Supplementary materials

<CHSH inequality analysis> Nonlocal and local causalities of lattice vibration analyzed by CHSH inequality

The local causality of lattice vibration is analyzed by CHSH inequality S instead of Bell's inequality in the classical regime as shown in the previous paper [1]. Then non-locality and entanglement conditions of the optical mode lattice vibration are examined.

CHSH inequality S is adopted to examine non-locality condition of the particle pairs located at separated locations x_A and x_B.

Here, we define that $(a,\ a')$ and $(b,\ b')$ are combinations of displacements and velocities of the particle pairs, which correspond to position and momentum of particles, located at x_A and x_B, respectively, as follow:

$$a = A\sin(\bar{k}\cdot\bar{x}_A + \theta - \bar{\omega}\cdot\bar{t}), \qquad b = B\sin(\bar{k}\cdot\bar{x}_B - \theta - \bar{\omega}\cdot\bar{t}),$$

$$a' = \frac{1}{\bar{\omega}}\cdot\frac{\partial a}{\partial\bar{t}} = -A\cos(\bar{k}\cdot\bar{x}_A + \theta - \bar{\omega}\cdot\bar{t}),\ b' = \frac{1}{\bar{\omega}}\cdot\frac{\partial b}{\partial\bar{t}} = -B\cos(\bar{k}\cdot\bar{x}_B - \theta - \bar{\omega}\cdot\bar{t}) \quad (1)$$

in which $\bar{\omega}$, $\bar{k}$, $\bar{t}$ and $\bar{x}$ are ω/ω_0, $k\cdot\bar{a}/2$, $\omega_0\cdot t$ and $x\cdot 2/\bar{a}$, respectively. Here, θ is the phase shift due to the amplitude modification.

In CHSH inequality S, the term $E(a,\ b)$ is the correlation of the particle pairs defined by the expectation values $\alpha(a,\ \bar{\lambda})$, $\beta(b,\ \bar{\lambda})$ based on the density function $\rho(\bar{\lambda})$ of the hidden variable $\bar{\lambda}$ under the local causality as follows:

$$E(a,\ b) = \int \alpha(a,\ \bar{\lambda})\cdot\beta(b,\ \bar{\lambda})\rho(\bar{\lambda})d\bar{\lambda} \qquad (2)$$

Considering the distance $n\bar{a}$ (n: integer) between the locations x_A and x_B, $E(a,\ b)$ in Eq. (2) is redefined under the theoretical analysis as follows:

$$\alpha(a,\ \bar{\lambda}) = a(\bar{x}_A,\bar{t})/(n\bar{a}),\ \ \beta(b,\ \bar{\lambda}) = b(\bar{x}_B,\bar{t})/(n\bar{a}), \frac{1}{2\pi}\int_0^{2\pi}\rho(\bar{\lambda})\,d\bar{\lambda} = 1,\ \because \rho(\bar{\lambda}) = 1$$

$$E(a,\ b) = \int \alpha(a,\ \bar{\lambda})\cdot\beta(b,\ \bar{\lambda})\varrho(\bar{\lambda})d\bar{\lambda} = \frac{1}{2\pi(n\bar{a})^2}\int_0^{2\pi} a(x_A,\ \bar{t})\cdot b(x_B,\bar{t})d\bar{t} \qquad (3)$$

Based on the conditions which are $|E(a,\ b)| \le 1$, $|E(a,\ b')| \le 1$, $|E(a',\ b)| \le 1$ and $|E(a',\ b')| \le 1$, following CHSH inequality S under the local causality never violates $|S| \le 2$.

$$S = E(a,\ b) - E(a,b') + E(a',b) + E(\,a',b')\ \ ,\ |S| \le 2 \qquad (4)$$

By introducing the non-dimensional wave amplitudes $\bar{A} \equiv A/(n\bar{a})$ and $\bar{B} \equiv B/(n\bar{a})$, $E(a,\ b)$ is derived from Eqs. (1) and (3) as follows:

$$E(a,\ b) = \frac{\bar{A}\cdot\bar{B}}{2\pi}\int_0^{2\pi}\sin(\bar{k}\cdot\bar{x}_A + \theta - \bar{\omega}\cdot\bar{t})\cdot\sin(\bar{k}\cdot\bar{x}_B - \theta - \bar{\omega}\cdot\bar{t})\cdot d\bar{t} \qquad (5)$$

