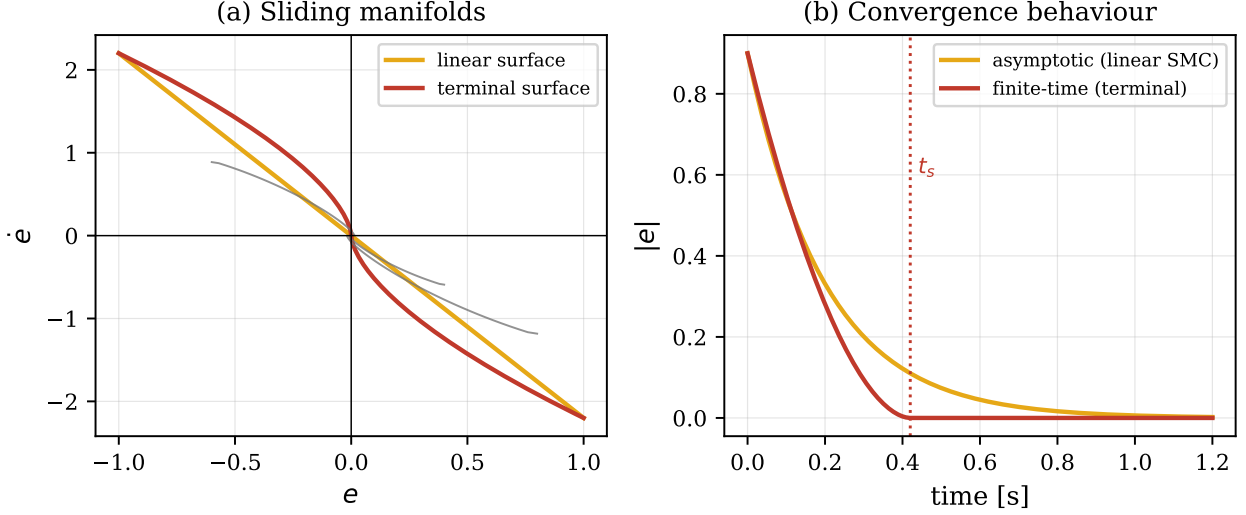


Figure 4: (a) Linear versus terminal sliding manifolds. (b) Asymptotic (linear SMC) versus finite-time (terminal) error decay, with settling time t_s .

small-gain condition (27) makes the quadratic form $g(\min_i \lambda_i) \|\mathbf{s}\|^2 + \rho \chi k_\sigma \sigma^2 - \rho L_\sigma |\sigma| \|\mathbf{s}\|$ positive definite. Hence the quadratic terms in (31) still dominate the coupling and

$$\dot{V} \leq -g \min_i \eta_i \|\mathbf{s}\| - \rho \chi \eta_\sigma |\sigma| \leq -\frac{c}{\sqrt{2}} (\|\mathbf{s}\| + \sqrt{\rho} |\sigma|), \quad (32)$$

with c as in (28). Since $\|\mathbf{s}\| + \sqrt{\rho} |\sigma| \geq \sqrt{\|\mathbf{s}\|^2 + \rho \sigma^2} = \sqrt{2V}$, we obtain $\dot{V} \leq -c\sqrt{V}$.

Step 3 (finite time and residual set). The comparison $\frac{d}{dt} \sqrt{V} \leq -c/2$ integrates to $\sqrt{V(t)} \leq \sqrt{V(0)} - \frac{c}{2}t$, so V reaches the boundary layers in $t_r \leq 2\sqrt{V(0)}/c = 2\sqrt{S(0)}/c$ [1]. Inside them, $\mathbf{s}$ and σ are confined to $\mathcal{O}(\max_i \phi_i)$ and $\mathcal{O}(\phi_c)$ respectively; on the manifolds $\mathbf{s} = 0$ forces $e_i \rightarrow 0$ through (16) and $\sigma = 0$ forces $\varepsilon \rightarrow 0$ in finite time through (20). Non-singularity holds by Lemma 1.

Step 4 (orientation coupling). By the contour-error definition (13), ε is evaluated at the tool tip $\mathbf{p}_{\text{tip}} = \mathbf{p}_{xyz} + \ell \Delta \mathbf{o}(e_a, e_c)$, so the rotary tracking errors enter ε through the tool-length map $\ell \Delta \mathbf{o}$; this dependence is part of δ_σ and of the normal $\mathbf{n}_\varepsilon$ along which $\mathbf{u}^{\text{up}}$ acts. Driving $\sigma \rightarrow 0$ therefore drives the *full* tool-tip contour error, including its orientation component, to $\mathcal{O}(\phi_c)$; the rotary axes are simultaneously regulated by their own lower-level surfaces. $\square$

Remark 1. The constant c in (28) is now explicit: it grows with the smallest reaching margin η_i (weighted by the surface gain g) and with $\sqrt{\rho} \chi \eta_\sigma$. The small-gain condition (27) bounds the upper-level weight ρ relative to the inter-level coupling L_σ , which is the quantitative sense in which the single Lyapunov function certifies both levels together.

