Supplementary Information for "Dark-mode theory for quadratic bosonic quantum networks"

Cheng Liu,[1] Jian Huang,[2] Le-Man Kuang,[1,3] Franco Nori,[4,5] and Jie-Qiao Liao[1,3,6,*]

[1]*Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Key Laboratory for Matter Microstructure and Function of Hunan Province, Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China*

[2]*Department of Physics, City University of Hong Kong, Kowloon 999077, Hong Kong SAR*

[3]*Hunan Research Center of the Basic Discipline for Quantum Effects and Quantum Technologies, Hunan Normal University, Changsha 410081, China*

[4]*Quantum Computing Center, RIKEN, Wakoshi, Saitama 351-0198, Japan*

[5]*Department of Physics, The University of Michigan, Ann Arbor, MI, 48109-1040, USA*

[6]*Institute of Interdisciplinary Studies, Hunan Normal University, Changsha 410081, China*

(Dated: July 5, 2026)

In this Supplemental Material, we present detailed derivations supporting the results reported in the main text. This document is composed of four parts:

Supplementary Note I: Bipartite-graph representation of the quadratic bosonic quantum networks;

Supplementary Note II: Dark-mode determination using the symplectic-singular-value decomposition method;

Supplementary Note III: Derivation of the linearized Hamiltonian of the optomechanical networks;

Supplementary Note IV: Collective cooling of the mechanical modes and generation of optomechanical entanglement in the optomechanical networks.

Supplementary Note I: Bipartite-graph representation of the quadratic bosonic quantum networks

In this section, we introduce a $(M+N)$-mode quadratic bosonic quantum network and present its Hamiltonian. We also present detailed derivations concerning the bipartite-graph representation for this $(M+N)$-mode quantum network. This quantum network consists of a type-a subnetwork with M nodes (each node is a type-a bosonic mode, $a_1, a_2, \cdots, a_M$) and a type-b subnetwork with N nodes (each node is a type-b bosonic mode, $b_1, b_2, \cdots, b_N$) [Fig. S1(a)]. The Hamiltonian of this $(M+N)$-mode quadratic bosonic quantum network consists of three parts,

$$\hat{H}_{\mathrm{nw}} = \hat{H}_a + \hat{H}_b + \hat{H}_{ab}, \tag{S1}$$

where $\hat{H}_a$, $\hat{H}_b$, and $\hat{H}_{ab}$ describe the Hamiltonians of the type-a subnetwork, type-b subnetwork, and inter-node coupling between the two subnetworks, respectively. Concretely, these Hamiltonians take the form ($\hbar = 1$)

$$\hat{H}_a = \sum_{k=1}^{M} \omega_{a_k} \hat{a}_k^\dagger \hat{a}_k + \sum_{k,k'=1,k<k'}^{M} (\xi_{kk'}^{(r)} \hat{a}_k^\dagger \hat{a}_{k'} + \xi_{kk'}^{(r)*} \hat{a}_{k'}^\dagger \hat{a}_k) + \sum_{k,k'=1,k<k'}^{M} (\xi_{kk'}^{(c)} \hat{a}_k^\dagger \hat{a}_{k'}^\dagger + \xi_{kk'}^{(c)*} \hat{a}_{k'} \hat{a}_k), \tag{S2a}$$

$$\hat{H}_b = \sum_{j=1}^{N} \omega_{b_j} \hat{b}_j^\dagger \hat{b}_j + \sum_{j,j'=1,j<j'}^{N} (\eta_{jj'}^{(r)} \hat{b}_j^\dagger \hat{b}_{j'} + \eta_{jj'}^{(r)*} \hat{b}_{j'}^\dagger \hat{b}_j) + \sum_{j,j'=1,j<j'}^{N} (\eta_{jj'}^{(c)} \hat{b}_j^\dagger \hat{b}_{j'}^\dagger + \eta_{jj'}^{(c)*} \hat{b}_{j'} \hat{b}_j), \tag{S2b}$$

$$\hat{H}_{ab} = \sum_{k=1}^{M} \sum_{j=1}^{N} (g_{kj}^{(r)} \hat{a}_k^\dagger \hat{b}_j + g_{kj}^{(r)*} \hat{b}_j^\dagger \hat{a}_k) + \sum_{k=1}^{M} \sum_{j=1}^{N} (g_{kj}^{(c)} \hat{a}_k^\dagger \hat{b}_j^\dagger + g_{kj}^{(c)*} \hat{b}_j \hat{a}_k), \tag{S2c}$$

where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are, respectively, the annihilation and creation operators of the kth mode in the type-a subnetwork, with resonance frequencies ω_{a_k}, and $\xi_{kk'}^{(r)}$ ($\xi_{kk'}^{(c)}$) is the coupling strength corresponding to the rotating-wave (counter-rotating) term between the kth and k'th type-a modes. In addition, $\hat{b}_j$ and $\hat{b}_j^\dagger$ are, respectively, the annihilation and creation operators of the jth mode in the type-b subnetwork, with resonance frequency ω_{b_j}, and $\eta_{jj'}^{(r)}$ ($\eta_{jj'}^{(c)}$) is the coupling strength of the rotating-wave (counter-rotating) term between the jth and j'th type-b modes. The variable $g_{kj}^{(r)}$ ($g_{kj}^{(c)}$) is the coupling strength of the rotating-wave (counter-rotating) term between the kth mode in the type-a subnetwork and the jth mode in the type-b subnetwork.

*Electronic address: Contact author: jqliao@hunnu.edu.cn

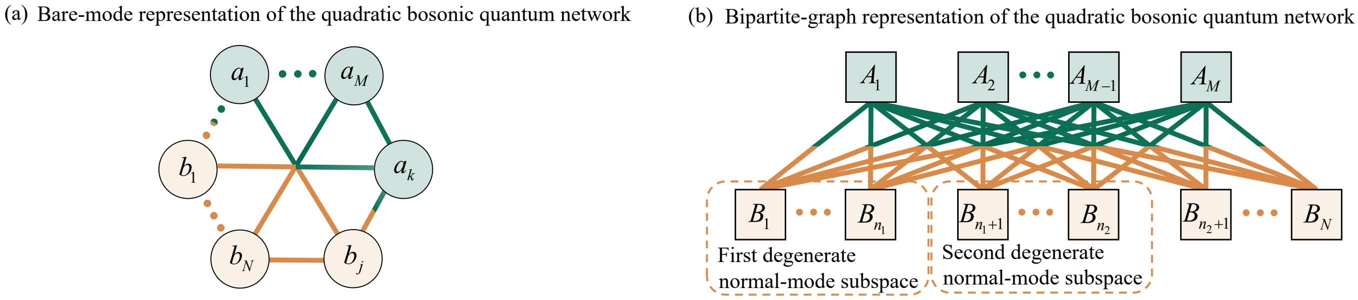

FIG. S1: (a) A quadratic bosonic quantum network composed of two coupled bosonic subnetworks: the type-a subnetwork consisting of M nodal modes $\{a_1, \cdots, a_k, \cdots, a_M\}$ and the type-b subnetwork consisting of N nodal modes $\{b_1, \cdots, b_j, \cdots, b_N\}$. (b) Bipartite-graph representation of the quadratic bosonic quantum network obtained by diagonalizing both the type-a and -b subnetworks. The nodes of the two subnetworks expressed in the bipartite-graph representation are these type-a and -b normal modes. Without loss of generality, here we assume that these normal modes $\{B_1, B_2, \cdots, B_{n_1}\}$ ($\{B_{n_1+1}, B_{n_1+1}, \cdots, B_{n_2}\}$) within the dashed box constitute the first (second) degenerate normal-mode subspace and share a common resonance frequency Ω_1 (Ω_2).

By introducing the operator vectors

$$\hat{\mathbf{r}}_a = (\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_M, \hat{a}_1^\dagger, \hat{a}_2^\dagger, \cdots, \hat{a}_M^\dagger)^{\mathrm{T}}, \tag{S3a}$$

$$\hat{\mathbf{r}}_b = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_N, \hat{b}_1^\dagger, \hat{b}_2^\dagger, \cdots, \hat{b}_N^\dagger)^{\mathrm{T}}, \tag{S3b}$$

with the notation "T" denoting the matrix transpose, the Hamiltonian of the whole network can be expressed as

$$\hat{H}_{\mathrm{nw}} = \frac{1}{2}(\hat{\mathbf{r}}_a^\dagger, \hat{\mathbf{r}}_b^\dagger)\left(\begin{array}{c|c} \mathbf{H}_a & \mathbf{H}_{ab} \\ \mathbf{H}_{ab}^\dagger & \mathbf{H}_b \end{array}\right)\begin{pmatrix} \hat{\mathbf{r}}_a \\ \hat{\mathbf{r}}_b \end{pmatrix} - \frac{1}{2}\mathrm{Tr}[\boldsymbol{\alpha}] - \frac{1}{2}\mathrm{Tr}[\boldsymbol{\beta}], \tag{S4}$$

where "Tr" denotes the trace operation of the involved matrices. In Eq. (S4), the block matrices $\mathbf{H}_a$, $\mathbf{H}_b$, and $\mathbf{H}_{ab}$ are expressed as

$$\mathbf{H}_a = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\xi} \\ \boldsymbol{\xi}^\dagger & \boldsymbol{\alpha}^{\mathrm{T}} \end{pmatrix}, \qquad \mathbf{H}_b = \begin{pmatrix} \boldsymbol{\beta} & \boldsymbol{\eta} \\ \boldsymbol{\eta}^\dagger & \boldsymbol{\beta}^{\mathrm{T}} \end{pmatrix}, \qquad \mathbf{H}_{ab} = \begin{pmatrix} \boldsymbol{g}_r & \boldsymbol{g}_c \\ \boldsymbol{g}_c^* & \boldsymbol{g}_r^* \end{pmatrix}, \tag{S5}$$

where we introduce the following submatrices

$$\boldsymbol{\alpha} = \begin{pmatrix} \omega_{a_1} & \xi_{12}^{(r)} & \cdots & \xi_{1M}^{(r)} \\ \xi_{12}^{(r)*} & \omega_{a_2} & \cdots & \xi_{2M}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{1M}^{(r)*} & \xi_{2M}^{(r)*} & \cdots & \omega_{a_M} \end{pmatrix}, \qquad \boldsymbol{\beta} = \begin{pmatrix} \omega_{b_1} & \eta_{12}^{(r)} & \cdots & \eta_{1N}^{(r)} \\ \eta_{12}^{(r)*} & \omega_{b_2} & \cdots & \eta_{2N}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1N}^{(r)*} & \eta_{2N}^{(r)*} & \cdots & \omega_{b_N} \end{pmatrix}, \qquad \boldsymbol{g}_r = \begin{pmatrix} g_{11}^{(r)} & g_{12}^{(r)} & \cdots & g_{1N}^{(r)} \\ g_{21}^{(r)} & g_{22}^{(r)} & \cdots & g_{2N}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{M1}^{(r)} & g_{M2}^{(r)} & \cdots & g_{MN}^{(r)} \end{pmatrix}, \tag{S6a}$$

$$\boldsymbol{\xi} = \begin{pmatrix} 0 & \xi_{12}^{(c)} & \cdots & \xi_{1M}^{(c)} \\ \xi_{12}^{(c)} & 0 & \cdots & \xi_{2M}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{1M}^{(c)} & \xi_{2M}^{(c)} & \cdots & 0 \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} 0 & \eta_{12}^{(c)} & \cdots & \eta_{1N}^{(c)} \\ \eta_{12}^{(c)} & 0 & \cdots & \eta_{2N}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{1N}^{(c)} & \eta_{2N}^{(c)} & \cdots & 0 \end{pmatrix}, \qquad \boldsymbol{g}_c = \begin{pmatrix} g_{11}^{(c)} & g_{12}^{(c)} & \cdots & g_{1N}^{(c)} \\ g_{21}^{(c)} & g_{22}^{(c)} & \cdots & g_{2N}^{(c)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{M1}^{(c)} & g_{M2}^{(c)} & \cdots & g_{MN}^{(c)} \end{pmatrix}. \tag{S6b}$$

The elements of these submatrices have been introduced in Eq. (S2).

To transform the network into a bipartite-graph network, we need to diagonalize the Hamiltonians of the two subnetworks in advance, and then express the network using the normal modes of the two subnetworks. To obtain the normal modes of the subnetwork beyond the rotating-wave approximation, we prefer to work in the quadrature-operator representation, in which the two subnetworks can be described by the quadrature-operator vectors

$$\hat{\tilde{\mathbf{r}}}_a = (\hat{x}_{a_1}, \hat{x}_{a_2}, \cdots, \hat{x}_{a_M}, \hat{p}_{a_1}, \hat{p}_{a_2}, \cdots, \hat{p}_{a_M})^{\mathrm{T}}, \tag{S7a}$$

$$\hat{\tilde{\mathbf{r}}}_b = (\hat{x}_{b_1}, \hat{x}_{b_2}, \cdots, \hat{x}_{b_N}, \hat{p}_{b_1}, \hat{p}_{b_2}, \cdots, \hat{p}_{b_N})^{\mathrm{T}}. \tag{S7b}$$

Here, these quadrature operators of these bosonic modes are defined by

$$\hat{x}_{a_k} = \frac{1}{\sqrt{2}}(\hat{a}_k^{\dagger} + \hat{a}_k), \qquad \hat{p}_{a_k} = i\frac{1}{\sqrt{2}}(\hat{a}_k^{\dagger} - \hat{a}_k), \tag{S8a}$$

$$\hat{x}_{b_j} = \frac{1}{\sqrt{2}}(\hat{b}_j^{\dagger} + \hat{b}_j), \qquad \hat{p}_{b_j} = i\frac{1}{\sqrt{2}}(\hat{b}_j^{\dagger} - \hat{b}_j), \tag{S8b}$$

for $k = 1, 2, \cdots, M$ and $j = 1, 2, \cdots, N$. The quadrature-operator vectors satisfy the commutation relations

$$[\hat{\tilde{\mathbf{r}}}_a, \hat{\tilde{\mathbf{r}}}_a^{\mathrm{T}}] = i\mathbf{J}_a, \qquad [\hat{\tilde{\mathbf{r}}}_b, \hat{\tilde{\mathbf{r}}}_b^{\mathrm{T}}] = i\mathbf{J}_b, \tag{S9}$$

where we introduce the symplectic matrices

$$\mathbf{J}_a = \begin{pmatrix} \mathbf{0} & \mathbf{I}_M \\ -\mathbf{I}_M & \mathbf{0} \end{pmatrix}, \qquad \mathbf{J}_b = \begin{pmatrix} \mathbf{0} & \mathbf{I}_N \\ -\mathbf{I}_N & \mathbf{0} \end{pmatrix}, \tag{S10}$$

with $\mathbf{I}_M$ ($\mathbf{I}_N$) being the M (N)-dimensional identity matrix. Note that the relations between the two operator-vector sets $\{\hat{\tilde{\mathbf{r}}}_a, \hat{\tilde{\mathbf{r}}}_b\}$ and $\{\hat{\mathbf{r}}_a, \hat{\mathbf{r}}_b\}$ are given by

$$\hat{\mathbf{r}}_a = \mathbf{L}_a\hat{\tilde{\mathbf{r}}}_a, \qquad \hat{\mathbf{r}}_b = \mathbf{L}_b\hat{\tilde{\mathbf{r}}}_b, \tag{S11}$$

where we introduce two unitary matrices

$$\mathbf{L}_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_M & i\mathbf{I}_M \\ \mathbf{I}_M & -i\mathbf{I}_M \end{pmatrix}, \qquad \mathbf{L}_b = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_N & i\mathbf{I}_N \\ \mathbf{I}_N & -i\mathbf{I}_N \end{pmatrix}. \tag{S12}$$

