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Frequency combs due to the magnetic square well scheme and dissipative exchange flows in one-dimensional ferromagnetic channels

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In this paper, we investigate the formation of frequency combs as a result of non-local interaction between supersonic spin-polarized currents at two different injection sites along a one-dimensional ferromagnetic channel. This injections scheme, which we termed *magnetic square well* ('well'), allows for dissipative exchange flows (DEFs), where DEFs are large-amplitude boundary value solutions to ferromagnetic channels. In the low-injection limit, DEFs reduce to spin superfluids while in the strong injection limit nonlinearities dominate close to the injection site and a contact soliton is formed. This spin superfluid and nonlinearity allows for this non-local interaction, resulting in interesting and novel magnetic textures, including frequency combs. Here, we numerically investigate a perturbed and unperturbed magnetic square well solutions in the high injection regime, where soliton dynamics are excited when two injection regions are separated by a certain distance, allowing for non-local interaction that we termed *DEF-interaction*. This interaction results in magnetic textures with the properties of frequency combs inside and outside the *well*.

I. INTRODUCTION

A modern holy grail of spintronics is achieving long-distance transport of spin states via spin waves and magnons, where magnons are defined as a quanta of magnetic excitation. The primary driver for this research is that spin transport over long spatial distances has the possibility of resolving one of the most intractable challenges in information processing and storage - joule heating. Each magnon carries an angular momentum $\hbar$ [1–3], and theoretically, this quantized angular momentum has the capacity to encode information for the purposes of processing and storage. However, magnons like their continuous counterpart, spin waves, experience exponential decay because of magnetic damping and where the decay length is material dependent. To maximize this length, materials like permalloy and Yttrium Iron Garnet (YIG) with decay lengths of approximately 1 and 20 μm respectively are used [2, 4–6]. However, other materials and methods are actively being explored to extend this decay length [1, 4–11].

This area has recently seen significant research interest because theoretical works have demonstrated that large-amplitude chiral magnetization states in one-dimensional ferromagnets can exhibit algebraic spatial decay. In general, these chiral magnetization states called *spin superfluids* [5, 6, 12] are solutions to the uniaxial Landau-Lifshitz equation of motion subject to spin injections. Moreover, and to delineate between the linear (weak injection) and nonlinear (strong injection) solutions/regimes, the term *dissipative exchange flow (DEF)* [10, 13] was introduced. In this context, *spin superfluid* are defined as solutions in the weak injection regime (e.g.,

from $\bar{\mu} = 0.01$ to $\bar{\mu} = 0.1$, where the linear limit is $\bar{\mu} \leq 0.4$, while the nonlinear regime is generally taken to be $\bar{\mu} > 0.4$).

It is also important to note that this algebraic decay results from the injection of spin polarized currents (torque) as a boundary value problem, making it necessary for the solution to span the full length of the channel.

As a general matter, the spin injections on the edges of a ferromagnetic channel induces an out of plane tilt in the magnetization, where this tilted vector then precesses due to the local dipole field in thin films (i.e, shape anisotropy). This precession then propagates along the channel because of the exchange force. Moreover, damping is compensated nonlocally by the torque injection (hence the term *dissipative exchange flow*); this results in a chiral state with a precessional frequency in the MHz range for a typical ferromagnet along the span of the channel. It is also important to note that the frequency, which is dependent on the injection magnitude, is linearly proportional for spin superfluids, where in the weak injection regime the magnetization dynamics are mostly in-plane. However, as the injection magnitude increases, nonlinearities dominate and a DEF solution predicts a saturation of the frequency [10, 14, 15], where this frequency becomes inversely proportional to the injection magnitude. This is because large magnitude injection results in the nucleation of a soliton. Moreover, as this magnitude is increased, the nucleated soliton becomes sharper in profile until it ceases to pump DEF into the channel, i.e., no dynamics outside the soliton [2, 16–18].

Another important property of the strong injection regime is that it allows for the nucleation of solitons at the injection site; these solitons termed *contact solitons dissipative exchange flows (CS-DEFs)* [10] or *self-stabilizing exchange-mediated spin transport* [19, 20] have profiles that depend on both the injection site and the environment [14, 21]. More importantly and for our

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purposes, these pumped DEFs allow for non-local interaction between nucleated solitons where the injection signs, distance between the nucleated solitons, and injection magnitudes collectively result in a kind of Zhang-Li like torque that includes a chiral handedness; the results of which were published in [22]. We extend and test this result by adding a second injection site within the spatial limits of the nonlocal interaction along the one-dimensional ferromagnetic channel; we termed this scheme *magnetic square well* ("well") - see Figure 1. In the strong injection regime, this *well* results in interesting and novel magnetic textures, including *frequency combs*.