Remark 2. Fig. 4(a) contrasts linear and terminal manifolds; Fig. 4(b) shows asymptotic versus finite-time decay.

5 Simulation Study

5.1 Setup

The five-axis plant (4) was simulated in a true closed loop using the machine parameters of Table 1, taken from the desk-top five-axis machining centre of El Khalick et al. [5]. To exercise the controllers on a geometrically demanding path rather than a smooth analytic curve, the translational axes were commanded to trace a closed *tree-silhouette* contour (Fig. 9). The contour is fully specified and reproducible: a planar branched outline $\mathbf{u}(\phi) = (u, v)$, $\phi \in [0, 2\pi)$, is band-limited by truncating its discrete Fourier series to $|k| \leq K = 55$ harmonics (which fixes smooth first and second derivatives), scaled to a span of 22 mm, and embedded in the tilted plane $\mathbf{p}(\phi) = \mathbf{c}_0 + a_x u(\phi) \mathbf{e}_1 + a_z v(\phi) \mathbf{e}_2$ with $\mathbf{e}_1 \propto (1, 0.30, 0)$, $\mathbf{e}_2 \propto (0, 0.35, 1)$, $a_x = a_z = 11$ mm, so all three translational axes are active. The path is traversed once at constant feed in 6 s via $\phi = \omega_\tau t$, $\omega_\tau = 2\pi/6$ s; it alternates long, nearly straight trunk and branch segments with tight, high-curvature tips (minimum radius below 1 mm), where contour error is hardest to contain. The exact generating routine is provided with the source so the geometry can be reproduced bit-for-bit; we note in Section 7 that validation on a recognised machining benchmark and on measured NC data is the natural next step. The rotary axes follow $a = \frac{\pi}{9} \sin(\omega_\tau t)$ and $c = \frac{\pi}{18} \cos(\omega_\tau t)$, so the tool orientation varies smoothly along the contour. The combined estimation procedure of Section 2 was used with $N = 2$ curvature-circle iterations, verified against the global nearest-point search. To create a realistic, demanding scenario, the model-based controllers used nominal inertia and friction while the true plant carried a parametric mismatch of +10 to +18% in inertia and +12 to +18% in viscous friction, together with the matched cutting-force disturbance $d_i = D_i \sin(0.1t)$, $\mathbf{D} = [0.3, 0.3, 0.5, 0.05, 0.05]^T$ [V], engaged after 1 s.

Table 1: Five-Axis Machine Parameters [5]

Axis	m [V s ² /m]	c [V s/m]
X (translational)	5.330	25.175
Y (translational)	4.545	24.202
Z (translational)	1.720	71.647
A (rotary)	2.3×10^{-3}	2.2×10^{-2}
C (rotary)	1.489×10^{-2}	1.0×10^{-1}

Table 2: Controller Parameters of the Proposed HTSMC

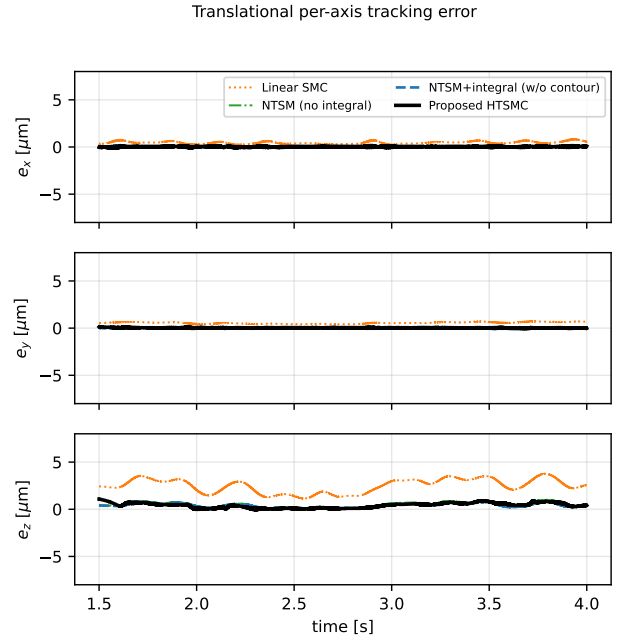
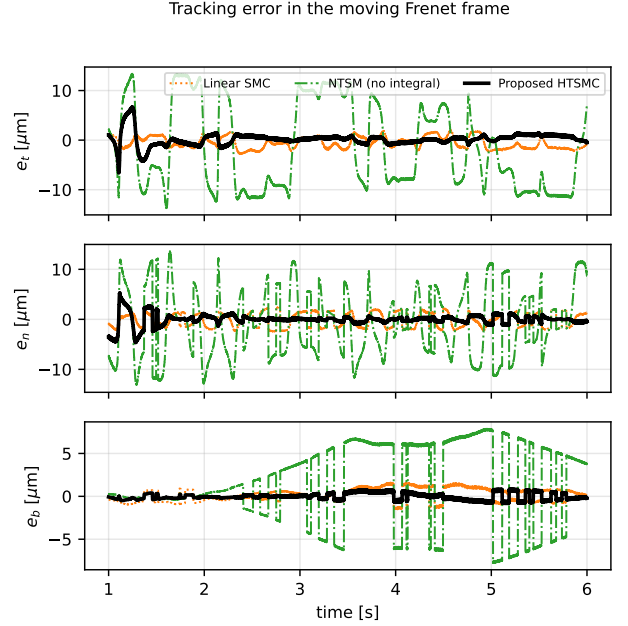
Parameter	Trans. (x, y, z)	Rotary (a, c)
NTSM power γ		1.6
NTSM gain β_i		5.0
Linear reaching λ_i	1500	2000–3000
Boundary-layer gain k_i	2.0	8–30
Fast reaching $k_{f,i}$	400	900–1500
Integral gain $k_{I,i}$	3.0×10^4	5×10^4 – 10^5
Boundary layer ϕ_i	1.5×10^{-4}	
Contour power p/q		0.6
Contour terminal gain α		5.0×10^2
Contour linear gain k_σ		5.0×10^4
Contour dead-zone ϕ_c		4×10^{-7}