In the quadrature-operator representation, the Hamiltonian (S4) can be expressed as

$$\hat{H}_{\mathrm{nw}} = \frac{1}{2}(\hat{\tilde{\mathbf{r}}}_a^{\mathrm{T}}, \hat{\tilde{\mathbf{r}}}_b^{\mathrm{T}})\left(\begin{array}{c|c} \tilde{\mathbf{H}}_a & \tilde{\mathbf{H}}_{ab} \\ \hline \tilde{\mathbf{H}}_{ab}^{\dagger} & \tilde{\mathbf{H}}_b \end{array}\right)\begin{pmatrix} \hat{\tilde{\mathbf{r}}}_a \\ \hat{\tilde{\mathbf{r}}}_b \end{pmatrix}, \tag{S13}$$

where these real symmetric coefficient matrices $\tilde{\mathbf{H}}_a$, $\tilde{\mathbf{H}}_b$, and $\tilde{\mathbf{H}}_{ab}$ are defined by

$$\begin{aligned} \tilde{\mathbf{H}}_a &= \frac{1}{2}\begin{pmatrix} \boldsymbol{\alpha} + \boldsymbol{\alpha}^{\mathrm{T}} + \boldsymbol{\xi}^{\dagger} + \boldsymbol{\xi} & i(\boldsymbol{\alpha} - \boldsymbol{\alpha}^{\mathrm{T}}) + i(\boldsymbol{\xi}^{\dagger} - \boldsymbol{\xi}) \\ -i(\boldsymbol{\alpha} - \boldsymbol{\alpha}^{\mathrm{T}}) + i(\boldsymbol{\xi}^{\dagger} - \boldsymbol{\xi}) & \boldsymbol{\alpha} + \boldsymbol{\alpha}^{\mathrm{T}} - (\boldsymbol{\xi}^{\dagger} + \boldsymbol{\xi}) \end{pmatrix} \\ &= \begin{pmatrix} \omega_{a_1} & \mathrm{Re}(\xi_{12}^{(r)} + \xi_{12}^{(c)}) & \cdots & \mathrm{Re}(\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) & 0 & \mathrm{Im}(-\xi_{12}^{(r)} + \xi_{12}^{(c)}) & \cdots & \mathrm{Im}(-\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) \\ \mathrm{Re}(\xi_{12}^{(r)} + \xi_{12}^{(c)}) & \omega_{a_2} & \cdots & \mathrm{Re}(\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) & \mathrm{Im}(\xi_{12}^{(r)} + \xi_{12}^{(c)}) & 0 & \cdots & \mathrm{Im}(-\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}(\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) & \mathrm{Re}(\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) & \cdots & \omega_{a_M} & \mathrm{Im}(\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) & \mathrm{Im}(\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) & \cdots & 0 \\ 0 & \mathrm{Im}(\xi_{12}^{(r)} + \xi_{12}^{(c)}) & \cdots & \mathrm{Im}(\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) & \omega_{a_1} & \mathrm{Re}(\xi_{12}^{(r)} - \xi_{12}^{(c)}) & \cdots & \mathrm{Re}(\xi_{1M}^{(r)} - \xi_{1M}^{(c)}) \\ \mathrm{Im}(-\xi_{12}^{(r)} + \xi_{12}^{(c)}) & 0 & \cdots & \mathrm{Im}(\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) & \mathrm{Re}(\xi_{12}^{(r)} - \xi_{12}^{(c)}) & \omega_{a_2} & \cdots & \mathrm{Re}(\xi_{2M}^{(r)} - \xi_{2M}^{(c)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Im}(-\xi_{1M}^{(r)} + \xi_{1M}^{(c)}) & \mathrm{Im}(-\xi_{2M}^{(r)} + \xi_{2M}^{(c)}) & \cdots & 0 & \mathrm{Re}(\xi_{1M}^{(r)} - \xi_{1M}^{(c)}) & \mathrm{Re}(\xi_{2M}^{(r)} - \xi_{2M}^{(c)}) & \cdots & \omega_{a_M} \end{pmatrix}, \end{aligned} \tag{S14}$$

,

$$\begin{aligned} \tilde{\mathbf{H}}_b &= \frac{1}{2}\begin{pmatrix} \boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\eta}^{\dagger} + \boldsymbol{\eta} & i(\boldsymbol{\beta} - \boldsymbol{\beta}^{\mathrm{T}}) + i(\boldsymbol{\eta}^{\dagger} - \boldsymbol{\eta}) \\ -i(\boldsymbol{\beta} - \boldsymbol{\beta}^{\mathrm{T}}) + i(\boldsymbol{\eta}^{\dagger} - \boldsymbol{\eta}) & \boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}} - (\boldsymbol{\eta}^{\dagger} + \boldsymbol{\eta}) \end{pmatrix} \\ &= \begin{pmatrix} \omega_{b_1} & \mathrm{Re}(\eta_{12}^{(r)} + \xi_{12}^{(c)}) & \cdots & \mathrm{Re}(\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) & 0 & \mathrm{Im}(-\eta_{12}^{(r)} + \eta_{12}^{(c)}) & \cdots & \mathrm{Im}(-\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) \\ \mathrm{Re}(\eta_{12}^{(r)} + \eta_{12}^{(c)}) & \omega_{b_2} & \cdots & \mathrm{Re}(\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) & \mathrm{Im}(\eta_{12}^{(r)} + \eta_{12}^{(c)}) & 0 & \cdots & \mathrm{Im}(-\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Re}(\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) & \mathrm{Re}(\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) & \cdots & \omega_{b_N} & \mathrm{Im}(\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) & \mathrm{Im}(\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) & \cdots & 0 \\ 0 & \mathrm{Im}(\eta_{12}^{(r)} + \eta_{12}^{(c)}) & \cdots & \mathrm{Im}(\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) & \omega_{b_1} & \mathrm{Re}(\eta_{12}^{(r)} - \eta_{12}^{(c)}) & \cdots & \mathrm{Re}(\eta_{1N}^{(r)} - \eta_{1N}^{(c)}) \\ \mathrm{Im}(-\eta_{12}^{(r)} + \eta_{12}^{(c)}) & 0 & \cdots & \mathrm{Im}(\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) & \mathrm{Re}(\eta_{12}^{(r)} - \eta_{12}^{(c)}) & \omega_{b_2} & \cdots & \mathrm{Re}(\eta_{2N}^{(r)} - \eta_{2N}^{(c)}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Im}(-\eta_{1N}^{(r)} + \eta_{1N}^{(c)}) & \mathrm{Im}(-\eta_{2N}^{(r)} + \eta_{2N}^{(c)}) & \cdots & 0 & \mathrm{Re}(\eta_{1N}^{(r)} - \eta_{1N}^{(c)}) & \mathrm{Re}(\eta_{2N}^{(r)} - \eta_{2N}^{(c)}) & \cdots & \omega_{b_N} \end{pmatrix}, \end{aligned} \tag{S15}$$

and

$$
\begin{aligned}
\tilde{\mathbf{H}}_{ab} &= \frac{1}{2}\begin{pmatrix} \boldsymbol{g}_r+\boldsymbol{g}_r^*+\boldsymbol{g}_c+\boldsymbol{g}_c^* & i(\boldsymbol{g}_r-\boldsymbol{g}_r^*)-i(\boldsymbol{g}_c-\boldsymbol{g}_c^*) \\ -i(\boldsymbol{g}_r-\boldsymbol{g}_r^*)-i(\boldsymbol{g}_c-\boldsymbol{g}_c^*) & \boldsymbol{g}_r+\boldsymbol{g}_r^*-(\boldsymbol{g}_c+\boldsymbol{g}_c^*) \end{pmatrix} \\
&= \begin{pmatrix}
\mathrm{Re}(g_{11}^{(r)}+g_{11}^{(c)}) & \mathrm{Re}(g_{12}^{(r)}+g_{12}^{(c)}) & \cdots & \mathrm{Re}(g_{1N}^{(r)}+g_{1N}^{(c)}) & \mathrm{Im}(-g_{11}^{(r)}+g_{11}^{(c)}) & \mathrm{Im}(-g_{12}^{(r)}+g_{12}^{(c)}) & \cdots & \mathrm{Im}(-g_{1N}^{(r)}+g_{1N}^{(c)}) \\
\mathrm{Re}(g_{21}^{(r)}+g_{21}^{(c)}) & \mathrm{Re}(g_{22}^{(r)}+g_{22}^{(c)}) & \cdots & \mathrm{Re}(g_{2N}^{(r)}+g_{2N}^{(c)}) & \mathrm{Im}(-g_{21}^{(r)}+g_{21}^{(c)}) & \mathrm{Im}(-g_{22}^{(r)}+g_{22}^{(c)}) & \cdots & \mathrm{Im}(-g_{2N}^{(r)}+g_{2N}^{(c)}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathrm{Re}(g_{M1}^{(r)}+g_{M1}^{(c)}) & \mathrm{Re}(g_{M2}^{(r)}+g_{M2}^{(c)}) & \cdots & \mathrm{Re}(g_{MN}^{(r)}+g_{MN}^{(c)}) & \mathrm{Im}(-g_{M1}^{(r)}+g_{M1}^{(c)}) & \mathrm{Im}(-g_{M2}^{(r)}+g_{M2}^{(c)}) & \cdots & \mathrm{Im}(-g_{MN}^{(r)}+g_{MN}^{(c)}) \\
\mathrm{Im}(g_{11}^{(r)}+g_{11}^{(c)}) & \mathrm{Im}(g_{12}^{(r)}+g_{12}^{(c)}) & \cdots & \mathrm{Im}(g_{1N}^{(r)}+g_{1N}^{(c)}) & \mathrm{Re}(g_{11}^{(r)}-g_{11}^{(c)}) & \mathrm{Re}(g_{12}^{(r)}-g_{12}^{(c)}) & \cdots & \mathrm{Re}(g_{1N}^{(r)}-g_{1N}^{(c)}) \\
\mathrm{Im}(g_{21}^{(r)}+g_{21}^{(c)}) & \mathrm{Im}(g_{22}^{(r)}+g_{22}^{(c)}) & \cdots & \mathrm{Im}(g_{2N}^{(r)}+g_{2N}^{(c)}) & \mathrm{Re}(g_{21}^{(r)}-g_{21}^{(c)}) & \mathrm{Re}(g_{22}^{(r)}-g_{22}^{(c)}) & \cdots & \mathrm{Re}(g_{2N}^{(r)}-g_{2N}^{(c)}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathrm{Im}(g_{M1}^{(r)}+g_{M1}^{(c)}) & \mathrm{Im}(g_{M2}^{(r)}+g_{M2}^{(c)}) & \cdots & \mathrm{Im}(g_{MN}^{(r)}+g_{MN}^{(c)}) & \mathrm{Re}(g_{M1}^{(r)}-g_{M1}^{(c)}) & \mathrm{Re}(g_{M2}^{(r)}-g_{M2}^{(c)}) & \cdots & \mathrm{Re}(g_{MN}^{(r)}-g_{MN}^{(c)})
\end{pmatrix}.
\end{aligned}
\tag{S16}
$$

Here, the notations "Re(·)" and "Im(·)" denote the real and imaginary parts of "(·)", respectively. In addition, we ignored the constant terms $-\mathrm{Tr}[\boldsymbol{\alpha}]/2 - \mathrm{Tr}[\boldsymbol{\beta}]/2$ in Eq. (S4).

Generally, it is a challenging task to clearly analyze the dark-mode effect of a coupled bosonic network in the bare-mode representation, since Eq. (S13) contains various coupling terms. Instead, we prefer to analyze the dark-mode effect in the normal-mode representation. To this end, we firstly apply a symplectic transformation $\mathbf{S}_a$ to the quadrature-operator vector $\hat{\mathbf{r}}_a$ for these bare modes of the type-a subnetwork,

$$
\hat{\mathbf{R}}_A = (\hat{X}_{A_1}, \hat{X}_{A_2}, \cdots, \hat{X}_{A_M}, \hat{P}_{A_1}, \hat{P}_{A_2}, \cdots, \hat{P}_{A_M})^{\mathrm{T}} = \mathbf{S}_a^{-1}\hat{\mathbf{r}}_a,
\tag{S17}
$$

where $\hat{X}_{A_k}$ and $\hat{P}_{A_k}$ are the quadrature operators (namely the dimensionless coordinate and momentum operators) for the kth normal mode A_k of the type-a normal-mode subnetwork. According to Williamson's theorem [S1], the positive definite real matrix $\tilde{\mathbf{H}}_a$ can be diagonalized with a symplectic transformation operator $\mathbf{S}_a$, namely,

$$
\mathbf{H}_A = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_a\mathbf{S}_a = \mathbf{I}_2 \otimes \mathrm{diag}(\Delta_1, \Delta_2, \cdots, \Delta_M),
\tag{S18}
$$

where the notation "$\otimes$" denotes the direct product of matrices, "diag(·)" gives a diagonal matrix with the elements of (·) on the main diagonal, and Δ_k is the resonance frequency of the kth normal mode for the type-a normal-mode subnetwork.

Similarly, we apply the symplectic transformation $\mathbf{S}_b$ to the quadrature-operator vector $\hat{\mathbf{r}}_b$ for these bare modes of the type-b subnetwork,

$$
\hat{\mathbf{R}}_B = (\hat{X}_{B_1}, \hat{X}_{B_2}, \cdots, \hat{X}_{B_N}, \hat{P}_{B_1}, \hat{P}_{B_2}, \cdots, \hat{P}_{B_N})^{\mathrm{T}} = \mathbf{S}_b^{-1}\hat{\mathbf{r}}_b,
\tag{S19}
$$

where $\hat{X}_{B_j}$ and $\hat{P}_{B_j}$ are the quadrature operators for the jth normal mode B_j of the type-b normal-mode subnetwork. In this case, the coefficient matrix $\tilde{\mathbf{H}}_b$ can be diagonalized as

$$
\mathbf{H}_B = \mathbf{S}_b^{\mathrm{T}}\tilde{\mathbf{H}}_b\mathbf{S}_b = \mathbf{I}_2 \otimes \mathrm{diag}(\Omega_1, \Omega_2, \cdots, \Omega_N),
\tag{S20}
$$

where Ω_j is the resonance frequency of the jth normal mode B_j for the type-b normal-mode subnetwork.

In the normal-mode representation, the coupling matrix between the two subnetworks is transformed into

$$
\mathbf{C}_{AB} = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_{ab}\mathbf{S}_b = \begin{pmatrix} \mathbf{C}^{(X_AX_B)} & \mathbf{C}^{(X_AP_B)} \\ \mathbf{C}^{(P_AX_B)} & \mathbf{C}^{(P_AP_B)} \end{pmatrix},
\tag{S21}
$$

with the element $\mathbf{C}_{kj}^{(\cdot)} \equiv G_{kj}^{(\cdot)}$. Here, the variable $G_{kj}^{(\cdot)}$ denotes the effective coupling strength between the normal modes A_k and B_j corresponding to the four coupling cases $\{X_AX_B, X_AP_B, P_AX_B, P_AP_B\}$.

Based on the fact that the dark modes only exist within the same degenerate normal-mode subspaces (all the normal modes have identical frequencies in this subspace), below we analyze the dark-mode effect within the degenerate normal-mode subspace. This is because the frequency mismatch among nondegenerate modes will induce cross couplings among the expected dark and bright modes, and then prevent the formation of the real dark modes.