Here, without loosing generality, we can settle on x_A being zero and x_B being $n\bar{a}$ (n: integer), i.e., $\bar{x}_B$ being 2n, respectively, therefore, $E(a,\ b)$ in Eq. (5) is obtained as follows:

$$E(a,\ b) = \frac{\bar{A}\cdot\bar{B}}{2\pi}\left\{\cos\left(2n\bar{k} - 2\theta\right) - \frac{\sin 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\cos 2n\bar{k} - \frac{1-\cos 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\sin 2n\bar{k}\right\} \qquad (6)$$

Similarly, $E(a, b')$, $E(a', b)$ and $E(a', b')$ are obtained as follows:

$$E(a, b') = -\frac{\bar{A}\cdot\bar{B}}{2\pi}\left\{-\sin\left(2n\bar{k} - 2\theta\right) + \frac{\sin 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\sin 2n\bar{k} - \frac{1-\cos 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\cos 2n\bar{k}\right\},$$

$$E(a', b) = -\frac{\bar{A}\cdot\bar{B}}{2\pi}\left\{\sin\left(2n\bar{k} - 2\theta\right) + \frac{\sin 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\sin 2n\bar{k} - \frac{1-\cos 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\cos 2n\bar{k}\right\} \qquad (7)$$

$$E(a', b') = \frac{\bar{A}\cdot\bar{B}}{2\pi}\left\{\cos\left(2n\bar{k} - 2\theta\right) + \frac{\sin 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\cos 2n\bar{k} + \frac{1-\cos 4\pi\bar{\omega}}{4\pi\bar{\omega}}\cdot\sin 2n\bar{k}\right\}$$

From Eqs. (4), (6) and (7), CHSH inequality S is obtained as follows:

$$S = \frac{\bar{A}\cdot\bar{B}}{2}\cdot 2\sqrt{2}\cos\left(2n\bar{k} - 2\theta + \frac{\pi}{4}\right) \qquad (8)$$

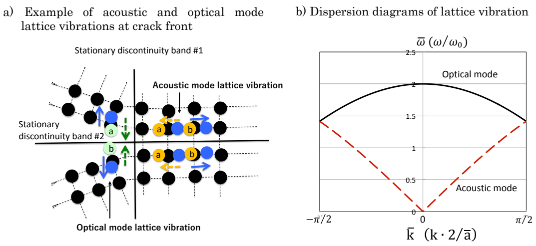

Fig. 1 Example of acoustic and optical mode lattice vibrations at crack front and dispersion diagrams of lattice vibration. (ω, ω_0, k and $\bar{a}$ are frequency, natural frequency, wave number and lattice constant, respectively)

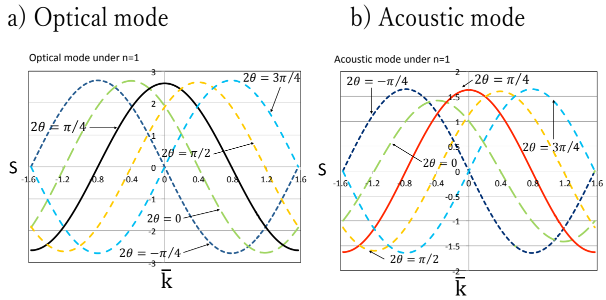

Fig. 2 The relations between CHSH inequality S and wave number $\bar{k}$ for optical and acoustic modes under several θ and n=1.

The relations between S and $\bar{k}$ for optical and acoustic modes of lattice vibration as shown in Fig.1 under several θ and n=1 are shown in Fig. 2. The values of $\bar{A}\cdot\bar{B}/2$ under several θ and n are determined to ensure $|E(a,\ b)| \leq 1$, $|E(a,\ b')| \leq 1$, $|E(a',\ b)| \leq 1$ and $|E(a',\ b')| \leq 1$.

Table 1 Values of $\bar{A}\cdot\bar{B}/2$ of optical and acoustic modes under n=1.