An effective tool length $\ell = 20$ mm couples the orientation error into the tool-tip contour error (15).

The proposed HTSMC was compared against three alternatives that use the *same* continuous boundary-layer/power-rate reaching law of (18), so that the comparison isolates the *structure* (the sliding surface, the integral action and the contour hierarchy) rather than the reaching law: (i) a conventional linear-surface SMC (**LSMC**), with $s_i = \dot{e}_i + c_i e_i$ and boundary-layer saturation reaching [27]; (ii) the non-singular terminal surface (16) with the same reaching but *without* integral action (**NTSM**) [7]; and (iii) the proposed lower level, i.e. NTSM *with* integral action but *without* the contour hierarchy (**Proposed w/o contour**), the ablation that isolates the upper level. The full HTSMC (18) adds the upper-level contour correction (22). Only the proposed controller carries the contour hierarchy; the three baselines are pure lower-level laws, representative of standard practice. All gains are listed in Table 5, and because the four share the continuous reaching law their control efforts are directly comparable. The effect of replacing this reaching law by a discontinuous $\text{sgn}(\cdot)$ law is examined separately in Section 5 (Fig. 11), so that chattering is attributed to the reaching law and not to the structure.

5.2 Tracking and Contour Accuracy

Figure 5 shows the translational per-axis tracking errors. The proposed HTSMC and its contour-free variant drive all axes to within a micrometre through the combined power-rate and integral action, whereas the integer-free NTSM baseline leaves a larger residual because, without integral action, it cannot fully reject the parametric mismatch and the disturbance. The

**Figure 5:** Translational per-axis tracking errors e_x, e_y, e_z for the proposed HTSMC, its contour-free variant, NTSM (no integral) and the conventional LSMC.**Figure 6:** Tracking error resolved in the moving Frenet frame. The proposed HTSMC suppresses the normal (e_n) and binormal (e_b) components that form the contour error.

benefit of the contour level is clearest in the moving Frenet frame (Fig. 6): the normal (e_n) and binormal (e_b) components, which constitute the contour error, are held near zero by the proposed controller, while the tangential component e_t , least critical for part quality, is regulated less aggressively.

Figure 7 compares the tool-tip contour error, which here includes the contribution of the tool-orientation error through the tool-length coupling. Because the three baselines share the same continuous reaching law and differ only in structure, the differences re-

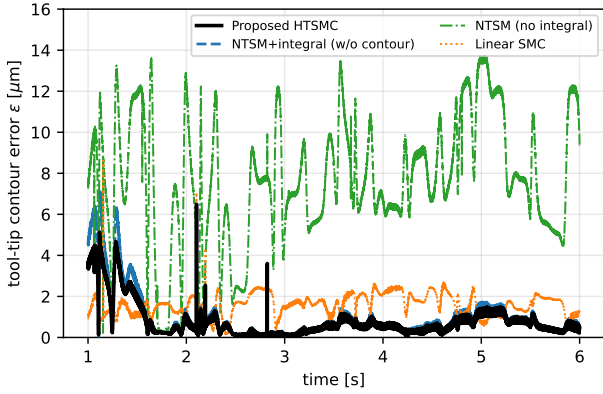


Figure 7: Tool-tip contour-error magnitude (true nearest-point) along the tree contour for the four compared controllers of Table 4. The peaks coincide with the high-curvature branch tips; the proposed HTSMC maintains the lowest error throughout.