Based on the above analyses, we assume that the type-b normal-mode subnetwork is composed of various groups of degenerate normal modes. Namely, we can divide all these modes into either degenerate normal-mode subspaces or nondegenerate normal-mode subspaces [Fig. S1(b)]. For convenience and without loss of generality, we assume that all these modes in the same degenerate normal-mode subspace are labeled with consecutive numbers. For example, we assume that the first n_1 normal modes are degenerate with the frequency Ω_1, and the next $(n_2 - n_1)$ normal modes are degenerate with the frequency Ω_2. The last degenerate normal-mode subspace has $(n_s - n_{s-1})$ normal modes with frequency Ω_s. Note that $n_s = N$ for the network. In this case, the Hamiltonian of the system can be expressed in the normal-mode representation as

$$
\begin{aligned}
\hat{H}_{\mathrm{nw}} &= \frac{1}{2}(\hat{\mathbf{R}}_A^{\mathrm{T}}, \hat{\mathbf{R}}_B^{\mathrm{T}})\left(\begin{array}{c|c} \mathbf{H}_A & \mathbf{C}_{AB} \\ \hline \mathbf{C}_{AB}^{\mathrm{T}} & \mathbf{H}_B \end{array}\right)\begin{pmatrix} \hat{\mathbf{R}}_A \\ \hat{\mathbf{R}}_B \end{pmatrix} \\
&= \frac{1}{2}\left(\hat{\mathbf{R}}_A^{\mathrm{T}}, (\hat{\mathbf{R}}_B^{[1\sim n_1]})^{\mathrm{T}}, (\hat{\mathbf{R}}_B^{[(n_1+1)\sim n_2]})^{\mathrm{T}}, \cdots, (\hat{\mathbf{R}}_B^{[(n_{s-1}+1)\sim n_s]})^{\mathrm{T}}\right)
\left(\begin{array}{c|c|c|c|c}
\mathbf{H}_A & \mathbf{C}_{AB}^{[1\sim n_1]} & \mathbf{C}_{AB}^{[(n_1+1)\sim n_2]} & \cdots & \mathbf{C}_{AB}^{[(n_{s-1}+1)\sim n_s]} \\ \hline
(\mathbf{C}_{AB}^{[1\sim n_1]})^{\mathrm{T}} & \mathbf{H}_B^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline
(\mathbf{C}_{AB}^{[(n_1+1)\sim n_2]})^{\mathrm{T}} & \mathbf{0} & \mathbf{H}_B^{(2)} & \cdots & \mathbf{0} \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
(\mathbf{C}_{AB}^{[(n_{s-1}+1)\sim n_s]})^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_B^{(s)}
\end{array}\right)
\begin{pmatrix} \hat{\mathbf{R}}_A \\ \hat{\mathbf{R}}_B^{[1\sim n_1]} \\ \hat{\mathbf{R}}_B^{[(n_1+1)\sim n_2]} \\ \cdots \\ \hat{\mathbf{R}}_B^{[(n_{s-1}+1)\sim n_s]} \end{pmatrix},
\end{aligned}
\tag{S22}
$$

where $\mathbf{H}_B^{(l)} = \Omega_l \mathbf{I}_{2(n_l-n_{l-1})}$ is the coefficient submatrix of the lth type-b degenerate normal-mode subnetwork. Here, the degenerate normal-mode frequency Ω_l is the common resonance frequency of these normal modes $\{B_{n_{l-1}+1}, B_{n_{l-1}+2}, \cdots, B_{n_l}\}$ in the lth degenerate normal-mode subspace, and the normal-mode quadrature-operator vector in this subspace can be expressed as

$$\hat{\mathbf{R}}_B^{[(n_{l-1}+1)\sim n_l]} = (\hat{X}_{B_{n_{l-1}+1}}, \hat{X}_{B_{n_{l-1}+2}}, \cdots, \hat{X}_{B_{n_l}}, \hat{P}_{B_{n_{l-1}+1}}, \hat{P}_{B_{n_{l-1}+2}}, \cdots, \hat{P}_{B_{n_l}})^{\mathrm{T}}. \tag{S23}$$

Notice that, to describe these mode spaces uniformly, we can treat the nondegenerate normal-mode subspace as a special degenerate-mode subspace, with degeneracy $f = 1$. We also point out that there is no dark mode in the nondegenerate normal-mode subspace because there is no quantum interference channel. In Eq. (S22), the submatrix $\mathbf{C}_{AB}^{[(n_{l-1}+1)\sim n_l]}$ is the coupling matrix between the type-a normal-mode subnetwork and all these normal modes in the l-th degenerate subspace of type-b normal-mode subnetwork, and it can be expressed as

$$\mathbf{C}_{AB}^{[(n_{l-1}+1)\sim n_l]} = \left(\begin{array}{cccc|cccc} G_{1(n_{l-1}+1)}^{(X_A,X_B)} & G_{1(n_{l-1}+2)}^{(X_A,X_B)} & \cdots & G_{1n_l}^{(X_A,X_B)} & G_{1(n_{l-1}+1)}^{(X_A,P_B)} & G_{1(n_{l-1}+2)}^{(X_A,P_B)} & \cdots & G_{1n_l}^{(X_A,P_B)} \\ G_{2(n_{l-1}+1)}^{(X_A,X_B)} & G_{2(n_{l-1}+2)}^{(X_A,X_B)} & \cdots & G_{2n_l}^{(X_A,X_B)} & G_{2(n_{l-1}+1)}^{(X_A,P_B)} & G_{2(n_{l-1}+2)}^{(X_A,P_B)} & \cdots & G_{2n_l}^{(X_A,P_B)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{M(n_{l-1}+1)}^{(X_A,X_B)} & G_{M(n_{l-1}+2)}^{(X_A,X_B)} & \cdots & G_{Mn_l}^{(X_A,X_B)} & G_{M(n_{l-1}+1)}^{(X_A,P_B)} & G_{M(n_{l-1}+2)}^{(X_A,P_B)} & \cdots & G_{Mn_l}^{(X_A,P_B)} \\ \hline G_{1(n_{l-1}+1)}^{(P_A,X_B)} & G_{1(n_{l-1}+2)}^{(P_A,X_B)} & \cdots & G_{1n_l}^{(P_A,X_B)} & G_{1(n_{l-1}+1)}^{(P_A,P_B)} & G_{1(n_{l-1}+2)}^{(P_A,P_B)} & \cdots & G_{1n_l}^{(P_A,P_B)} \\ G_{2(n_{l-1}+1)}^{(P_A,X_B)} & G_{2(n_{l-1}+2)}^{(P_A,X_B)} & \cdots & G_{2n_l}^{(P_A,X_B)} & G_{2(n_{l-1}+1)}^{(P_A,P_B)} & G_{2(n_{l-1}+2)}^{(P_A,P_B)} & \cdots & G_{2n_l}^{(P_A,P_B)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{M(n_{l-1}+1)}^{(P_A,X_B)} & G_{M(n_{l-1}+2)}^{(P_A,X_B)} & \cdots & G_{Mn_l}^{(P_A,X_B)} & G_{M(n_{l-1}+1)}^{(P_A,P_B)} & G_{M(n_{l-1}+2)}^{(P_A,P_B)} & \cdots & G_{Mn_l}^{(P_A,P_B)} \end{array}\right). \tag{S24}$$

In principle, the dark-mode effect in the whole network can be investigated by analyzing the existence of dark modes in all these degenerate normal-mode subspaces.

Supplementary Note II: Dark-mode determination using the symplectic-singular-value decomposition method

For a quadratic bosonic quantum network consisting of a type-a subnetwork and a type-b subnetwork, the coupling Hamiltonian between the two subnetworks can be expressed in the normal-mode representation as

$$\hat{H}_{ab} = \hat{\mathbf{R}}_A^{\mathrm{T}} \mathbf{C}_{AB} \hat{\mathbf{R}}_B, \tag{S25}$$

where the operator vectors $\hat{\mathbf{R}}_A$ and $\hat{\mathbf{R}}_B$ are given in Eqs. (S17) and (S19), and the coupling matrix $\mathbf{C}_{AB}$ is described by Eq. (S22). Since the dark modes are confined within the same degenerate normal-mode subspace, we can determine the number of type-b bright and dark modes within each of these degenerate normal-mode subspaces of the network. As an example, we analyze the dark-mode effect in the first degenerate normal-mode subspace composed of these type-b normal modes B_l, described by the quadrature operators $\{\hat{X}_{B_l}, \hat{P}_{B_l}\}$ for $l = 1, 2, \cdots, n_1$, which share the degenerate normal-mode frequency Ω_1. In this degenerate normal-mode subspace, the coupling matrix can be expressed as

$$\mathbf{C}_{AB}^{[1\sim n_1]} = \left(\begin{array}{cccc|cccc} G_{11}^{(X_A,X_B)} & G_{12}^{(X_A,X_B)} & \cdots & G_{1n_1}^{(X_A,X_B)} & G_{11}^{(X_A,P_B)} & G_{12}^{(X_A,P_B)} & \cdots & G_{1n_1}^{(X_A,P_B)} \\ G_{21}^{(X_A,X_B)} & G_{22}^{(X_A,X_B)} & \cdots & G_{2n_1}^{(X_A,X_B)} & G_{21}^{(X_A,P_B)} & G_{22}^{(X_A,P_B)} & \cdots & G_{2n_1}^{(X_A,P_B)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{M1}^{(X_A,X_B)} & G_{M2}^{(X_A,X_B)} & \cdots & G_{Mn_1}^{(X_A,X_B)} & G_{M1}^{(X_A,P_B)} & G_{M2}^{(X_A,P_B)} & \cdots & G_{Mn_1}^{(X_A,P_B)} \\ \hline G_{11}^{(P_A,X_B)} & G_{12}^{(P_A,X_B)} & \cdots & G_{1n_1}^{(P_A,X_B)} & G_{11}^{(P_A,P_B)} & G_{12}^{(P_A,P_B)} & \cdots & G_{1n_1}^{(P_A,P_B)} \\ G_{21}^{(P_A,X_B)} & G_{22}^{(P_A,X_B)} & \cdots & G_{2n_1}^{(P_A,X_B)} & G_{21}^{(P_A,P_B)} & G_{22}^{(P_A,P_B)} & \cdots & G_{2n_1}^{(P_A,P_B)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{M1}^{(P_A,X_B)} & G_{M2}^{(P_A,X_B)} & \cdots & G_{Mn_1}^{(P_A,X_B)} & G_{M1}^{(P_A,P_B)} & G_{M2}^{(P_A,P_B)} & \cdots & G_{Mn_1}^{(P_A,P_B)} \end{array}\right). \tag{S26}$$

The matrix element $G_{kl}^{(\cdot)}$ in Eq. (S26) denotes the effective coupling strength between the normal modes A_k and B_l related to these four coupling cases $\{X_A X_B, X_A P_B, P_A X_B, P_A P_B\}$. Based on the above analyses, the coupled bosonic network associated with the first type-b degenerate normal-mode subspace can be described by the coupling configuration shown in Fig. S2(a). In addition, the Hamiltonian of the network including all these type-a modes and these n_1 type-b modes $\{B_{l=1\text{-}n_1}\}$ can be expressed as

$$\begin{aligned} \hat{H}_{\mathrm{nw}}^{[1\sim n_1]} &= \frac{1}{2}\left(\hat{\mathbf{R}}_A^{\mathrm{T}}, (\hat{\mathbf{R}}_B^{[1\sim n_1]})^{\mathrm{T}}\right)\begin{pmatrix} \mathbf{H}_A & \mathbf{C}_{AB}^{[1\sim n_1]} \\ (\mathbf{C}_{AB}^{[1\sim n_1]})^{\mathrm{T}} & \mathbf{H}_B^{(1)} \end{pmatrix}\begin{pmatrix} \hat{\mathbf{R}}_A \\ \hat{\mathbf{R}}_B^{[1\sim n_1]} \end{pmatrix} \\ &= \frac{1}{2}\sum_{k=1}^{M} \Delta_k (\hat{X}_{A_k}^2 + \hat{P}_{A_k}^2) + \frac{1}{2}\Omega_1 \sum_{j=1}^{n_1} (\hat{X}_{B_j}^2 + \hat{P}_{B_j}^2) + \sum_{k=1}^{M}\sum_{j=1}^{n_1} (G_{kj}^{(X_A X_B)} \hat{X}_{A_k}\hat{X}_{B_j} + G_{kj}^{(X_A P_B)} \hat{X}_{A_k}\hat{P}_{B_j} + G_{kj}^{(P_A X_B)} \hat{P}_{A_k}\hat{X}_{B_j} + G_{kj}^{(P_A P_B)} \hat{P}_{A_k}\hat{P}_{B_j}), \end{aligned} \tag{S27}$$

where the operators $\{\hat{X}_{A_k}, \hat{P}_{A_k}\}$ and $\{\hat{X}_{B_j}, \hat{P}_{B_j}\}$ are given in Eqs. (S17) and (S19).

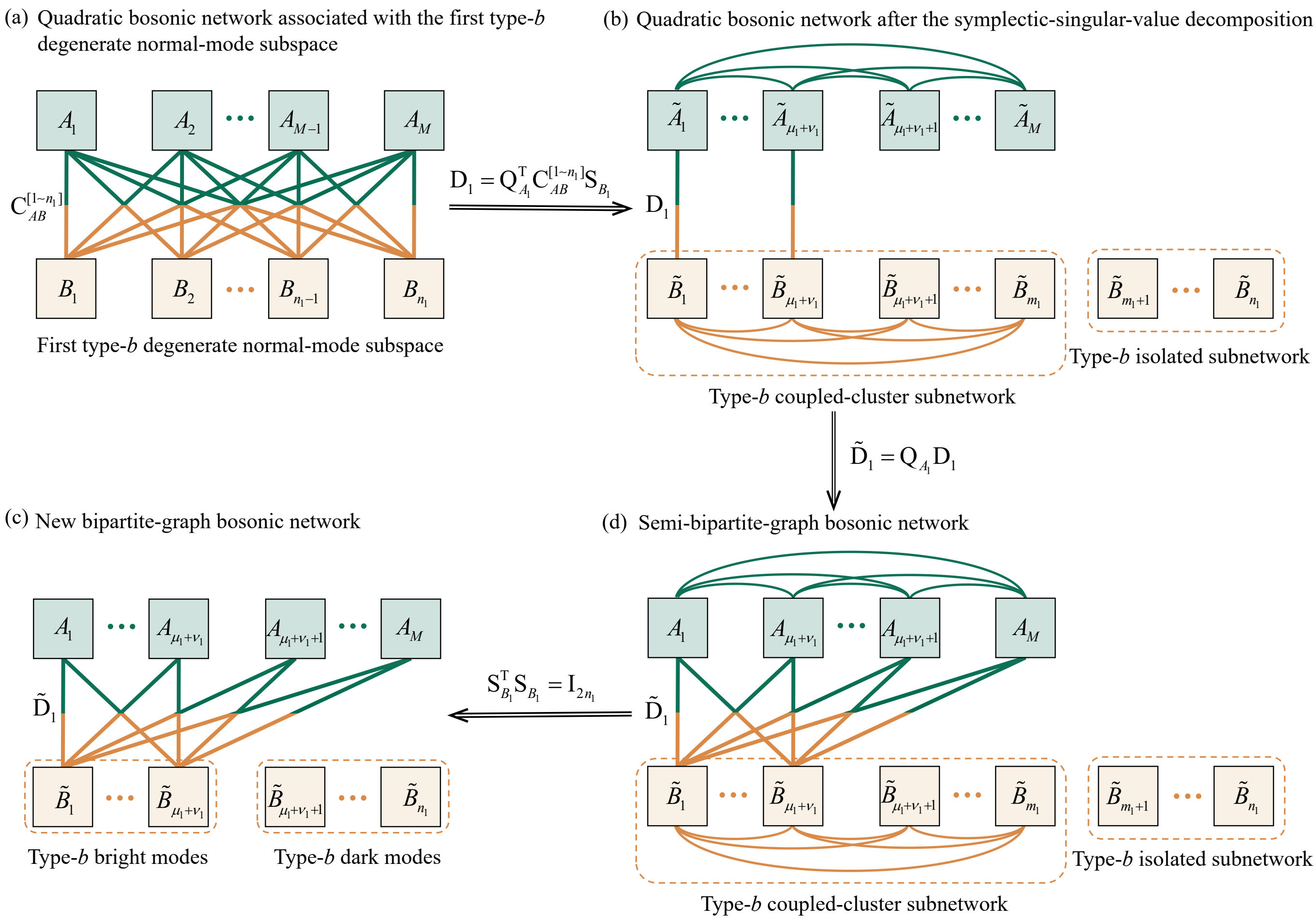

FIG. S2: Deformation of the quadratic bosonic network for determining the dark modes. (a) Quadratic bosonic associated with the first type-b degenerate normal-mode subspace. (b) Quadratic bosonic network after the symplectic-singular-value decomposition. The straight lines represent connections between the two subnetworks described by the Hamiltonian $\hat{\tilde{\mathbf{R}}}_{\tilde{A}}^{\mathrm{T}}\mathbf{D}_1\hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]}/2$, with $\hat{\tilde{\mathbf{R}}}_{\tilde{A}} = \mathbf{Q}_{A_1}^{\mathrm{T}}\hat{\mathbf{R}}_A$; while the curved lines denote the internal connections within the two subnetworks. Here, these modes $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_M$ might be non-orthogonal, while these modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n_1}$ form a set of coupled modes. Within this set, the subset $\{\tilde{B}_{m_1+1}, \tilde{B}_{m_1+2}, \ldots, \tilde{B}_{n_1}\}$ is completely decoupled from the type-a subnetwork, forming a type-b isolated subnetwork, while the modes in the complementary subset $\{\tilde{B}_1, \tilde{B}_2 \ldots, \tilde{B}_{m_1}\}$ are directly or indirectly coupled to the type-a subnetwork and form a coupled-cluster subnetwork. (c) Semi-bipartite-graph bosonic network formed by diagonalizing the type-a non-normal-mode subnetwork in the quadratic bosonic network described by panel (b). (d) New bipartite-graph quantum network formed by considering the symplectic matrix $\mathbf{S}_{B_1}$ to be an orthogonal-symplectic matrix, i.e., $\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1} = \mathbf{I}_{2n_1}$. In this case, all these normal modes $\tilde{B}_j$ for $j = 1, 2, \cdots, n_1$ are orthogonal.