2θ	$-\pi/4$	0	$\pi/4$	$\pi/2$	$3\pi/4$
Optical mode	0.95827	0.95276	0.92545	0.93793	0.95827
Acoustic mode	0.58176	0.50109	0.57605	0.56627	0.58176

Therefore, the values of S in Fig. 2 were calculated using the values $\bar{A}\cdot\bar{B}/2$ in Table 1 determined by the maximum $E(\cdot,\cdot)$ values. Figures 3 and 4 show S, $E(a,\ b)$, $E(a', b)$, $E(a, b')$ and $E(a', b')$ of optical and acoustic mode under $2\theta = \pi/4$ and n=1, respectively.

It is recognized that CHSH inequality S in Fig. 2 must be symmetrical with respect to the wave number such as the symmetric character of the dispersion diagrams of the lattice vibration with respect to the wave number. Therefore, 2θ may be selected as $2\theta = \pi/4$. Figure 5 shows the relations between CHSH inequality S and wave number $\bar{k}$ for optical and acoustic modes under $2\theta = \pi/4$ and n=1.

In the case of $\bar{k} = 0$ and $2\theta = \pi/4$ of optical mode, S in Eq. (8) is as follow:

$$S = \frac{\bar{A} \cdot \bar{B}}{2} 2\sqrt{2} \qquad (9)$$

If $\bar{A} \cdot \bar{B}$ is 2, then S is identical to the maximum violation of polarization-entangled photons $S = 2\sqrt{2}$ [3].

The maximum value of $|E(a,\ b)|$, $|E(a,\ b')|$, $|E(a',\ b)|$ and $|E(a',\ b')|$ in the case of optical mode is $1.081 \cdot \bar{A} \cdot \bar{B}/2$, therefore, according to maximum value being equal or less than 1, $|\bar{A} \cdot \bar{B}| \leq 1.85$ must be satisfied. Hence the cross-correlation of the optical mode vibration between particle pairs located at x_A and x_B is kept under the condition $|\bar{A} \cdot \bar{B}| \leq 1.85$. In the case of acoustic mode, the maximum value is $1.736 \cdot \bar{A} \cdot \bar{B}/2$. As similarly the optical mode, the cross-correlation of the acoustic mode vibration between particle pairs located at x_A and x_B is kept under the condition $\bar{A} \cdot \bar{B} \leq 1.152$.

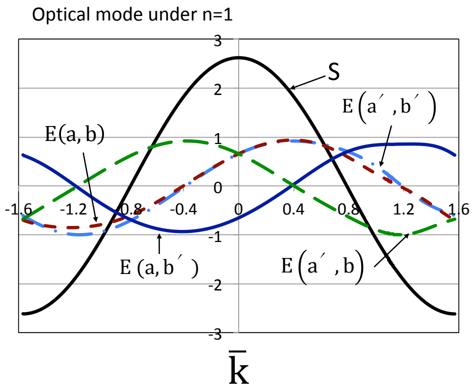

Fig. 3 The correlative diagrams for CHSH inequality S and the terms $E(a,\ b)$, $E(a, b')$, $E(a', b)$ and $E(a', b')$ of optical mode.

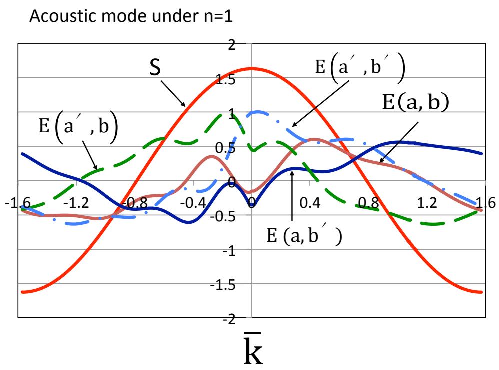

Fig. 4 The correlative diagrams for CHSH inequality S and the terms $E(a,\ b)$, $E(a, b')$, $E(a', b)$ and $E(a', b')$ of acoustic mode.