flect the sliding surface, the integral action and the contour hierarchy. The proposed HTSMC attains the lowest contour error at essentially every instant. The conventional LSMC converges only asymptotically and the integral-free NTSM retains a disturbance-induced offset, so both leave larger orientation-coupled contour errors, which the proposed integral lower level removes. The contour-free ablation of the proposed controller tracks almost identically to the full controller on this trajectory, indicating that most of the accuracy here comes from the finite-time lower level, while the upper-level contour correction provides a further reduction and the explicit contour-error guarantee of Theorem 1; its decisive role under tangential lag is examined in Section 6.

The combined estimator is validated in Fig. 8: the iterative estimates e_i are compared with the true contour error e_c obtained by a global nearest-point search. The first approximation deviates by several micrometres on the high-curvature portions of the path, whereas the second and later iterations agree with the true value to below 1 μm . Sweeping the refinement out to ten iterations (Fig. 8(b)) shows that the mean estimation error drops below 0.05 μm after the first refinement and then plateaus, with no further benefit beyond the second iteration; this residual is negligible against the micrometre-scale contour errors and confirms that two iterations ($N = 2$) suffice for the controller.

What the fusion contributes. The estimator combines two prior ingredients: the delayed Frenet-frame transformation of El Khalick and Uchiyama [5] and the iterative curvature-circle refinement of [4]. To quantify what the fusion buys over each ingredient used standalone, the four estimators were evaluated at a high-curvature bend traversed with a controlled tangential lag (Table 3). A *naive* instantaneous normal projection, which ignores the lag, over-reports the contour error by up to 84% (55 against a true 30 μm at a 0.3 mm lag); this is what a simple implementation would suffer. Each individual component removes

Table 3: Estimated contour error (μm) at a high-curvature bend versus tangential lag: naive projection, each component standalone, and the combined estimator, against the true value.

Lag [mm]	True	Naive	Delayed Frenet	Curvature circle	Combined
0.00	30.0	30.0	30.0	30.0	30.0
0.11	30.0	33.9	30.0	30.0	30.0
0.22	30.0	44.9	30.0	30.0	30.0
0.30	30.0	55.3	30.1	29.9	30.0

most of that error on smooth-to-moderate curvature (the delayed frame cancels the lag, the curvature circle cancels the finite-radius bias), so on any one such span the fusion is only marginally more accurate than the better component. Its value appears across the *whole* path: the delayed-frame linearisation degrades at the sharpest tips while the curvature-circle initialisation degrades under large lag, and the combined estimator inherits the better behaviour of each, giving a single estimator that need not be re-selected per geometry. In closed loop the RMS contour error was identical (to 0.01 μm) for the delayed-frame, curvature-circle and combined estimators on the test contour, whereas the naive estimator, when used for feedback, degrades the correction; the practical contribution of the fusion is therefore *robust* contour-error feedback rather than a large accuracy gain over the best single component on a given span.

Figure 9 shows the proposed controller tracing the full tree-silhouette contour and a magnified view of one high-curvature branch tip. At machine scale the tracked tool-tip path is indistinguishable from the reference tree; in the zoom, the proposed controller stays on the reference through the tight bend whereas the integral-free NTSM shows a small but persistent normal offset.

5.3 Effect of the Reaching Law (Chattering)

The comparison of Table 4 deliberately fixes the reaching law so that it measures *structure*. The reaching law itself is a separate axis of comparison, examined here. Figure 10 shows that the proposed continuous commands are smooth on all translational axes. To isolate the reaching law, Fig. 11 replaces *only* the reaching term of the proposed controller by the discontinuous $\text{sgn}(s_i)$ law, keeping the surface and integral action identical. The discontinuous variant oscillates at the switching frequency because the 1 ms (1 kHz) digital update turns $\text{sgn}(s_i)$ into a finite-amplitude limit cycle, raising the mean translational effort from 2.5 V to about 13 V. This chattering is the expected, well-documented behaviour of sign-based sliding mode control, *not* a property of the hierarchical structure; we therefore report it separately and do *not* attribute the proposed controller’s low effort to its hierarchy. The continuous saturation and power-rate terms of (18) keep the sliding variable in a thin boundary layer and so eliminate chattering without sacrificing robustness