To analyze the dark-mode effect within the degenerate normal-mode subspace, we introduce the symplectic-singular-value decomposition for the coupling matrix $\mathbf{C}_{AB}^{[1\sim n_1]}$. Since the $\mathbf{C}_{AB}^{[1\sim n_1]}$ is a real matrix, there always exists a symplectic-singular-value decomposition [S2],

$$\mathbf{C}_{AB}^{[1\sim n_1]} = \mathbf{Q}_{A_1}\mathbf{D}_1\mathbf{S}_{B_1}^{-1}, \tag{S28}$$

where $\mathbf{Q}_{A_1}$ is an orthogonal matrix with dimension $2M \times 2M$, and $\mathbf{S}_{B_1}$ is a symplectic matrix with dimension $2n_1 \times 2n_1$. In addition, the matrix $\mathbf{D}_1$ with dimension $2M \times 2n_1$ takes the form

$$\mathbf{D}_1 = \begin{pmatrix} \Sigma^{[\mu_1,\mu_1]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}^{[\nu_1,\nu_1]} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}^{[\mu_1,n_1-\mu_1-\nu_1]} & \Sigma^{[\mu_1,\mu_1]} & \mathbf{0}^{[\nu_1,\nu_1]} & \mathbf{0}^{[\mu_1,n_1-\mu_1-\nu_1]} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}^{[2M-2\mu_1-\nu_1,n_1-\mu_1-nu_1]} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{S29}$$

where $\Sigma^{[\mu_1,\mu_1]} = \mathrm{diag}(\sigma_1^{(1)}, \sigma_2^{(1)}, \cdots, \sigma_{\mu_1}^{(1)})$, and the superscripts "$[l, l']$" in these submatrices denote their row and column numbers. In addition, the symplectic singular value $\sigma_{\mu_1}^{(1)}$ is determined by both the skew-symmetric matrix $\mathbf{C}_{AB}^{[1\sim n_1]}\mathbf{J}_b^{[2n_1,2n_1]}\mathbf{C}_{AB}^{[1\sim n_1]^{\mathrm{T}}}$ and the orthogonal matrix $\mathbf{Q}_{A_1}$ [S2].

Accordingly, the operator vectors in Eq. (S22) become

$$\hat{\tilde{\mathbf{R}}}_{\tilde{A}} = (\hat{\tilde{X}}_{\tilde{A}_1}, \hat{\tilde{X}}_{\tilde{A}_2}, \cdots, \hat{\tilde{X}}_{\tilde{A}_{n_M}}, \hat{\tilde{P}}_{\tilde{A}_1}, \hat{\tilde{P}}_{\tilde{A}_2}, \cdots, \hat{\tilde{P}}_{\tilde{A}_M})^{\mathrm{T}} = \mathbf{Q}_A^{\mathrm{T}}\hat{\mathbf{R}}_A, \tag{S30a}$$

$$\hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]} = (\hat{\tilde{X}}_{\tilde{B}_1}, \hat{\tilde{X}}_{\tilde{B}_2}, \cdots, \hat{\tilde{X}}_{\tilde{B}_{p_1+q_1}}, \hat{\tilde{X}}_{\tilde{B}_{p_1+q_1+1}}, \cdots, \hat{\tilde{X}}_{\tilde{B}_{n_1}}, \hat{\tilde{P}}_{\tilde{B}_1}, \hat{\tilde{P}}_{\tilde{B}_2}, \cdots, \hat{\tilde{P}}_{\tilde{B}_{p+q}}, \hat{\tilde{P}}_{\tilde{B}_{p_1+q_1+1}}, \cdots, \hat{\tilde{P}}_{\tilde{B}_{n_1}})^{\mathrm{T}} = \mathbf{S}_{B_1}^{-1}\hat{\mathbf{R}}_B^{[1\sim n_1]}. \tag{S30b}$$

Based on Eqs. (S30), the Hamiltonian of the system (including the type-a subnetwork and those type-b modes restricted within the first type-b degenerate normal-mode subspace) can be expressed as

$$\hat{H}_{\mathrm{nw}}^{[1\sim n_1]} = \frac{1}{2}\left(\hat{\tilde{\mathbf{R}}}_{\tilde{A}}^{\mathrm{T}}, (\hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]})^{\mathrm{T}}\right)\left(\begin{array}{c|c} \tilde{\mathbf{H}}_{\tilde{A}} & \mathbf{D}_1 \\ \hline \mathbf{D}_{AB}^{\mathrm{T}} & \tilde{\mathbf{H}}_{\tilde{B}}^{(1)} \end{array}\right)\left(\begin{array}{c} \hat{\tilde{\mathbf{R}}}_{\tilde{A}} \\ \hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]} \end{array}\right), \tag{S31}$$

with the coefficient submatrices

$$\tilde{\mathbf{H}}_{\tilde{A}} = \mathbf{Q}_{A_1}^{\mathrm{T}}\mathbf{H}_A\mathbf{Q}_{A_1}, \tag{S32a}$$

$$\tilde{\mathbf{H}}_{\tilde{B}}^{(1)} = \mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{H}_B^{(1)}\mathbf{S}_{B_1} = \Omega_1\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1}. \tag{S32b}$$

In Eq. (S29), the rank of $\mathbf{D}_1$ is $(2\mu_1 + \nu_1)$, and both the submatrices corresponding to the third and sixth columns are zero matrices, thus these type-b modes $\tilde{B}_{\mu_1+\nu_1+1}, \tilde{B}_{\mu_1+\nu_1+2}, \cdots, \tilde{B}_{n_1}$ are not directly coupled to any modes in the type-a subnetwork. Since the matrix $\mathbf{S}_{B_1}$ is the symplectic matrix, the coefficient matrix $\mathbf{H}_B^{[1\sim n_1]}$ may contain the off-diagonal elements, which indicates that there may be internal couplings among these modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n_1}$. Therefore, some modes in the subset $\{\tilde{B}_{\mu_1+\nu_1+1}, \tilde{B}_{\mu_1+\nu_1+2}, \cdots, \tilde{B}_{n_1}\}$ may either couple to the type-a subnetwork indirectly via these modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{\mu_1+\nu_1}$ or remain decoupled from it entirely. In this case, we can divide these modes in the subset $\{\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n_1}\}$ into two subsets: type-b coupled-cluster subset (named type-b coupled-cluster subnetwork containing directly-coupled modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{\mu_1+\nu_1}$ and indirectly-coupled modes $\tilde{B}_{\mu_1+\nu_1+1}, \tilde{B}_{\mu_1+\nu_1+2}, \cdots, \tilde{B}_{m_1}$) and type-$b$ isolated subset (named type-b isolated subnetwork containing modes $\tilde{B}_{m_1+1}, \tilde{B}_{m_1+2}, \cdots, \tilde{B}_{n_1}$). These modes in the isolated subnetwork are always decoupled from type-a subnetwork and the type-b coupled-cluster subnetwork. In addition, since the matrix $\mathbf{Q}_{A_1}$ is merely orthogonal, it cannot guarantee that the operators in the vector (S30a) satisfy the canonical commutation relation $[\hat{\tilde{X}}_{A_k}, \hat{\tilde{P}}_{A_{k'}}] = i\delta_{k,k'}$, which means that these modes $\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_M\}$ might be non-orthogonal. Based on the above analyses, the coupled bosonic network after the symplectic-singular-value decomposition can be described by the coupling configuration as shown in Fig. S2(b).

In Fig. S2(b), we cannot obtain the explicit expression for the real couplings between the bright modes $\{\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{\mu_1+\nu_1}\}$ and the normal modes in type-a subnetwork. To clearly analyze the real couplings between the type-b bright modes and the modes in type-a normal-mode subnetwork, we need to transform these non-normal modes $\{\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_M\}$ into normal modes. To this end, we reintroduce the orthogonal matrix $\mathbf{Q}_{A_1}$ to diagonalize the coefficient matrix $\tilde{\mathbf{H}}_{\tilde{A}}$, namely

$$\mathbf{H}_A = \mathbf{Q}_{A_1}\tilde{\mathbf{H}}_{\tilde{A}}\mathbf{Q}_{A_1}^{\mathrm{T}}, \tag{S33}$$

then the operator vector $\hat{\tilde{\mathbf{R}}}_{\tilde{A}}$ becomes $\hat{\mathbf{R}}_A$. In this case, this network after the symplectic-singular-value decomposition becomes a semi-bipartite-graph bosonic network in Fig. S2(c). Consequently, the Hamiltonian of the semi-bipartite-graph bosonic network can be expressed as

$$\hat{H}_{\mathrm{nw}}^{[1\sim n_1]} = \frac{1}{2}\left(\hat{\mathbf{R}}_A^{\mathrm{T}}, (\hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]})^{\mathrm{T}}\right)\left(\begin{array}{c|c} \mathbf{H}_A & \tilde{\mathbf{D}}_1 \\ \hline \tilde{\mathbf{D}}_{AB}^{\mathrm{T}} & \tilde{\mathbf{H}}_{\tilde{B}}^{(1)} \end{array}\right)\left(\begin{array}{c} \hat{\mathbf{R}}_A \\ \hat{\tilde{\mathbf{R}}}_{\tilde{B}}^{[1\sim n_1]} \end{array}\right), \tag{S34}$$

where we introduce the new coupling matrix

$$\tilde{\mathbf{D}}_1 = \mathbf{Q}_{A_1}\mathbf{D}_1. \tag{S35}$$

Since $\mathbf{Q}_{A_1}$ is an orthogonal matrix, i.e., $\mathbf{Q}_{A_1}\mathbf{Q}_{A_1}^{\mathrm{T}} = \mathbf{I}$, we have $\mathbf{Q}_{A_1}^{-1} = \mathbf{Q}_{A_1}^{\mathrm{T}}$. Therefore, left-multiplying $\mathbf{D}_1$ by $\mathbf{Q}_{A_1}$ does not change the rank of $\mathbf{D}_1$ [Rank($\tilde{\mathbf{D}}_1$)=Rank($\mathbf{D}_1$)], which means that the nonzero column vectors in $\tilde{\mathbf{D}}_1$ are linearly independent, and the transformation preserves the null subspace without introducing new ones. When the directly-coupled set $\{\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{\mu_1+\nu_1}\}$ and indirectly-coupled set $\{\tilde{B}_{\mu_1+\nu_1+1}, \tilde{B}_{\mu_1+\nu_1+2}, \cdots, \tilde{B}_{m_1}\}$ are decoupled, the Hamiltonian of the network can be expressed as

$$\begin{aligned}
\hat{H}_{\mathrm{nw}}^{[1\sim n_1]} = {} & \frac{1}{2}\sum_{k=1}^{M}\Delta_k(\hat{X}_{A_k}^2 + \hat{P}_{A_k}^2) + \frac{1}{2}\Omega_1\sum_{j,j'=1}^{\mu_1+\nu_1}[(\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,n_1+j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{P}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,j'}\hat{\tilde{P}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,n_1+j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}}] \\
& + \frac{1}{2}\Omega_1\sum_{j,j'=\mu_1+\nu_1+1}^{m_1}[(\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,n_1+j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{P}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,j'}\hat{\tilde{P}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,n_1+j'}\hat{\tilde{P}}_{\tilde{B}_j}\hat{\tilde{P}}_{\tilde{B}_{j'}}] \\
& + \frac{1}{2}\Omega_1\sum_{j,j'=m_1+1}^{n_1}[(\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{j,n_1+j'}\hat{\tilde{X}}_{\tilde{B}_j}\hat{\tilde{P}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,j'}\hat{\tilde{P}}_{\tilde{B}_j}\hat{\tilde{X}}_{\tilde{B}_{j'}} + (\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1})_{n_1+j,n_1+j'}\hat{\tilde{P}}_{\tilde{B}_j}\hat{\tilde{P}}_{\tilde{B}_{j'}}] \\
& + \sum_{k=1}^{M}\sum_{j=1}^{\mu_1+\nu_1}(\tilde{G}_{kj}^{(X_A\tilde{X}_{\tilde{B}})}\hat{X}_{A_k}\hat{\tilde{X}}_{\tilde{B}_j} + \tilde{G}_{kj}^{(X_A\tilde{P}_{\tilde{B}})}\hat{X}_{A_k}\hat{\tilde{P}}_{\tilde{B}_j} + \tilde{G}_{kj}^{(P_A\tilde{X}_{\tilde{B}})}\hat{P}_{A_k}\hat{\tilde{X}}_{\tilde{B}_j} + \tilde{G}_{kj}^{(P_A\tilde{P}_{\tilde{B}})}\hat{P}_{A_k}\hat{\tilde{P}}_{\tilde{B}_j}),
\end{aligned} \tag{S36}$$

where the variable $\tilde{G}_{kj}^{(\cdot)}$ denotes the effective coupling strength between the normal modes A_k and $\tilde{B}_j$ related to these four coupling cases

$\{X_A\tilde{X}_{\tilde{B}}, X_A\tilde{P}_{\tilde{B}}, P_A\tilde{X}_{\tilde{B}}, P_A\tilde{P}_{\tilde{B}}\}$. Thus, there exist $(\mu_1+\nu_1)$ type-b bright modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{\mu_1+\nu_1}$ and $(n_1-\mu_1-\nu_1)$ type-b dark modes $\tilde{B}_{\mu_1+\nu_1+1}, \tilde{B}_{\mu_1+\nu_1+2}, \cdots, \tilde{B}_{n_1}$.

Although the number of dark modes can be determined under the current condition, their explicit forms remain inaccessible because the modes within the dark-mode subnetwork may still be coupled. To clearly identify the forms of the dark modes, we need to impose an additional condition on the symplectic matrix $\mathbf{S}_{B_1}$. As required, when $\mathbf{S}_{B_1}$ is also an orthogonal matrix, i.e.,

$$\mathbf{S}_{B_1}^{\mathrm{T}}\mathbf{S}_{B_1} = \mathbf{I}_{2n_1}, \tag{S37}$$

the coefficient matrix $\tilde{\mathbf{H}}_{\tilde{B}}^{(1)}$ becomes a diagonal matrix with the diagonal elements $\Omega_1\mathbf{I}_{2n_1}$, which means that all these normal modes $\tilde{B}_1, \tilde{B}_2, \cdots, \tilde{B}_{n_1}$ are orthogonal. Thus, there are no couplings between the bright-mode and dark-mode subnetworks, and then the semi-bipartite-graph network becomes a bipartite-graph quantum network [Fig. S2(d)]. In this case, the Hamiltonian (S36) becomes

$$\hat{H}_{\mathrm{nw}}^{[1\sim n_1]} = \frac{1}{2}\sum_{k=1}^{M}\Delta_k(\hat{X}_{A_k}^2+\hat{P}_{A_k}^2)+\frac{1}{2}\Omega_1\sum_{j=1}^{n_1}(\hat{\tilde{X}}_{\tilde{B}_j}^2+\hat{\tilde{P}}_{\tilde{B}_j}^2)+\sum_{k=1}^{M}\sum_{j=1}^{\mu_1+\nu_1}(\tilde{G}_{kj}^{(X_A\tilde{X}_{\tilde{B}})}\hat{X}_{A_k}\hat{\tilde{X}}_{\tilde{B}_j}+\tilde{G}_{kj}^{(X_A\tilde{P}_{\tilde{B}})}\hat{X}_{A_k}\hat{\tilde{P}}_{\tilde{B}_j}+\tilde{G}_{kj}^{(P_A\tilde{X}_{\tilde{B}})}\hat{P}_{A_k}\hat{\tilde{X}}_{\tilde{B}_j}+\tilde{G}_{kj}^{(P_A\tilde{P}_{\tilde{B}})}\hat{P}_{A_k}\hat{\tilde{P}}_{\tilde{B}_j}). \tag{S38}$$

Using the same method, we can determine the dark modes in all these degenerate normal-mode subspaces. Based on the above analyses, we can determine the total number of the type-b dark modes in the quadratic bosonic quantum network, which equals to the sum of the dark mode number in all degenerate-mode subspaces.