Figures 6 and 7 show CHSH inequality S of optical and acoustic modes under $2\theta = \pi/4$ and n=1, 2, 3 and 10, respectively. It is clear from Fig. 6 in the case of the optical mode that violation of CHSH inequalities severally happens around several $\bar{k}$ ranges which satisfy the steady wave condition $m\lambda_h = n\bar{a}$ (λ_h: half − wave length and $-n \leq m \leq n$, m: integer, $\bar{k} = \pi/2 \cdot m/n$ $\because \lambda_h = \pi/2 \cdot \bar{a}/\bar{k}$) under which the distances between the particles at the nodes are saved to be constant; CHSH inequalities are violated commonly around $\bar{k} = 0\ (m = 0)$ and $\bar{k} = \pm\pi/2\ (m = \pm n)$ under all n. On the other hand, it is clear from Fig. 7 in the case of acoustic

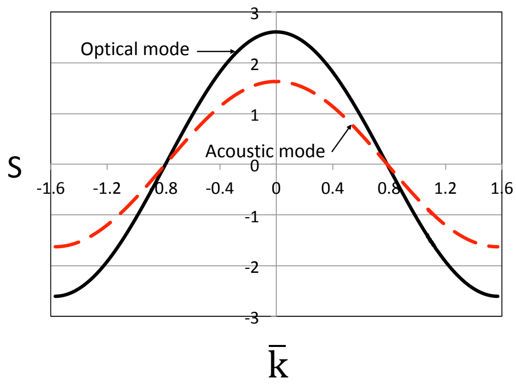

Fig. 5 The relations between CHSH inequality S and wave number $\bar{k}$ for optical and acoustic modes under $2\theta = \pi/4$ and n=1.

mode that CHSH inequalities are never violated independently on n values.

Transmission of phase information as a wave with the phase velocity, that is possible to be infinity in wavelength/wave packet, can induce nonlocal causality. Based on the foregoing results, i.e., the phase velocity V_{ph} and wavelength λ going to infinity under $\bar{k}$ being to 0, it is easily recognized that the entanglement between the neighboring particles is deeply dependent on the phase information due to the group velocity V_{gr} going to 0. However, even if CHSH inequality is violated, in the case of the phase velocity being finite, the entanglement between moving particles with mass becomes less complete due to the delay of synchronization and the finite wavelength under steady wave states that save the distances between the nodes to be constant, therefore, such incomplete entanglement under the finite phase velocity and the finite wavelength is dependent on locations, distance/space and time.

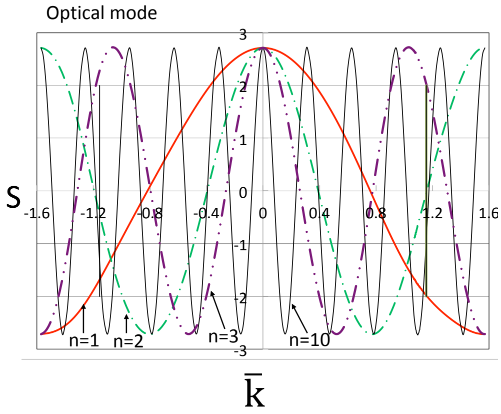

Fig. 6 CHSH inequalities of optical mode under n=1, 2, 3 and 10.

While it is cleared by Fig. 6 that the condition S=$2\sqrt{2}/1.081 = 2.616$ and $\bar{k} = 0$, under which both the phase velocity and the wavelength are infinity, violates CHSH inequality at every point and satisfies nonlocal causality; namely both the phase velocity and the wavelength being infinity is non-locality condition. Therefore, it is concluded that the entanglement of optical mode is established completely under the condition of S=$2\sqrt{2}/1.081 = 2.616$ and $\bar{k} = 0$ and such complete entanglement is not dependent on locations, distance/space and time due to instantaneously simultaneous synchronization.

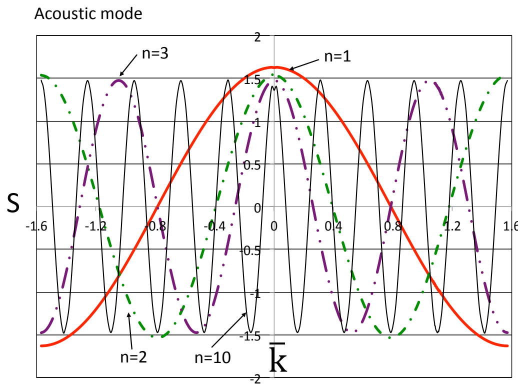

Fig. 7 CHSH inequalities of acoustic mode under n=1, 2, 3 and 10.

1. M. Kobayashi, Nonlocal vibration of crack front particles entangled by phase transformation caused by fracture as shock-wave state, Eur. Phys. J. Plus. 139:962 (2024). https://doi.org/10.11.40/epjp/s13360-024-05721-y