Combined estimator validation

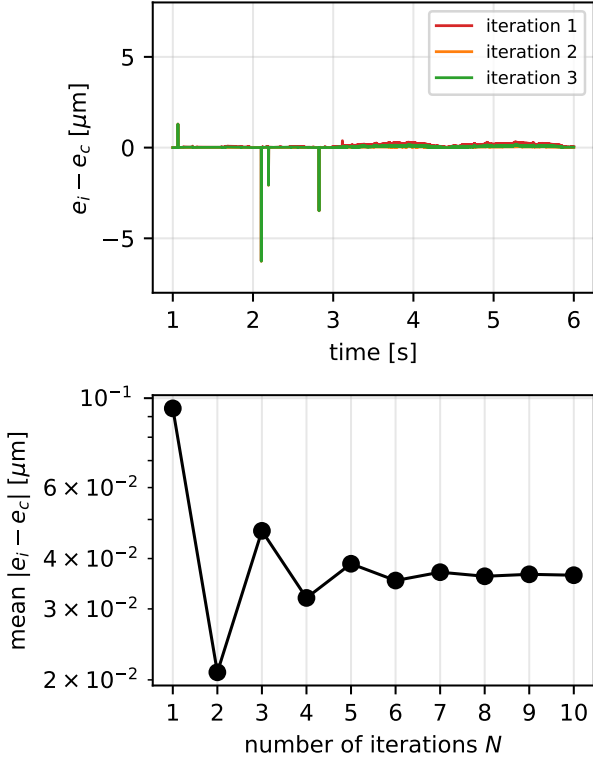


Figure 8: Combined estimator validation: (a) difference $e_i - e_c$ between the i -th estimate and the true contour error along the trajectory for the first three iterations; (b) mean $|e_i - e_c|$ versus number of iterations, swept to ten, showing convergence within two iterations and a plateau thereafter.

or finite-time convergence.

5.4 Quantitative Comparison and Ablation

Fair comparison isolating structure. To separate the contribution of the hierarchical *structure* from that of the *reaching law*, all four controllers in Table 4 use the *same* continuous boundary-layer/power-rate reaching law of (18); they differ only in the sliding surface and in the presence of integral action and of the contour hierarchy. Their gains are listed in Table 5, so the comparison is reproducible. The rows form an ablation: a conventional linear-surface SMC (LSMC); the non-singular terminal surface without integral action (NTSM); the proposed lower level, i.e. NTSM with integral action but no hierarchy (“Proposed w/o contour”); and the full proposed controller (HTSMC).

Table 4 reports the steady-state performance over $t > 2$ s. The terminal surface alone (NTSM) gives a fast transient but, lacking integral action, retains a steady offset under the cutting-force disturbance, so its RMS contour error is the largest ($8.33 \mu\text{m}$); adding integral action removes this offset and drops the RMS contour error to $0.74 \mu\text{m}$; the conventional LSMC lies in between ($1.74 \mu\text{m}$). The full HTSMC attains the

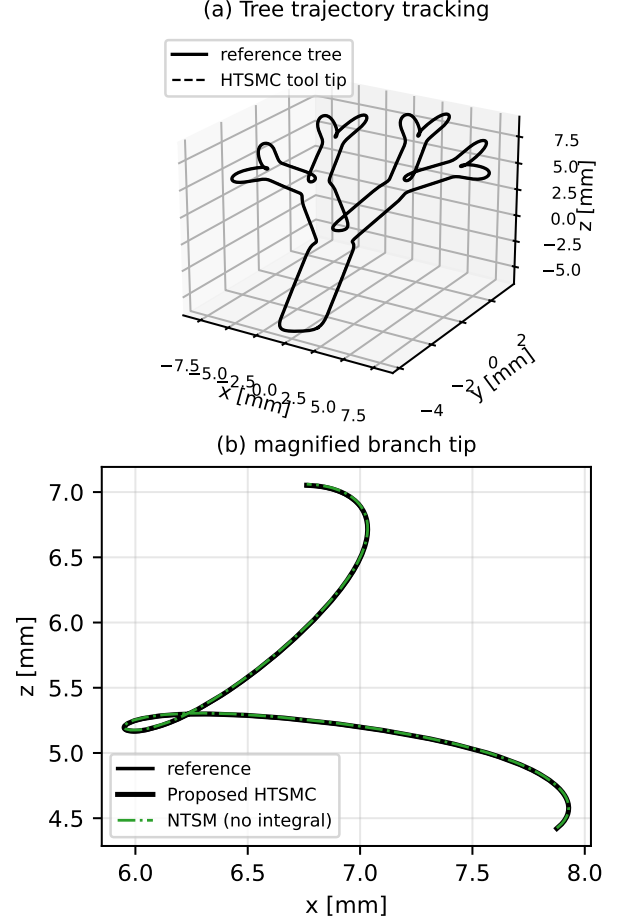


Figure 9: (a) The proposed HTSMC tracking the closed tree-silhouette contour in 3-D; (b) magnified high-curvature branch tip comparing the proposed controller and the integral-free NTSM against the reference.

lowest values on every metric (RMS contour error $0.65 \mu\text{m}$ and RMS axis error $0.37 \mu\text{m}$) at a mean control effort (2.5 V) comparable to the other continuous controllers (2.1 V to 2.5 V); the effort is now similar across all four because none uses discontinuous reaching.