A. Reduced to the beam-splitter-type coupled bosonic networks under the rotating-wave approximation

In this section, we show that our present method can be reduced to the arrowhead-matrix method for determining the dark-mode effect in the beam-splitter-type coupled bosonic networks [S3], which can be obtained by discarding all the counter-rotating coupling terms in the present networks. For the $(M+N)$-mode beam-splitter-type coupled bosonic network, the Hamiltonian can be expressed in the quadrature-operator representation as

$$\hat{H} = \frac{1}{2}(\hat{\tilde{\mathbf{r}}}_a^{\mathrm{T}}, \hat{\tilde{\mathbf{r}}}_b^{\mathrm{T}})\left(\begin{array}{c|c}\tilde{\mathbf{H}}_a & \tilde{\mathbf{H}}_{ab}\\ \hline \tilde{\mathbf{H}}_{ab}^{\dagger} & \tilde{\mathbf{H}}_b\end{array}\right)\begin{pmatrix}\hat{\tilde{\mathbf{r}}}_a\\ \hat{\tilde{\mathbf{r}}}_b\end{pmatrix}, \tag{S39}$$

where the operator vectors $\tilde{\mathbf{r}}_a$ and $\tilde{\mathbf{r}}_b$ are given in Eqs. (S7), and the coefficient matrices are given by

$$\tilde{\mathbf{H}}_a = \frac{1}{2}\begin{pmatrix}\boldsymbol{\alpha}+\boldsymbol{\alpha}^{\mathrm{T}} & i(\boldsymbol{\alpha}-\boldsymbol{\alpha}^{\mathrm{T}})\\ -i(\boldsymbol{\alpha}-\boldsymbol{\alpha}^{\mathrm{T}}) & \boldsymbol{\alpha}+\boldsymbol{\alpha}^{\mathrm{T}}\end{pmatrix}, \tag{S40a}$$

$$\tilde{\mathbf{H}}_b = \frac{1}{2}\begin{pmatrix}\boldsymbol{\beta}+\boldsymbol{\beta}^{\mathrm{T}} & i(\boldsymbol{\beta}-\boldsymbol{\beta}^{\mathrm{T}})\\ -i(\boldsymbol{\beta}-\boldsymbol{\beta}^{\mathrm{T}}) & \boldsymbol{\beta}+\boldsymbol{\beta}^{\mathrm{T}}\end{pmatrix}, \tag{S40b}$$

$$\tilde{\mathbf{H}}_{ab} = \frac{1}{2}\begin{pmatrix}\boldsymbol{g}_r+\boldsymbol{g}_r^{*} & i(\boldsymbol{g}_r-\boldsymbol{g}_r^{*})\\ -i(\boldsymbol{g}_r-\boldsymbol{g}_r^{*}) & \boldsymbol{g}_r+\boldsymbol{g}_r^{*}\end{pmatrix}, \tag{S40c}$$

with $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{g}_r$ given in Eq. (S6a). Namely, the submatrices $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{g}_c$ in Eq. (S6b) become zero submatrices. To diagonalize the coefficient matrices $\tilde{\mathbf{H}}_a$ and $\tilde{\mathbf{H}}_b$, we introduce two symplectic matrices

$$\mathbf{S}_a = \begin{pmatrix}\mathrm{Re}(\mathbf{U}_a) & -\mathrm{Im}(\mathbf{U}_a)\\ \mathrm{Im}(\mathbf{U}_a) & \mathrm{Re}(\mathbf{U}_a)\end{pmatrix}, \tag{S41a}$$

$$\mathbf{S}_b = \begin{pmatrix}\mathrm{Re}(\mathbf{U}_b) & -\mathrm{Im}(\mathbf{U}_b)\\ \mathrm{Im}(\mathbf{U}_b) & \mathrm{Re}(\mathbf{U}_b)\end{pmatrix}, \tag{S41b}$$

with $\mathbf{U}_a$ and $\mathbf{U}_b$ being the unitary matrices. In this case, the coefficient matrices $\tilde{\mathbf{H}}_a$ and $\tilde{\mathbf{H}}_b$ can be diagonalized as

$$\mathbf{H}_A = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_a\mathbf{S}_a = \mathbf{I}_2\otimes\mathrm{diag}(\Delta_1, \Delta_2, \cdots, \Delta_M), \tag{S42a}$$

$$\mathbf{H}_B = \mathbf{S}_b^{\mathrm{T}}\tilde{\mathbf{H}}_b\mathbf{S}_b = \mathbf{I}_2\otimes\mathrm{diag}(\Omega_1, \Omega_2, \cdots, \Omega_N), \tag{S42b}$$

respectively, where we use the two relations

$$\mathbf{U}_a^{\dagger}\boldsymbol{\alpha}\mathbf{U}_a = \mathrm{diag}(\Delta_1, \Delta_2, \cdots, \Delta_M), \tag{S43a}$$

$$\mathbf{U}_b^{\dagger}\boldsymbol{\beta}\mathbf{U}_b = \mathrm{diag}(\Omega_1, \Omega_2, \cdots, \Omega_N). \tag{S43b}$$

Note that these notations have been introduced in Supplementary Note I of this supplementary material. In addition, the coupling matrix between the two normal-mode subnetworks becomes

$$\mathbf{C}_{AB} = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_{ab}\mathbf{S}_b = \begin{pmatrix}\mathrm{Re}(\mathbf{G}_r) & -\mathrm{Im}(\mathbf{G}_r)\\ \mathrm{Im}(\mathbf{G}_r) & \mathrm{Re}(\mathbf{G}_r)\end{pmatrix}, \tag{S44}$$

where we introduce the effective coupling matrix

$$\mathbf{G}_r = \mathbf{U}_a^{\dagger}\boldsymbol{g}_r\mathbf{U}_b. \tag{S45}$$

Accordingly, the new operator vectors in the normal-mode representation can be expressed as

$$\hat{\mathbf{R}}_A = (\hat{X}_{A_1}, \hat{X}_{A_2}, \cdots, \hat{X}_{A_M}, \hat{P}_{A_1}, \hat{P}_{A_2}, \cdots, \hat{P}_{A_M})^{\mathrm{T}} = \mathbf{S}_a^{-1}\hat{\mathbf{r}}_a, \tag{S46a}$$

$$\hat{\mathbf{R}}_B = (\hat{X}_{B_1}, \hat{X}_{B_2}, \cdots, \hat{X}_{B_N}, \hat{P}_{B_1}, \hat{P}_{B_2}, \cdots, \hat{P}_{B_N})^{\mathrm{T}} = \mathbf{S}_b^{-1}\hat{\mathbf{r}}_b. \tag{S46b}$$

To clarify the idea for analyzing the dark-mode effect in the beam-splitter-type coupled bosonic network, we now focus on the case with one degenerate normal-mode subspace composed of $n_1 \leq N$ type-b normal modes B_j ($j = 1, 2, \cdots, n_1$). In this degenerate normal-mode subspace, the corresponding coupling matrix reads

$$\mathbf{C}_{AB}^{[1\sim n_1]} = \begin{pmatrix} \mathrm{Re}(\mathbf{G}_r^{[1\sim n_1]}) & -\mathrm{Im}(\mathbf{G}_r^{[1\sim n_1]}) \\ \mathrm{Im}(\mathbf{G}_r^{[1\sim n_1]}) & \mathrm{Re}(\mathbf{G}_r^{[1\sim n_1]}) \end{pmatrix}. \tag{S47}$$

Based on Eq. (S28), the symplectic-singular-value decomposition of the coupling matrix in Eq. (S47) can be obtained as

$$\mathbf{C}_{AB}^{[1\sim n_1]} = \mathbf{Q}_A \mathbf{D} \mathbf{S}_B^{-1}, \tag{S48}$$

where the orthogonal and symplectic matrices are given by

$$\mathbf{Q}_A = \begin{pmatrix} \mathrm{Re}(\mathbf{U}_A) & -\mathrm{Im}(\mathbf{U}_A) \\ \mathrm{Im}(\mathbf{U}_A) & \mathrm{Re}(\mathbf{U}_A) \end{pmatrix}, \tag{S49a}$$

$$\mathbf{S}_B = \begin{pmatrix} \mathrm{Re}(\mathbf{U}_B) & -\mathrm{Im}(\mathbf{U}_B) \\ \mathrm{Im}(\mathbf{U}_B) & \mathrm{Re}(\mathbf{U}_B) \end{pmatrix}. \tag{S49b}$$

After some algebra, we obtain

$$\mathbf{D} = \mathbf{Q}_A^{\mathrm{T}} \mathbf{C}_{AB}^{[1\sim n_1]} \mathbf{S}_B = \begin{pmatrix} \boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma} \end{pmatrix}, \tag{S50}$$

where the matrix $\boldsymbol{\Sigma}$ takes the form

$$\boldsymbol{\Sigma} = \mathbf{U}_A^{\dagger} \mathrm{G}_r \mathbf{U}_B = \begin{pmatrix} \Sigma^{[\mu_1,\mu_1]} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^{[\nu_1,\nu_1]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}^{[M-\mu_1-\nu_1, n_1-\mu_1-\nu_1]} \end{pmatrix}, \tag{S51}$$

with the diagonal matrix $\Sigma^{[\mu_1,\mu_1]} = \mathrm{diag}(\sigma_1^{(1)}, \sigma_2^{(1)}, \cdots, \sigma_{\mu_1}^{(1)})$ and $\mu_1 = \min(M, n_1)$. Here, $(\sigma_{\mu_1}^{(1)})^2$ is the p-th eigenvalue of the matrix $\mathbf{G}_r\mathbf{G}_r^{\dagger}$, and the unitary matrix $\mathbf{U}_A$ ($\mathbf{U}_B$) is composed of the eigenvectors of $\mathbf{G}_r\mathbf{G}_r^{\dagger}$ ($\mathbf{G}_r^{\dagger}\mathbf{G}_r$). Based on Eq. (S51), the symplectic-singular-value-decomposition method is reduced to the singular-value-decomposition method in the beam-splitter-type quantum networks. Thus, our method can cover the arrowhead-matrix method as a special case for the beam-splitter-type coupled bosonic networks.

B. The dark-mode determination in the (2 + 3)-mode linearized optomechanical networks

We consider a (2 + 3)-mode optomechanical network, which is composed of two optical modes and three mechanical modes. For convenience, we consider the case where the frequencies of the two optical modes (three mechanical modes) are degenerate, i.e., $\delta_{a_1} = \delta_{a_2} = \delta_a$ ($\omega_{b_1} = \omega_{b_2} = \omega_{b_3} = \omega_b$), and the real coupling strengths between the mechanical oscillators are equal, namely $\eta_{12} = \eta_{13} = \eta_{23} = \eta$. In this case, the linearized Hamiltonian (the linearized Hamiltonian of the optomechanical network will be presented in Supplementary Note III) in the normal-mode representation can be expressed as

$$\hat{H}_{\mathrm{lin}} = \frac{1}{2}(\hat{\mathbf{r}}_a^{\mathrm{T}}, \hat{\mathbf{r}}_b^{\mathrm{T}}) \left(\begin{array}{c|c} \tilde{\mathbf{H}}_a^{[4,4]} & \tilde{\mathbf{H}}_{ab}^{[4,6]} \\ \hline \tilde{\mathbf{H}}_{ab}^{[4,6]\mathrm{T}} & \tilde{\mathbf{H}}_b^{[6,6]} \end{array}\right) \begin{pmatrix} \hat{\mathbf{r}}_a \\ \hat{\mathbf{r}}_b \end{pmatrix}, \tag{S52}$$

where we introduce the coefficient matrices

$$\tilde{\mathbf{H}}_a^{[4,4]} = \begin{pmatrix} \tilde{\boldsymbol{\alpha}}^{[2,2]} & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{\alpha}}^{[2,2]} \end{pmatrix}, \qquad \tilde{\mathbf{H}}_b^{[6,6]} = \begin{pmatrix} \tilde{\boldsymbol{\beta}}^{[3,3]} & \mathbf{0} \\ \mathbf{0} & \omega_b \mathbf{I}_3 \end{pmatrix}, \qquad \tilde{\mathbf{H}}_{ab}^{[4,6]} = \begin{pmatrix} \mathbf{g}^{[2,3]} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}^{[2,3]} \end{pmatrix}, \tag{S53}$$

with

$$\tilde{\boldsymbol{\alpha}}^{[2,2]} = \begin{pmatrix} \delta_{a_1} & \xi_{12} \\ \xi_{12} & \delta_{a_2} \end{pmatrix}, \qquad \tilde{\boldsymbol{\beta}}^{[3,3]} = \begin{pmatrix} \omega_b & 2\eta & 2\eta \\ 2\eta & \omega_b & 2\eta \\ 2\eta & 2\eta & \omega_b \end{pmatrix}, \qquad \mathbf{g}^{[2,3]} = \begin{pmatrix} 2g_{11} & 2g_{12} & 2g_{13} \\ 2g_{21} & 2g_{22} & 2g_{23} \end{pmatrix}. \tag{S54}$$

Note that here we introduced the superscript $[s, s']$ to mark the dimension of the submatrices. The fluctuation operator vectors are defined as

$$\hat{\mathbf{r}}_a = (\hat{x}_{a_1}, \hat{x}_{a_2}, \hat{p}_{a_1}, \hat{p}_{a_2})^{\mathrm{T}}, \tag{S55a}$$

$$\hat{\mathbf{r}}_b = (\hat{x}_{b_1}, \hat{x}_{b_2}, \hat{x}_{b_3}, \hat{p}_{b_1}, \hat{p}_{b_2}, \hat{p}_{b_3})^{\mathrm{T}}. \tag{S55b}$$

To diagonalize the two coefficient matrices $\tilde{\mathbf{H}}_a^{[4,4]}$ and $\tilde{\mathbf{H}}_b^{[6,6]}$, we introduce two symplectic matrices $\mathbf{S}_a$ and $\mathbf{S}_b$,

$$\mathbf{S}_a = \begin{pmatrix} \mathbf{U}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_a \end{pmatrix}, \tag{S56a}$$

$$\mathbf{S}_b = \begin{pmatrix} \omega_b^{1/4} \mathbf{U}_b \boldsymbol{\Xi}_b^{-1/4} & \mathbf{0} \\ \mathbf{0} & \omega_b^{-1/4} \mathbf{U}_b \boldsymbol{\Xi}_b^{1/4} \end{pmatrix}, \tag{S56b}$$

with the orthogonal matrices

$$\mathbf{U}_a = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{U}_b = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \tag{S57}$$

and the diagonal matrix

$$\mathbf{\Xi}_b = \mathbf{U}_b^{\mathrm{T}}\tilde{\boldsymbol{\beta}}^{[3,3]}\mathbf{U}_b = \begin{pmatrix} \omega_b - 2\eta & 0 & 0 \\ 0 & \omega_b - 2\eta & 0 \\ 0 & 0 & \omega_b + 4\eta \end{pmatrix}. \tag{S58}$$

Therefore, the two coefficient matrices $\tilde{\mathbf{H}}_a^{[4,4]}$ and $\tilde{\mathbf{H}}_a^{[6,6]}$ can be diagonalized as

$$\mathbf{H}_A^{[4,4]} = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_a^{[4,4]}\mathbf{S}_a = \mathbf{I}_2 \otimes \begin{pmatrix} \delta_a - \xi_{12} & 0 \\ 0 & \delta_a + \xi_{12} \end{pmatrix}, \tag{S59a}$$

$$\mathbf{H}_B^{[6,6]} = \mathbf{S}_b^{\mathrm{T}}\tilde{\mathbf{H}}_b^{[6,6]}\mathbf{S}_b = \mathbf{I}_2 \otimes \begin{pmatrix} \sqrt{\omega_b(\omega_b - 2\eta)} & 0 & 0 \\ 0 & \sqrt{\omega_b(\omega_b - 2\eta)} & 0 \\ 0 & 0 & \sqrt{\omega_b(\omega_b + 4\eta)} \end{pmatrix}. \tag{S59b}$$