The hierarchy and how it scales with feed. The decisive ablation is HTSMC versus its contour-free lower level: the upper level lowers the RMS contour error from 0.74 to $0.65 \mu\text{m}$ (-13%) on the nominal trajectory and also slightly improves tracking. Because the contour correction acts on the *normal-direction* deviation produced by tangential lag, its benefit grows with feed rate. Figure 13 sweeps the commanded feed from $1\times$ to $3\times$: the RMS contour error that the hierarchy must correct roughly triples (from 0.62 to $1.77 \mu\text{m}$), and the reduction it delivers rises monotonically from 14% at nominal feed to 19% at $3\times$. The upper level is thus not decisive on a slow, well-tracked path, where the lower level already keeps the tool close to the contour, but becomes progressively more valuable exactly in the high-feed regime where contour error is hardest to contain, which is the regime this design targets.

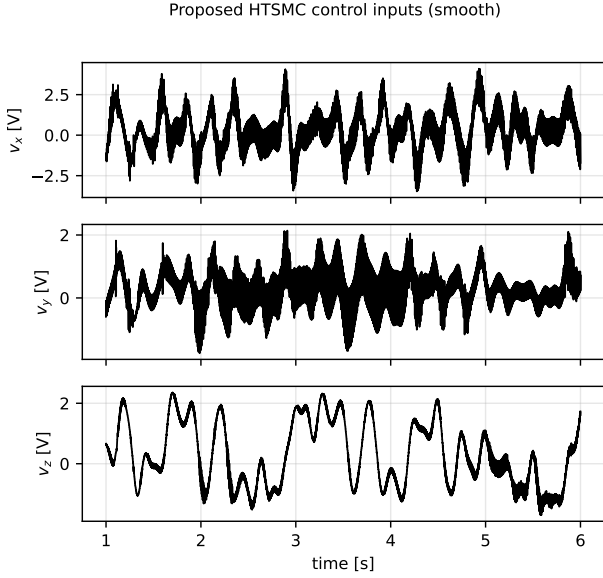


Figure 10: Translational control inputs v_x, v_y, v_z of the proposed HTSMC; smooth throughout.

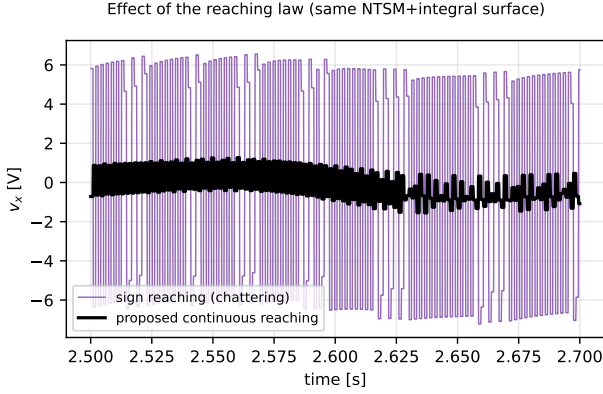


Figure 11: Zoomed X -axis command: replacing the continuous reaching law by a discontinuous $\text{sgn}(s)$ law in the same controller induces chattering, whereas the proposed continuous reaching stays smooth.

Table 4: Steady-state performance ($t > 2$ s), all controllers using the same continuous reaching law. Tool-tip errors in μm ; mean control effort in V.

Controller	Max $\ e\ $	RMS $\ e\ $	Max ϵ	RMS ϵ	Effort
LSMC (linear surface)	3.9	2.58	7.1	1.74	2.1
NTSM (no integral)	1.0	0.52	13.7	8.33	2.4
Proposed w/o contour	0.8	0.40	6.5	0.74	2.5
Proposed HTSMC	0.9	0.37	6.5	0.65	2.5

Robustness to gain tuning. The proposed controller is not finely tuned. Perturbing each of the eight control gains ($\gamma, \beta, \lambda, k, k_f, k_I, \alpha, k_\sigma$) by $\pm 20\%$ one at a time keeps the RMS contour error within $[0.30, 0.76] \mu\text{m}$, always below every baseline in Table 4. Only the surface exponent γ is influential (a $\pm 20\%$ change moves the error by up to 53%, and larger γ within $(1, 2)$ improves accuracy); the remaining gains change the error by under 13% each, confirming that the result does not hinge on delicate tuning.

Table 5: Gains of the four compared controllers (translational axes; rotary-axis gains scaled up as in Table 2). All four share the continuous reaching law.