In this case, the coupling matrix $\tilde{\mathbf{H}}_{ab}^{[4,6]}$ becomes

$$\mathbf{C}_{AB}^{[4,6]} = \mathbf{S}_a^{\mathrm{T}}\tilde{\mathbf{H}}_{ab}^{[4,6]}\mathbf{S}_b = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \end{pmatrix} \oplus \mathbf{0}^{[2,3]}, \tag{S60}$$

with the effective coupling strengths

$$G_{11} = \frac{\omega_b^{1/4}}{(\omega_b - 2\eta)^{1/4}}(g_{11} - g_{13} - g_{21} + g_{23}), \tag{S61a}$$

$$G_{12} = \frac{\omega_b^{1/4}}{\sqrt{3}(\omega_b - 2\eta)^{1/4}}(g_{11} - 2g_{12} + g_{13} - g_{21} + 2g_{22} - g_{23}), \tag{S61b}$$

$$G_{13} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b + 4\eta)^{1/4}}(-g_{11} - g_{12} - g_{13} + g_{21} + g_{22} + g_{23}), \tag{S61c}$$

$$G_{21} = \frac{\omega_b^{1/4}}{(\omega_b - 2\eta)^{1/4}}(-g_{11} + g_{13} - g_{21} + g_{23}), \tag{S61d}$$

$$G_{22} = -\frac{\omega_b^{1/4}}{\sqrt{3}(\omega_b - 2\eta)^{1/4}}(g_{11} - 2g_{12} + g_{13} + g_{21} - 2g_{22} + g_{23}), \tag{S61e}$$

$$G_{23} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b + 4\eta)^{1/4}}(g_{11} + g_{12} + g_{13} + g_{21} + g_{22} + g_{23}). \tag{S61f}$$

The Hamiltonian (S52) can be expressed in the normal-mode representation as

$$\hat{H}_{\mathrm{lin}} = \frac{1}{2}(\hat{\mathbf{R}}_A^{\mathrm{T}}, \hat{\mathbf{R}}_B^{\mathrm{T}}) \left(\begin{array}{c|c} \mathbf{H}_A^{[4,4]} & \mathbf{C}_{AB}^{[4,6]} \\ \hline \mathbf{C}_{AB}^{[4,6]\mathrm{T}} & \mathbf{H}_B^{[6,6]} \end{array}\right) \begin{pmatrix} \hat{\mathbf{R}}_A \\ \hat{\mathbf{R}}_B \end{pmatrix}, \tag{S62}$$

where we introduce the new fluctuation operator vectors

$$\hat{\mathbf{R}}_A = (\hat{X}_{A_1}, \hat{X}_{A_2}, \hat{P}_{A_1}, \hat{P}_{A_2})^{\mathrm{T}} = \mathbf{S}_a^{-1}\hat{\mathbf{r}}_a, \tag{S63a}$$

$$\hat{\mathbf{R}}_B = (\hat{X}_{B_1}, \hat{X}_{B_2}, \hat{X}_{B_3}, \hat{P}_{B_1}, \hat{P}_{B_2}, \hat{P}_{B_3})^{\mathrm{T}} = \mathbf{S}_b^{-1}\hat{\mathbf{r}}_b, \tag{S63b}$$

with

$$\hat{X}_{B_1} = -\frac{1}{\sqrt{2}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_1} + \frac{1}{\sqrt{2}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_3}, \tag{S64a}$$

$$\hat{X}_{B_2} = -\frac{1}{\sqrt{6}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_1} + \frac{\sqrt{2}}{\sqrt{3}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_2} - \frac{1}{\sqrt{6}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_3}, \tag{S64b}$$

$$\hat{X}_{B_3} = \frac{1}{\sqrt{3}}\frac{(\omega_b+4\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_1} + \frac{1}{\sqrt{3}}\frac{(\omega_b+4\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_2} + \frac{1}{\sqrt{3}}\frac{(\omega_b+4\eta)^{1/4}}{\omega_b^{1/4}}\hat{x}_{b_3}, \tag{S64c}$$

$$\hat{P}_{B_1} = -\frac{1}{\sqrt{2}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}\hat{p}_{b_1} + \frac{1}{\sqrt{2}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}\hat{p}_{b_3}, \tag{S64d}$$

$$\hat{P}_{B_2} = -\frac{1}{\sqrt{6}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}\hat{p}_{b_1} + \frac{\sqrt{2}}{\sqrt{3}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}\hat{p}_{b_2} - \frac{1}{\sqrt{6}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}\hat{p}_{b_3}, \tag{S64e}$$

$$\hat{P}_{B_3} = \frac{1}{\sqrt{3}}\frac{\omega_b^{1/4}}{(\omega_b+4\eta)^{1/4}}\hat{p}_{b_1} + \frac{1}{\sqrt{3}}\frac{\omega_b^{1/4}}{(\omega_b+4\eta)^{1/4}}\hat{p}_{b_2} + \frac{1}{\sqrt{3}}\frac{\omega_b^{1/4}}{(\omega_b+4\eta)^{1/4}}\hat{p}_{b_3}. \tag{S64f}$$

We can see from Eq. (S59b) that there exists two degenerate type-b normal modes: B_1 with the operators $\{\hat{X}_{B_1}, \hat{P}_{B_1}\}$ and B_2 with the operators $\{\hat{X}_{B_2}, \hat{P}_{B_2}\}$, thus the dark modes of the $(2+3)$-mode linearized optomechanical networks only exist in this degenerate normal-mode subspace. Here, the coupling matrix can be expressed as

$$\tilde{\mathbf{C}}_{AB}^{[4,4]} = \begin{pmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{S65}$$

To further analyze the dark modes in the $(2+3)$-mode linearized optomechanical network, we introduce an orthogonal matrix $\mathbf{Q}_A = \mathbf{U}_A \oplus \mathbf{U}_A$ and a real symplectic matrix $\mathbf{S}_B = \mathbf{V}_B \oplus \mathbf{V}_B$, with the orthogonal matrices $\mathbf{U}_A$ and $\mathbf{V}_B$. Then the symplectic-singular-value decomposition of the coupling matrix $\mathbf{C}_{AB}^{[4,4]}$ can be obtained as

$$\mathbf{C}_{AB}^{[4,4]} = \mathbf{Q}_A \mathbf{D}_{AB}^{[4,4]} \mathbf{S}_B^{-1}, \tag{S66}$$

with the diagonal matrix

$$\mathbf{D}_{AB}^{[4,4]} = \begin{pmatrix} \sigma_1^{(1)} & 0 & 0 & 0 \\ 0 & \sigma_2^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{S67}$$

In Eq. (S67), the symplectic singular values $\sigma_1^{(1)}$ and $\sigma_2^{(1)}$ can be calculated as

$$\sigma_{1(2)}^{(1)} = \sqrt{\frac{(G_{11}^2+G_{21}^2+G_{12}^2+G_{22}^2) \pm \sqrt{(G_{11}^2+G_{21}^2-G_{12}^2-G_{22}^2)^2 + 4(G_{11}G_{12}+G_{21}G_{22})^2}}{2}}. \tag{S68}$$

In addition, the orthogonal matrices $\mathbf{U}_A$ and $\mathbf{V}_B$ are composed of the eigenstates of the matrices $\mathbf{C}_{AB}^{[4,4]}\mathbf{C}_{AB}^{[4,4]\mathrm{T}}$ and $\mathbf{C}_{AB}^{[4,4]\mathrm{T}}\mathbf{C}_{AB}^{[4,4]}$, respectively. Note that the dark modes appear when either of the singular values $\sigma_1^{(1)}$ or $\sigma_2^{(1)}$ vanishes.

Based on the above decomposition, we will analyze several cases according to the tunable coupling strengths g_{22} and g_{23}, where other coupling strengths are equal ($g_{11} = g_{12} = g_{13} = g_{21} = g$).

(i) In the case $g_{22} = g_{23} = g$, the effective coupling strengths can be expressed as $G_{11} = G_{21} = G_{12} = G_{13} = G_{22} = 0$ and $G_{23} = 2\sqrt{6}[\omega_b/(\omega_b+4\eta)]^{1/4}$, and then the symplectic singular value can be obtained as $\sigma_{1,2}^{(1)} = 0$. In this case, there exist two dark modes $\{\hat{X}_{B_1}, \hat{P}_{B_1}\}$ and $\{\hat{X}_{B_2}, \hat{P}_{B_2}\}$.

(ii) In the case $g_{22} = g_{23} \neq g$, the effective coupling strengths in Eqs. (S64f) become

$$G_{12} = G_{22} = \frac{1}{\sqrt{3}}G_{11} = \frac{1}{\sqrt{3}}G_{21} = \frac{\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g_{23}-g), \tag{S69a}$$

$$G_{13} = \frac{2\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(g_{23}-g), \tag{S69b}$$

$$G_{23} = \frac{2\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(2g+g_{23}). \tag{S69c}$$

Thus we can obtain the symplectic singular values

$$\sigma_1^{(1)} = \frac{2\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g_{23}-g), \qquad \sigma_2^{(1)} = 0, \tag{S70}$$

which means that there exists one dark mode

$$\hat{X}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}(\hat{x}_{b_2}-\hat{x}_{b_3}), \qquad \hat{P}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}(\hat{p}_{b_2}-\hat{p}_{b_3}). \tag{S71}$$

(iii) In the case $g_{22}=g$ and $g_{23}\neq g$, the effective coupling strengths G_{kj} can be expressed as

$$G_{12} = G_{22} = -\frac{1}{\sqrt{3}}G_{11} = -\frac{1}{\sqrt{3}}G_{21} = \frac{\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g-g_{23}), \tag{S72a}$$

$$G_{13} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(g_{23}-g), \tag{S72b}$$

$$G_{23} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(5g+g_{23}). \tag{S72c}$$

Substituting Eqs. (S72) into Eq. (S68), we can obtain the symplectic singular values

$$\sigma_1^{(1)} = \frac{2\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g_{23}-g), \qquad \sigma_2^{(1)} = 0, \tag{S73}$$

which indicates that the following normal mode becomes a dark mode

$$\hat{X}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}(\hat{x}_{b_1}-\hat{x}_{b_2}), \qquad \hat{P}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}(\hat{p}_{b_1}-\hat{p}_{b_2}). \tag{S74}$$

(iv) In the case $g_{23}=g$ and $g_{22}\neq g$, based on Eq. (S61), we can obtain the effective coupling strengths

$$G_{21} = G_{11} = 0, \tag{S75a}$$

$$G_{12} = G_{22} = \frac{2\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g_{22}-g), \tag{S75b}$$

$$G_{13} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(g_{22}-g), \tag{S75c}$$

$$G_{23} = \frac{\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b+4\eta)^{1/4}}(5g+g_{22}). \tag{S75d}$$

In this case, the symplectic singular values are obtained as

$$\sigma_1^{(1)} = \frac{2\sqrt{2}\omega_b^{1/4}}{\sqrt{3}(\omega_b-2\eta)^{1/4}}(g_{22}-g), \qquad \sigma_2^{(1)} = 0. \tag{S76}$$

Therefore, there exists one dark mode

$$\hat{X}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{(\omega_b-2\eta)^{1/4}}{\omega_b^{1/4}}(\hat{x}_{b_3}-\hat{x}_{b_1}), \qquad \hat{P}_{\tilde{B}_2} = \frac{1}{\sqrt{2}}\frac{\omega_b^{1/4}}{(\omega_b-2\eta)^{1/4}}(\hat{p}_{b_3}-\hat{p}_{b_1}). \tag{S77}$$

By analyzing the symplectic singular values $\sigma_{1,2}^{(1)}$ for various values of g_{22} and g_{23} within the degenerate-mode subspace, we can obtain the conditions for the existence of the dark modes.

Supplementary Note III: Derivation of the linearized Hamiltonian of the optomechanical networks

In this section, we present the detailed derivation of the linearized Hamiltonian of the optomechanical networks. Concretely, we consider an optomechanical network consisting of M optical modes and N mechanical modes, where these optical modes are coupled to each other through the photon-hopping interactions, while these mechanical modes are coupled to each other through the phonon-hopping interactions. The optical modes are coupled to the mechanical modes via the radiation-pressure interaction. In addition, each optical mode is driven by a monochromatic field with the driving strength Λ_k and the same driving frequency ω_L. In a rotating frame defined by the transformation operator $\hat{V}=\exp(-i\omega_L t\sum_{k=1}^{N}\hat{c}_k^{\dagger}\hat{c}_k)$, the Hamiltonian of the optomechanical network can be expressed as

$$\begin{aligned}\hat{H}_R = & \sum_{k=1}^{M}\tilde{\delta}_{c_k}\hat{c}_k^{\dagger}\hat{c}_k + \sum_{k=1,k<k'}^{M}\xi_{kk'}(\hat{c}_k^{\dagger}\hat{c}_{k'}+\hat{c}_{k'}^{\dagger}\hat{c}_k) + \sum_{j=1}^{N}\omega_{d_j}\hat{d}_j^{\dagger}\hat{d}_j + \sum_{j=1,j<j'}^{N}\eta_{jj'}(\hat{d}_j^{\dagger}+\hat{d}_j)(\hat{d}_{j'}^{\dagger}+\hat{d}_{j'}) \\ & + \sum_{k=1}^{M}\sum_{j=1}^{N}\tilde{g}_{kj}\hat{c}_k^{\dagger}\hat{c}_k(\hat{d}_j^{\dagger}+\hat{d}_j) + \sum_{k=1}^{M}\Lambda_k(\hat{c}_k^{\dagger}+\hat{c}_k),\end{aligned} \tag{S78}$$

where $\hat{c}_k$ ($\hat{c}_k^\dagger$) and $\hat{d}_j$ ($\hat{d}_j^\dagger$) are, respectively, the annihilation (creation) operators of the kth driving optical mode and the jth mechanical mode in the optomechanical network, with the corresponding driving detuning $\tilde{\delta}_{c_k}$ and resonance frequency ω_{d_j}, and $\xi_{kk'}$ ($\eta_{jj'}$) is the photon-hopping (phonon-hopping) coupling strength between the kth (jth) and k'th (j'th) optical (mechanical) modes. The variable $\tilde{g}_{kj}$ is the single-photon optomechanical coupling strength between the kth optical mode and the jth mechanical mode.

Due to the inevitable interactions between the system and the environments, we assume that each optical mode is coupled to a vacuum bath, while each mechanical mode is coupled to a heat bath. Within the Markovian-dissipation framework, we can obtain the Langevin equations of the system operators as

$$\dot{\hat{c}}_1 = -(i\tilde{\delta}_{c_1} + \kappa_1)\hat{c}_1 - i\sum_{k>1}^{M}\xi_{1k}\hat{c}_k - i\sum_{j=1}^{N}\tilde{g}_{1j}\hat{c}_1(\hat{d}_j^\dagger + \hat{d}_j) - i\Lambda_1 + \sqrt{2\kappa_1}\hat{c}_{1,\mathrm{in}}, \tag{S79a}$$

$$\dot{\hat{c}}_2 = -(i\tilde{\delta}_{c_2} + \kappa_2)\hat{c}_2 - i\xi_{12}\hat{c}_1 - i\sum_{k>2}^{M}\xi_{2k}\hat{c}_k - i\sum_{j=1}^{N}\tilde{g}_{2j}\hat{c}_2(\hat{d}_j^\dagger + \hat{d}_j) - i\Lambda_2 + \sqrt{2\kappa_2}\hat{c}_{2,\mathrm{in}}, \tag{S79b}$$

$$\vdots$$

$$\dot{\hat{c}}_M = -(i\tilde{\delta}_{c_M} + \kappa_M)\hat{c}_M - i\sum_{k=1}^{M-1}\xi_{kM}\hat{c}_k - i\sum_{j=1}^{N}\tilde{g}_{Mj}\hat{c}_M(\hat{d}_j^\dagger + \hat{d}_j) - i\Lambda_M + \sqrt{2\kappa_M}\hat{c}_{M,\mathrm{in}}, \tag{S79c}$$

$$\dot{\hat{d}}_1 = -(i\omega_{d_1} + \gamma_1)\hat{d}_1 - i\sum_{j>1}^{N}\eta_{1j}(\hat{d}_j^\dagger + \hat{d}_j) - i\sum_{k=1}^{M}\tilde{g}_{k1}\hat{c}_k^\dagger\hat{c}_k + \sqrt{2\gamma_1}\hat{d}_{1,\mathrm{in}}, \tag{S79d}$$

$$\dot{\hat{d}}_2 = -(i\omega_{d_2} + \gamma_2)\hat{d}_2 - i\eta_{12}(\hat{d}_1^\dagger + \hat{d}_1) - i\sum_{j>2}^{N}\eta_{2j}(\hat{d}_j^\dagger + \hat{d}_j) - i\sum_{k=1}^{M}\tilde{g}_{k2}\hat{c}_k^\dagger\hat{c}_k + \sqrt{2\gamma_2}\hat{d}_{2,\mathrm{in}}, \tag{S79e}$$

$$\vdots$$

$$\dot{\hat{d}}_N = -(i\omega_{d_N} + \gamma_N)\hat{d}_N - i\sum_{j=1}^{N-1}\eta_{jN}(\hat{d}_j^\dagger + \hat{d}_j) - i\sum_{k=1}^{M}\tilde{g}_{kN}\hat{c}_k^\dagger\hat{c}_k + \sqrt{2\gamma_N}\hat{d}_{N,\mathrm{in}}, \tag{S79f}$$

where κ_k and γ_j are, respectively, the decay rates of the kth optical mode c_k and the jth mechanical mode d_j. In addition, the operators $\hat{c}_{k,\mathrm{in}}$ ($\hat{c}_{k,\mathrm{in}}^\dagger$) and $\hat{d}_{j,\mathrm{in}}$ ($\hat{d}_{j,\mathrm{in}}^\dagger$) are the noise operators associated with the kth optical mode and jth mechanical mode, respectively. These noise operators have zero average values and satisfy the correlation functions [S4]

$$\langle\hat{c}_{k,\mathrm{in}}^\dagger(t)\hat{c}_{k',\mathrm{in}}(t')\rangle = 0, \tag{S80a}$$
$$\langle\hat{c}_{k,\mathrm{in}}(t)\hat{c}_{k',\mathrm{in}}^\dagger(t')\rangle = \delta_{k,k'}\delta(t-t'), \tag{S80b}$$
$$\langle\hat{d}_{j,\mathrm{in}}(t)\hat{d}_{j',\mathrm{in}}^\dagger(t')\rangle = (\bar{n}_j + 1)\delta_{j,j'}\delta(t-t'), \tag{S80c}$$
$$\langle\hat{d}_{j,\mathrm{in}}^\dagger(t)\hat{d}_{j',\mathrm{in}}(t')\rangle = \bar{n}_j\delta_{j,j'}\delta(t-t'), \tag{S80d}$$

where $\bar{n}_j$ is the mean thermal phonon number associated with the jth mechanical mode.

In the strong-driving regime, we can linearize the optomechanical Hamiltonians by expanding the operators $\hat{c}_k$ and $\hat{d}_j$ as a summation of their steady-state mean values and quantum fluctuation operators,

$$\hat{c}_k = \langle\hat{c}_k\rangle_{\mathrm{ss}} + \hat{a}_k, \qquad \hat{c}_k^\dagger = \langle\hat{c}_k\rangle_{\mathrm{ss}}^* + \hat{a}_k^\dagger, \tag{S81a}$$
$$\hat{d}_j = \langle\hat{d}_j\rangle_{\mathrm{ss}} + \hat{b}_j, \qquad \hat{d}_j^\dagger = \langle\hat{d}_j\rangle_{\mathrm{ss}}^* + \hat{b}_j^\dagger. \tag{S81b}$$

Note that for keeping the notation consistent, we consider the operators $\hat{a}_k$ and $\hat{b}_j$ to represent the fluctuation operators of the optical mode c_k and the mechanical mode d_j, respectively. Further, for working in the quadrature-operator representation, we introduce the quadrature operators corresponding to the linearized fluctuation operators of the system. The expressions of these quadrature operators are given in Eqs. (S8). Accordingly, we introduce the quadrature noise operators associated with these linearized fluctuation operators as

$$\hat{x}_{a_k}^{\mathrm{in}} = \frac{1}{\sqrt{2}}(\hat{c}_{k,\mathrm{in}}^\dagger + \hat{c}_{k,\mathrm{in}}), \qquad \hat{p}_{a_k}^{\mathrm{in}} = i\frac{1}{\sqrt{2}}(\hat{c}_{k,\mathrm{in}}^\dagger - \hat{c}_{k,\mathrm{in}}), \tag{S82a}$$
$$\hat{x}_{b_j}^{\mathrm{in}} = \frac{1}{\sqrt{2}}(\hat{d}_{j,\mathrm{in}}^\dagger + \hat{d}_{j,\mathrm{in}}), \qquad \hat{p}_{b_j}^{\mathrm{in}} = i\frac{1}{\sqrt{2}}(\hat{d}_{j,\mathrm{in}}^\dagger - \hat{d}_{j,\mathrm{in}}). \tag{S82b}$$

Then the linearized Langevin equations of the system can be expressed in the quadrature-operator representation as

$$\dot{\hat{x}}_{a_1} = -\kappa_1 \hat{x}_{a_1} + \delta_{a_1} \delta \hat{p}_{a_1} + \sum_{k>1}^{M} \xi_{1k} \hat{p}_{a_k} + \sqrt{2\kappa_1} \hat{x}_{a_1}^{\mathrm{in}}, \tag{S83a}$$

$$\dot{\hat{p}}_{a_1} = -\delta_{a_1} \hat{x}_{a_1} - \kappa_1 \hat{p}_{a_1} - \sum_{k>1}^{M} \xi_{1k} \hat{x}_{a_k} - 2 \sum_{j=1}^{N} g_{1j} \hat{x}_{b_j} + \sqrt{2\kappa_1} \hat{p}_{a_1}^{\mathrm{in}}, \tag{S83b}$$

$$\vdots$$

$$\dot{\hat{x}}_{a_M} = \delta_{a_M} \hat{p}_{a_M} - \kappa_M \hat{x}_{a_M} + \sum_{k=1}^{M-1} \xi_{kM} \hat{p}_{a_k} + \sqrt{2\kappa_M} \hat{x}_{a_M}^{\mathrm{in}}, \tag{S83c}$$

$$\dot{\hat{p}}_{a_M} = -\delta_{a_M} \hat{x}_{a_M} - \kappa_M \hat{p}_{a_M} - \sum_{k=1}^{M-1} \xi_{kM} \hat{x}_{a_k} - 2 \sum_{j=1}^{N} g_{Mj} \hat{x}_{b_j} + \sqrt{2\kappa_M} \hat{p}_{a_M}^{\mathrm{in}}, \tag{S83d}$$

$$\dot{\hat{x}}_{b_1} = \omega_{b_1} \hat{p}_{b_1} - \gamma_1 \hat{x}_{b_1} + \sqrt{2\gamma_1} \hat{x}_{b_1}^{\mathrm{in}}, \tag{S83e}$$

$$\dot{\hat{p}}_{b_1} = -\omega_{b_1} \hat{x}_{b_1} - \gamma_1 \hat{p}_{b_1} - 2 \sum_{j>1}^{N} \eta_{1j} \hat{x}_{b_j} - 2 \sum_{k=1}^{M} g_{k1} \hat{x}_{a_k} + \sqrt{2\gamma_1} \hat{p}_{b_1}^{\mathrm{in}}, \tag{S83f}$$

$$\vdots$$

$$\dot{\hat{x}}_{b_N} = \omega_{b_N} \hat{p}_{b_N} - \gamma_N \hat{x}_{b_N} + \sqrt{2\gamma_N} \hat{x}_{b_N}^{\mathrm{in}}, \tag{S83g}$$

$$\dot{\hat{p}}_{b_N} = -\omega_{b_N} \hat{x}_{b_N} - \gamma_N \hat{p}_{b_N} - 2 \sum_{j=1}^{N-1} \eta_{jN} \hat{x}_{b_j} - 2 \sum_{k=1}^{M} g_{kN} \hat{x}_{a_k} + \sqrt{2\gamma_N} \hat{p}_{b_N}^{\mathrm{in}}. \tag{S83h}$$

Here, we introduce the effective detunings $\delta_{a_k} = \tilde{\delta}_{c_k} + \sum_{j=1}^{N} \tilde{g}_{kj}(\langle \hat{d}_j \rangle_{\mathrm{ss}}^{*} + \langle \hat{d}_j \rangle_{\mathrm{ss}})$, the resonance frequencies $\omega_{b_j} = \omega_{d_j}$, and the coupling strengths $g_{kj} = \tilde{g}_{kj} \langle \hat{c}_k \rangle_{\mathrm{ss}}$, where the corresponding $\langle \hat{c}_k \rangle_{\mathrm{ss}}$ and $\langle \hat{d}_j \rangle_{\mathrm{ss}}$ are the steady-state mean values of the operators $\hat{c}_k$ and $\hat{d}_j$, which are governed by

$$\langle \hat{c}_1 \rangle_{\mathrm{ss}} = \frac{-i \sum_{k>1}^{M} \xi_{1k} \langle \hat{c}_k \rangle_{\mathrm{ss}} - i\Lambda_1}{(\kappa_1 + i\delta_{c_1})}, \tag{S84a}$$

$$\langle \hat{c}_2 \rangle_{\mathrm{ss}} = \frac{-i\xi_{12} \langle \hat{c}_1 \rangle_{\mathrm{ss}} - i \sum_{k>2}^{M} \xi_{2k} \langle \hat{c}_k \rangle_{\mathrm{ss}} - i\Lambda_2}{(\kappa_2 + i\delta_{c_2})}, \tag{S84b}$$

$$\vdots$$

$$\langle \hat{c}_M \rangle_{\mathrm{ss}} = \frac{-i \sum_{k=1}^{M-1} \xi_{kM} \langle \hat{c}_k \rangle_{\mathrm{ss}} - i\Lambda_M}{(\kappa_N + i\delta_{c_M})}, \tag{S84c}$$

$$\langle \hat{d}_1 \rangle_{\mathrm{ss}} = \frac{-i \sum_{j>1}^{N} \eta_{1j} (\langle \hat{d}_j \rangle_{\mathrm{ss}}^{*} + \langle \hat{d}_j \rangle_{\mathrm{ss}}) - i \sum_{k=1}^{M} \tilde{g}_{k1} \langle \hat{c}_k \rangle_{\mathrm{ss}} \langle \hat{c}_k \rangle_{\mathrm{ss}}^{*}}{(\gamma_1 + i\omega_{d_1})}, \tag{S84d}$$

$$\langle \hat{d}_2 \rangle_{\mathrm{ss}} = \frac{-i\eta_{12} (\langle \hat{d}_1 \rangle_{\mathrm{ss}}^{*} + \langle \hat{d}_1 \rangle_{\mathrm{ss}}) - i \sum_{j>2}^{N} \eta_{2j} (\langle \hat{d}_j \rangle_{\mathrm{ss}}^{*} + \langle \hat{d}_j \rangle_{\mathrm{ss}}) - i \sum_{k=1}^{M} \tilde{g}_{k2} \langle \hat{c}_k \rangle_{\mathrm{ss}} \langle \hat{c}_k \rangle_{\mathrm{ss}}^{*}}{(\gamma_2 + i\omega_{d_2})}, \tag{S84e}$$

$$\vdots$$

$$\langle \hat{d}_N \rangle_{\mathrm{ss}} = \frac{-i \sum_{j=1}^{N-1} \eta_{jN} (\langle \hat{d}_j \rangle_{\mathrm{ss}}^{*} + \langle \hat{d}_j \rangle_{\mathrm{ss}}) - i \sum_{k=1}^{M} \tilde{g}_{kN} \langle \hat{c}_k \rangle_{\mathrm{ss}} \langle \hat{c}_k \rangle_{\mathrm{ss}}^{*}}{(\gamma_M + i\omega_{d_N})}. \tag{S84f}$$

By ignoring the damping and noise terms in Eq. (S83h), we can obtain the linearized optomechanical Hamiltonian in the quadrature-operator representation,

$$\hat{H}_{\mathrm{lin}} = \sum_{k=1}^{M} \frac{\delta_{a_k}}{2} (\hat{x}_{a_k}^2 + \hat{p}_{a_k}^2) + \sum_{k=1,k<k'}^{M} \xi_{kk'} (\hat{x}_{a_k} \hat{x}_{a_{k'}} + \hat{p}_{a_k} \hat{p}_{a_{k'}}) + \sum_{j=1}^{N} \frac{\omega_{b_j}}{2} (\hat{x}_{b_j}^2 + \hat{p}_{b_j}^2) + 2 \sum_{j=1,j<j'}^{N} \eta_{jj'} \hat{x}_{b_j} \hat{x}_{b_{j'}} + 2 \sum_{k=1}^{M} \sum_{j=1}^{N} g_{kj} \hat{x}_{a_k} \hat{x}_{b_j}. \tag{S85}$$

The Hamiltonian (S85) is the starting point for our study of the dark-mode effect for the quadratic continuous-variable quantum networks in the main text.

Supplementary Note IV: Collective cooling of the mechanical modes and generation of optomechanical entanglement in the optomechanical networks

In optomechanical networks, the presence of the dark modes will strongly suppress the simultaneous ground-state cooling of the mechanical modes involved in the dark modes, thereby the generation of the optomechanical entanglement between the optical modes and the mechanical modes will be destroyed by the residual thermal phonon number. Therefore, the existence of the dark modes can be examined by analyzing the collective cooling of the mechanical modes and the generation of optomechanical entanglement in the optomechanical networks. Below, we will examine the collective cooling of these collective mechanical modes by calculating the mean phonon numbers. We also characterize the optomechanical entanglement by introducing the logarithmic negativity.

C. Mean phonon number of the collective mechanical modes

For the linearized optomechanical network, the mean phonon number of the collective mechanical modes can be calculated by using the covariance matrix of the system. For a Gaussian initial state, the evolution of the linearized optomechanical network can be described by the covariance matrix. To derive the covariance matrix, we first introduce the quadrature fluctuation operator vector

$$\hat{\mathbf{r}}(t) = \left(\hat{\mathbf{r}}_a^{\mathrm{T}}(t), \hat{\mathbf{r}}_b^{\mathrm{T}}(t)\right)^{\mathrm{T}} \tag{S86}$$

with

$$\hat{\mathbf{r}}_a(t) = \left(\hat{x}_{a_1}(t), \hat{x}_{a_2}(t), \cdots, \hat{x}_{a_M}(t), \hat{p}_{a_1}(t), \hat{p}_{a_2}(t), \cdots \hat{p}_{a_M}(t)\right)^{\mathrm{T}}, \tag{S87a}$$

$$\hat{\mathbf{r}}_b(t) = \left(\hat{x}_{b_1}(t), \hat{x}_{b_2}(t), \cdots, \hat{x}_{b_N}(t), \hat{p}_{b_1}(t), \hat{p}_{b_2}(t), \cdots, \hat{p}_{b_N}(t)\right)^{\mathrm{T}}, \tag{S87b}$$

and the quadrature noise operator vector

$$\hat{\mathbf{N}}(t) = \left(\hat{\mathbf{r}}_{a,\mathrm{in}}^{\mathrm{T}}(t), \hat{\mathbf{r}}_{b,\mathrm{in}}^{\mathrm{T}}(t)\right)^{\mathrm{T}}, \tag{S88}$$

with

$$\hat{\mathbf{r}}_{a,\mathrm{in}}(t) = \sqrt{2}\left(\sqrt{\kappa_1}\hat{x}_{a_1}^{\mathrm{in}}(t), \sqrt{\kappa_2}\hat{x}_{a_2}^{\mathrm{in}}(t), \cdots, \sqrt{\kappa_M}\hat{x}_{a_M}^{\mathrm{in}}(t), \sqrt{\kappa_1}\hat{p}_{a_1}^{\mathrm{in}}(t), \sqrt{\kappa_2}\hat{p}_{a_2}^{\mathrm{in}}(t), \cdots, \sqrt{\kappa_M}\hat{p}_{a_M}^{\mathrm{in}}(t)\right)^{\mathrm{T}}, \tag{S89a}$$

$$\hat{\mathbf{r}}_{b,\mathrm{in}}(t) = \sqrt{2}\left(\sqrt{\gamma_1}\hat{x}_{b_1}^{\mathrm{in}}(t), \sqrt{\gamma_2}\hat{x}_{b_2}^{\mathrm{in}}(t), \cdots, \sqrt{\gamma_N}\hat{x}_{b_N}^{\mathrm{in}}(t), \sqrt{\gamma_1}\hat{p}_{b_1}^{\mathrm{in}}(t), \sqrt{\gamma_2}\hat{p}_{b_2}^{\mathrm{in}}(t), \cdots, \sqrt{\gamma_N}\hat{p}_{b_N}^{\mathrm{in}}(t)\right)^{\mathrm{T}}. \tag{S89b}$$