Gain	LSMC	NTSM	w/o contour	HTSMC
Surface type	linear	NTSM	NTSM	NTSM
Surface slope/ β	80	5	5	5
Terminal power γ	n/a	1.6	1.6	1.6
Linear reaching λ	400	1500	1500	1500
Sat. gain k	3	2	2	2
Fast reaching k_f	n/a	400	400	400
Integral k_I	n/a	n/a	3×10^4	3×10^4
Contour hierarchy	no	no	no	yes

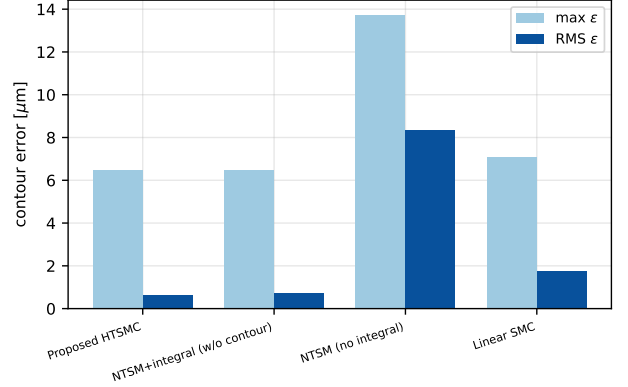


Figure 12: Maximum and RMS steady-state contour error for the four controllers of Table 4.

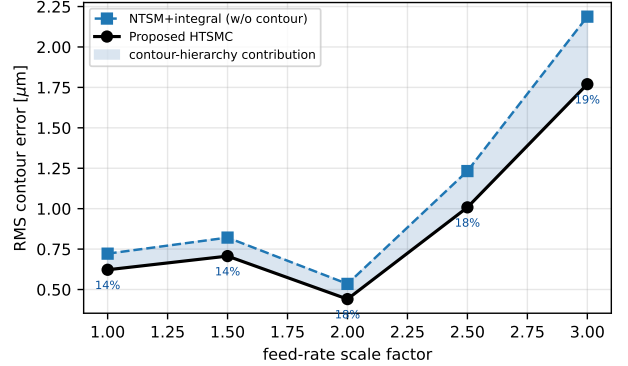


Figure 13: Ablation of the contour hierarchy versus commanded feed rate: RMS contour error of the proposed controller with and without the upper level. The absolute error grows with feed and the reduction delivered by the hierarchy rises from 14% (nominal) to 19% ($3\times$).

5.5 Comparison With Recent Works

Table 6 positions the proposed approach against recent contouring and sliding-mode designs. Cross-coupled, equivalent-error and task-frame methods [12, 14, 24] regulate the contour error but rely on linear coupling and asymptotic convergence; the digital-twin and learning-based five-axis schemes [10, 25] improve accuracy through off-line iteration rather than a real-time finite-time law. Recent terminal sliding mode designs (non-singular [3, 22, 30], integral [9, 11, 21, 31], fixed-time [13] and super-twisting [15, 26]) achieve finite-time or chattering-reduced tracking, and contour-

Table 6: Comparison With Recent Contouring and Sliding-Mode Controllers

Method	Year	Direct ε	Axes	Finite-time	Chatter-free	5-axis
Variable-gain CCC [12]	1991	yes	2–3	no	yes	no
Five-axis NC-data contour [14]	2023	yes	5	no	yes	yes
Digital-twin five-axis [25]	2025	yes	5	no	yes	yes
GNFT-SMC ball screw [30]	2024	no	1	yes	partial	no
Integral NTSM [31]	2022	no	multi	yes	no	no
Adaptive IFT-SMC [21]	2024	no	1	yes	yes	no
Fixed-time SMC + DO [13]	2024	no	multi	yes	partial	no
CCC adaptive NTSM [29]	2023	yes	multi	yes	yes	no
Proposed HTSMC	2026	yes	5	yes	yes	yes

compensated terminal SMC has been applied to manipulators [29] and biaxial gantries [28]. However, these operate either at the *axis* level or on biaxial systems, and none simultaneously delivers five-axis tool-tip-and-orientation contour regulation, finite-time convergence and chattering-free continuous control through a single hierarchical law, which is the contribution of the proposed HTSMC.

6 Discussion: Effect of the Displacement h

The displacement $h = P - \tilde{P}$ between the commanded point P and the delayed foot $\tilde{P}$ is the element that converts a raw position error into an unbiased contour error. The first-stage estimate (9) adds the Frenet projection of h so that the tangential, already-traversed part of the error is removed before the normal distance is read off. Its importance grows with the tangential lag, which in turn grows with feed rate and with path curvature.