Then the linearized Langevin equations (S83h) can be expressed as a compact form

$$\dot{\hat{\mathbf{r}}}(t) = \mathbf{A}\hat{\mathbf{r}}(t) + \hat{\mathbf{N}}(t), \tag{S90}$$

where the drift matrix is introduced as

$$\mathbf{A} = \begin{pmatrix}
-\kappa_1 & 0 & \cdots & 0 & \delta_{a_1} & \xi_{12} & \cdots & \xi_{1M} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & -\kappa_2 & \cdots & 0 & \xi_{12} & \delta_{a_2} & \cdots & \xi_{2M} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -\kappa_M & \xi_{1M} & \xi_{2M} & \cdots & \delta_{a_M} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
-\delta_{a_1} & -\xi_{12} & \cdots & -\xi_{1M} & -\kappa_1 & 0 & \cdots & 0 & -2g_{11} & -2g_{12} & -2g_{13} & \cdots & -2g_{1N} & 0 & 0 & 0 & \cdots & 0 \\
-\xi_{12} & -\delta_{a_2} & \cdots & -\xi_{2M} & 0 & -\kappa_2 & \cdots & 0 & -2g_{21} & -2g_{22} & -2g_{23} & \cdots & -2g_{2N} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
-\xi_{1M} & -\xi_{2M} & \cdots & -\delta_{a_M} & 0 & 0 & \cdots & -\kappa_M & -2g_{M1} & -2g_{M2} & -2g_{M3} & \cdots & -2g_{MN} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -\gamma_1 & 0 & 0 & \cdots & 0 & \omega_{b_1} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -\gamma_2 & 0 & \cdots & 0 & 0 & \omega_{b_2} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\gamma_3 & \cdots & 0 & 0 & 0 & \omega_{b_3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\gamma_N & 0 & 0 & 0 & \cdots & \omega_{b_N} \\
-2g_{11} & -2g_{21} & \cdots & -2g_{M1} & 0 & 0 & \cdots & 0 & -\omega_{b_1} & -2\eta_{1,2} & -2\eta_{13} & \cdots & -2\eta_{1N} & -\gamma_1 & 0 & 0 & \cdots & 0 \\
-2g_{12} & -2g_{22} & \cdots & -2g_{M2} & 0 & 0 & \cdots & 0 & -2\eta_{12} & -\omega_{b_2} & -2\eta_{23} & \cdots & -2\eta_{2N} & 0 & -\gamma_2 & 0 & \cdots & 0 \\
-2g_{13} & -2g_{23} & \cdots & -2g_{M3} & 0 & 0 & \cdots & 0 & -2\eta_{13} & -2\eta_{23} & -\omega_{b_3} & \cdots & -2\eta_{3N} & 0 & 0 & -\gamma_3 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-2g_{1N} & -2g_{2N} & \cdots & -2g_{MN} & 0 & 0 & \cdots & 0 & -2\eta_{1N} & -2\eta_{2N} & -2\eta_{3N} & \cdots & -\omega_{b_N} & 0 & 0 & 0 & \cdots & -\gamma_N
\end{pmatrix}. \tag{S91}$$

The stability of this linearized optomechanical network can be analyzed using the Routh-Hurwitz criterion [S5]. Note that all the numerical results presented in this work are within the stable regime.

To study the collective cooling of the mechanical modes, we introduce the steady-state covariance matrix $\mathbf{V}$ of the optomechanical network, which is defined by the matrix elements

$$\mathbf{V}_{i,j} = \frac{1}{2}\left(\langle\hat{\mathbf{r}}_i(\infty)\hat{\mathbf{r}}_j(\infty)\rangle + \langle\hat{\mathbf{r}}_j(\infty)\hat{\mathbf{r}}_i(\infty)\rangle\right) \tag{S92}$$

for i, j = 1-2$(M+N)$. The covariance matrix $\mathbf{V}$ is governed by the Lyapunov equation

$$\mathbf{AV} + \mathbf{VA}^{\mathrm{T}} = -\mathbf{Q}, \tag{S93}$$

where we introduce the diffusion matrix

$$\mathbf{Q} \;=\; \mathrm{diag}[\kappa_1, \kappa_2,\cdots, \kappa_M, \kappa_1, \kappa_2,\cdots, \kappa_M, (2\bar{n}_1+1)\gamma_1, (2\bar{n}_2+1)\gamma_2,\cdots, (2\bar{n}_N+1)\gamma_N, (2\bar{n}_1+1)\gamma_1, (2\bar{n}_2+1)\gamma_2,\cdots, (2\bar{n}_N+1)\gamma_N]. \tag{S94}$$

By solving the Lyapunov equation, we can obtain the expression of the covariance matrix $\mathbf{V}$. Since the mean occupations of the collective mechanical modes are directly governed by the average value of the second-order moments for these mechanical modes, we can express the covariance matrix $\mathbf{V}$ in the block form as

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_a & \mathbf{V}_{ab} \\ \mathbf{V}_{ab}^{\mathrm{T}} & \mathbf{V}_b \end{pmatrix}, \tag{S95}$$

where the submatrices $\mathbf{V}_a$ and $\mathbf{V}_b$ are, respectively, the correlation matrices of the type-a and -b subnetworks, while the submatrix $\mathbf{V}_{ab}$ is the cross correlation matrix between the two subnetworks. Concretely, the elements of these submatrices take the form as $(\mathbf{V}_a)_{k,k'} = \mathbf{V}_{k,k'}$ for k, k' = 1-2M, $(\mathbf{V}_b)_{j,j'} = \mathbf{V}_{2M+j,2M+j'}$ for j, j' = 1-2N, and $(\mathbf{V}_{ab})_{k,j} = \mathbf{V}_{k,2M+j}$ for k = 1-2M and j = 1-2N. Based on Eqs. (S11) and (S87b), the average value of the second-order moments for these mechanical modes can be expressed in the matrix form as

$$\begin{aligned}
\langle\hat{\mathbf{r}}_b\hat{\mathbf{r}}_b^{\dagger}\rangle \;&=\; \begin{pmatrix}
\langle\hat{b}_1\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_1\hat{b}_2^{\dagger}\rangle & \cdots & \langle\hat{b}_1\hat{b}_N^{\dagger}\rangle & \langle\hat{b}_1\hat{b}_1\rangle & \langle\hat{b}_1\hat{b}_2\rangle & \cdots & \langle\hat{b}_1\hat{b}_N\rangle \\
\langle\hat{b}_2\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_2\hat{b}_2^{\dagger}\rangle & \cdots & \langle\hat{b}_2\hat{b}_N^{\dagger}\rangle & \langle\hat{b}_2\hat{b}_1\rangle & \langle\hat{b}_2\hat{b}_2\rangle & \cdots & \langle\hat{b}_2\hat{b}_N\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle\hat{b}_N\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_N\hat{b}_2^{\dagger}\rangle & \cdots & \langle\hat{b}_N\hat{b}_N^{\dagger}\rangle & \langle\hat{b}_N\hat{b}_1\rangle & \langle\hat{b}_N\hat{b}_2\rangle & \cdots & \langle\hat{b}_N\hat{b}_N\rangle \\
\langle\hat{b}_1^{\dagger}\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_1^{\dagger}\hat{b}_2^{\dagger}\rangle & \cdots & \langle\hat{b}_1^{\dagger}\hat{b}_N^{\dagger}\rangle & \langle\hat{b}_1^{\dagger}\hat{b}_1\rangle & \langle\hat{b}_1^{\dagger}\hat{b}_2\rangle & \cdots & \langle\hat{b}_1^{\dagger}\hat{b}_N\rangle \\
\langle\hat{b}_2^{\dagger}\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_2^{\dagger}\hat{b}_2^{\dagger}\rangle & \cdots & \langle\hat{b}_2^{\dagger}\hat{b}_N^{\dagger}\rangle & \langle\hat{b}_2^{\dagger}\hat{b}_1\rangle & \langle\hat{b}_2^{\dagger}\hat{b}_2\rangle & \cdots & \langle\hat{b}_2^{\dagger}\hat{b}_N\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle b_N^{\dagger}\hat{b}_1^{\dagger}\rangle & \langle\hat{b}_N^{\dagger}\hat{b}_2^{\dagger}\rangle & \cdots & \langle\delta\hat{b}_N^{\dagger}\hat{b}_N^{\dagger}\rangle & \langle\delta\hat{b}_N^{\dagger}\hat{b}_1\rangle & \langle\hat{b}_N^{\dagger}\hat{b}_2\rangle & \cdots & \langle\hat{b}_N^{\dagger}\hat{b}_N\rangle
\end{pmatrix} \\
&=\; \mathbf{L}_b\langle\hat{\tilde{\mathbf{r}}}_b\hat{\tilde{\mathbf{r}}}_b^{\mathrm{T}}\rangle\mathbf{L}_b^{\dagger} \\
&=\; \mathbf{L}_b\mathbf{V}_b\mathbf{L}_b^{\dagger} + \frac{1}{2}i\mathbf{J}_b,
\end{aligned} \tag{S96}$$

where the symplectic matrix $\mathbf{J}_b$ and the unitary matrix $\mathbf{L}_b$ are given in Eqs. (S10) and (S12).

To analyze the collective cooling of the mechanical modes, we introduce the collective-fluctuation-operator vector in analogy to Eq. (S3b) as

$$\hat{\mathbf{r}}_{\tilde{B}} = (\hat{\tilde{B}}_1, \hat{\tilde{B}}_2, \cdots \hat{\tilde{B}}_N, \hat{\tilde{B}}_1^{\dagger}, \hat{\tilde{B}}_2^{\dagger}, \cdots, \hat{\tilde{B}}_N^{\dagger})^{\mathrm{T}} = \boldsymbol{\mathcal{U}}\mathbf{L}_b^{\dagger}\hat{\mathbf{r}}_b, \tag{S97a}$$

$$\hat{\mathbf{r}}_{\tilde{B}}^{\dagger} = (\hat{\tilde{B}}_1^{\dagger}, \hat{\tilde{B}}_2^{\dagger}, \cdots, \hat{\tilde{B}}_N^{\dagger}, \hat{\tilde{B}}_1, \hat{\tilde{B}}_2, \cdots \hat{\tilde{B}}_N) = \hat{\mathbf{r}}_b^{\dagger}\mathbf{L}_b\boldsymbol{\mathcal{U}}^{\dagger}, \tag{S97b}$$

where the matrix $\boldsymbol{\mathcal{U}}$ is introduced as

$$\boldsymbol{\mathcal{U}} = \mathbf{L}_b\mathbf{S}_B^{-1}\mathbf{S}_b^{-1}. \tag{S98}$$

Therefore, we can obtain the average values of these second-order moments for these collective-mechanical-fluctuation operators $\tilde{B}_j$ for j = 1-N as

$$\begin{aligned}
\langle\hat{\mathbf{r}}_{\tilde{B}}\hat{\mathbf{r}}_{\tilde{B}}^{\dagger}\rangle \;&=\; \begin{pmatrix}
\langle\hat{\tilde{B}}_1\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_1\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_1\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_1\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_1\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_1\hat{\tilde{B}}_N\rangle \\
\langle\hat{\tilde{B}}_2\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_2\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_2\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_2\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_2\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_2\hat{\tilde{B}}_N\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle\hat{\tilde{B}}_N\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_N\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_N\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_N\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_N\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_N\hat{\tilde{B}}_N\rangle \\
\langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_1^{\dagger}\hat{\tilde{B}}_N\rangle \\
\langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_2^{\dagger}\hat{\tilde{B}}_N\rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_1^{\dagger}\rangle & \langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_2^{\dagger}\rangle & \cdots & \langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_N^{\dagger}\rangle & \langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_1\rangle & \langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_2\rangle & \cdots & \langle\hat{\tilde{B}}_N^{\dagger}\hat{\tilde{B}}_N\rangle
\end{pmatrix} \\
&=\; \boldsymbol{\mathcal{U}}\mathbf{L}_b^{\dagger}\langle\hat{\mathbf{r}}_b\hat{\mathbf{r}}_b^{\dagger}\rangle\mathbf{L}_b\boldsymbol{\mathcal{U}}^{\dagger} \\
&=\; \boldsymbol{\mathcal{U}}\mathbf{V}_b\boldsymbol{\mathcal{U}}^{\dagger} + \frac{1}{2}\boldsymbol{\mathcal{U}}(\sigma_x\otimes\mathbf{I}_N)\boldsymbol{\mathcal{U}}^{\dagger},
\end{aligned} \tag{S99}$$

with $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Based on Eq. (S99), we can obtain the mean phonon number of the jth (j = 1-N) collective mechanical mode as

$$\langle \hat{\tilde{B}}_j^\dagger \hat{\tilde{B}}_j \rangle = \langle \hat{\mathbf{r}}_{\tilde{B}} \hat{\mathbf{r}}_{\tilde{B}}^\dagger \rangle_{N+j,N+j}. \tag{S100}$$

Equation (S100) gives the expression of the thermal occupation for each collective mechanical mode, which can be used to evaluate the collective cooling performance.

D. Logarithmic negativity between the optical and mechanical modes

To analyze the optomechanical entanglement in the optomechanical network, we adopt the logarithmic negativity to quantify the optomechanical entanglement. Concretely, to characterize the optomechanical entanglement between the kth optical mode a_k and jth mechanical mode b_k, we investigate the logarithmic negativity

$$E_{\mathcal{N}}^{(k,j)} = \max[0, -\ln(2\varsigma_{k,j}^-)], \tag{S101}$$

where we introduce the smallest eigenvalue of the reduced correlation matrix $\mathbf{V}'_{k,j}$ [obtained by extracting the submatrices associated with the kth optical mode a_k and the jth mechanical mode b_j from the full covariance matrix $\mathbf{V}$ in Eq. (S92)] as

$$\varsigma_{k,j}^- = \frac{1}{\sqrt{2}} \left[\Sigma(\mathbf{V}'_{k,j}) - \sqrt{\Sigma(\mathbf{V}'_{k,j})^2 - 4\det(\mathbf{V}'_{k,j})} \right]^{1/2}, \tag{S102}$$

with

$$\Sigma(\mathbf{V}'_{k,j}) = \det(\mathcal{A}_k) + \det(\mathcal{B}_j) - 2\det(C_{k,j}). \tag{S103}$$

In Eq. (S102), the reduced correlation matrix $\mathbf{V}'_{k,j}$ takes the form as

$$\mathbf{V}'_{k,j} = \begin{pmatrix} \mathcal{A}_k & C_{k,j} \\ C_{k,j}^{\mathrm{T}} & \mathcal{B}_j \end{pmatrix}, \tag{S104}$$

where $\mathcal{A}_k$ and $\mathcal{B}_j$ are, respectively, 2×2 block matrices associated with the modes a_k and b_j, while $C_{k,j}$ is related to the correlation between the modes a_k and b_j. Concretely, these submatrices take the form as

$$\mathcal{A}_k = \begin{pmatrix} \mathbf{V}_{k,k} & \mathbf{V}_{k,M+k} \\ \mathbf{V}_{M+k,k} & \mathbf{V}_{M+k,M+k} \end{pmatrix}, \tag{S105a}$$

$$\mathcal{B}_j = \begin{pmatrix} \mathbf{V}_{2M+j,2M+j} & \mathbf{V}_{2M+j,2M+N+j} \\ \mathbf{V}_{2M+N+j,2M+j} & \mathbf{V}_{2M+N+j,2M+N+j} \end{pmatrix}, \tag{S105b}$$

$$C_{k,j} = \begin{pmatrix} \mathbf{V}_{k,2M+j} & \mathbf{V}_{k,2M+N+j} \\ \mathbf{V}_{M+k,2M+j} & \mathbf{V}_{M+k,2M+N+j} \end{pmatrix}. \tag{S105c}$$

We can further quantify the optomechanical entanglement between the kth optical mode a_k and the jth mechanical mode b_j in the optomechanical network by evaluating the logarithmic negativity $E_{\mathcal{N}}^{(k,j)}$ given by Eq. (S101).

[S1] J. Williamson, On the Algebraic Problem Concerning the Normal Forms of Linear Dynamical Systems, American J. Math. **58**, 141 (1936).
[S2] H. Xu, An SVD-like matrix decomposition and its applications, Linear Algebra Appl. **368**, 1 (2003).
[S3] J. Huang, C. Liu, X.-W. Xu, and J.-Q. Liao, Dark-Mode Theorems for Quantum Networks, arXiv:2312.06274.
[S4] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer, Berlin, 2000).
[S5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 2014).