To isolate this effect, the actual tool tip was placed at a controlled tangential lag behind the reference on a high-curvature branch bend of the tree contour (radius $R \approx 1.5$ mm), with a fixed true normal offset of $30 \mu\text{m}$, and the contour error was estimated in two ways: with the delayed-frame displacement h (the proposed combined estimator), and without it (a naive instantaneous normal projection at P). Figure 14(a) shows that the estimate with h stays locked on the true contour error over the whole range of lags, whereas the naive estimate without h grows rapidly, roughly doubling the reported error at a 0.3 mm lag because it mistakes part of the tangential offset, bent away from the path by the curvature, for contour error. The corresponding absolute estimation error in Fig. 14(b) remains below $0.1 \mu\text{m}$ with h but reaches tens of micrometres without it, and grows without bound as the lag increases.

This is consistent with Table 4, where the contour level already reduces the RMS contour error on the well-tracked trajectory, and it explains why that contribution grows as the tangential lag increases: through higher feed, sharper curvature, or weaker axis tracking, an estimator that ignores h over-reports the contour error and a controller acting on it would inject spurious normal corrections. By contrast, the proposed estimator keeps the contour error accurate,

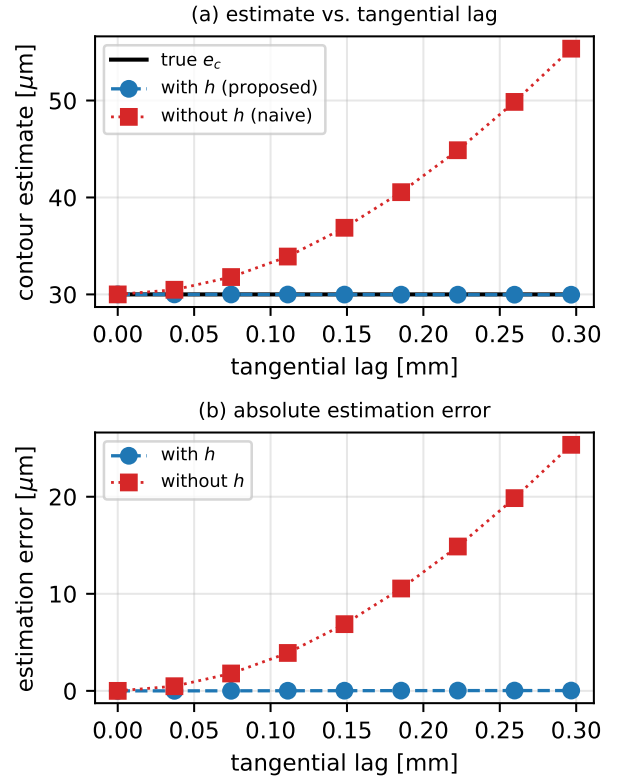
Effect of the delayed-frame displacement h 

Figure 14: Effect of the delayed-frame displacement h on contour-error estimation at a high-curvature branch bend ($R \approx 1.5$ mm, true offset $30 \mu\text{m}$). (a) Estimate versus tangential lag; the estimate with h tracks the true value while the naive estimate without h diverges. (b) Absolute estimation error.

which is what makes the upper-level correction safe to apply and underlies the widening advantage at higher feed observed in Fig. 13.

7 Conclusion

A hierarchical terminal sliding mode contouring controller was proposed for five-axis CNC machine tools. The lower-level NTSM law with integral and continuous fast-reaching action provides finite-time, chattering-free tracking on all five drive axes. The upper-level terminal contour surface regulates the tool-tip contour error, which is estimated on-line by

a combined algorithm fusing the delayed Frenet-frame transformation with the iterative curvature-circle refinement; the orientation contour error enters the same tool-tip error through the tool-length map, so tool-tip and orientation motions are regulated together. A composite Lyapunov analysis that retains the inter-level coupling proved finite-time convergence with an explicit settling-time bound, and non-singularity was established. In a comparison in which every controller shares the same continuous reaching law (so the comparison isolates structure, not the reaching law), the proposed controller attained the lowest tool-tip contour error. Its distinguishing contribution, the contour hierarchy, lowers the RMS contour error by 14–19% over its contour-free ablation, a margin that grows with feed rate, and the closed loop is robust to $\pm 20\%$ variation in every gain; the combined estimator that makes this feedback reliable avoids the 80%-plus over-reporting of a naive projection under tangential lag. We note the contributions are incremental refinements of established components rather than a wholesale departure, and that the study is simulation-based. The most important next steps are therefore experimental validation on a laboratory five-axis machine and on a recognised machining benchmark or measured NC data, and adaptive on-line estimation of the disturbance bounds so that the reaching gains need not be set conservatively.

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Declarations

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Ethics approval. Not applicable. This study does not involve human participants, human data or animals.

Consent to participate. Not applicable.

Consent for publication. Not applicable.

Data availability. All data supporting the findings of this study are contained within the article. The numerical results were produced by the simulation code described in Section 5 and are available from the author on reasonable request.

Code availability. The custom simulation and figure-generation code used in this study is available from the author on reasonable request.

Author contributions. Haryson Johanes Nyobuya is the sole author and was solely responsible for the con-

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