Frequency combs (also referred to as *spectral comb*) are a spectrum of discrete and regularly spaced spectral lines where the combs are Dirac delta like spikes (in Fourier space) that result from a low frequency signal interacting non-linearly with a higher frequency signal [23–28]. In this regard, some of their properties mimic those of harmonics (more on this later). Moreover, frequency combs are most often discussed in the context of optics, where they are generated using laser sources. However, other methods and experimental models for producing these combs, including magnetic systems, are actively being explored.

In this paper, we report on our observation of frequency combs as a result of super sonic spin-polarized torque injections at two or more sites with spatial separation that allows for *DEF-Interactions* - i.e., the *magnetic square well* (*MSW*) scheme. We numerically investigate the properties of this *well* in the strong injection regime, where soliton dynamics are excited and interact non-locally (via DEFs - hence the term *DEF-Interaction*). This results in novel magnetic textures inside and outside the well. Moreover, our numerical approach was confirmed to be approximately valid in the weak injection regime (i.e., analytical) as discussed in publication [22].

The parameters we varied for our analysis include the magnitude of injection, the relative sign of the injected torques, the distance between the injection sites, channel lengths, spin vs. spin-free conditions at the edges of the channels (i.e., spin sink vs. no spin sink conditions), as well as perturbation at points inside and outside the square well - see Figure 4. We also varied the time for dynamics to ensure stability; we ran the simulation for a time range between 6 and 1,000 nanoseconds, where the stabilization time appears to also depend on the value of the input parameters listed above. The trials resulted in interesting and often unexpected dynamics, including those published in [22].

That said and in the following sections, the focus is on the observed frequency spectra as a consequence of the nonlinear and non-local interaction between nucleated solitons. We also report on the results of applied perturbations and/or the self-interaction that a single soliton exhibits as a consequence of its intrinsic and resonant frequency.

II. CONFINED HYDRODYNAMIC SOLUTIONS

Our numerical investigation of the dynamics of the magnetic square well are given by the Landau-Lifshitz equation

$$\frac{\partial}{\partial t} \mathbf{m} = -\mathbf{m} \times \mathbf{H}_{\text{eff}} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \quad (1)$$

where the first term in the equation is the Larmor torque, and the second is the dissipative term scaled by the Gilbert damping parameter, which is valid for $\alpha \ll 1$. Equation (1) is then normalized by the magnetization vector $\mathbf{m} = (m_x, m_y, m_z)$ in order to maintain its norm in time and space. The equation is also made dimensionless by the following: $x \rightarrow x/\lambda_{\text{ex}}$, where λ_{ex} is the exchange length, and $t \rightarrow \gamma\mu_0 M_s t$, where γ is the gyromagnetic ratio, μ_0 is the vacuum permeability, and M_s is the saturation magnetization. Moreover, the effective field $\mathbf{H}_{\text{eff}} = \Delta \mathbf{m} - m_z \hat{z}$ includes the ferromagnetic exchange and a negative uniaxial anisotropy, which approximates the demagnetization field of thin films (i.e., in-plane ferromagnetic material) [14].

We also derive a hydrodynamical representation of Equation (1) by invoking the canonical transformations of $n = m_z$ and $\mathbf{u} = -\nabla [\arctan(m_y/m_x)]$, where n is the fluid density and $\mathbf{u}$ is the fluid velocity [15, 29]. Moreover, the gradient in the definition of $\mathbf{u}$ follows a Cartesian coordinate system, where if $\mathbf{u} > 0$, it is oriented along $+\hat{x}$, and if $\mathbf{u} < 0$, it is oriented along $-\hat{x}$. This is convenient for a one-dimensional ferromagnetic channel (e.g., $\mathbf{u} = u\hat{x}$) scheme because it allows us to consider the velocity as a signed scalar u . The resulting dispersive hydrodynamic equations are as follows:

$$\begin{aligned} \partial_t n &= \partial_x [(1 - n^2)u] + \alpha(1 - n^2)\partial_t \Phi, \\ \partial_t u &= -\partial_x [(1 + u^2)n] - \left[\frac{\partial_{xx} n}{1 - n^2} + \frac{(\partial_x n)^2}{(1 - n^2)^2} \right] \\ &\quad + \alpha \partial_x \left[\frac{1}{1 - n^2} \partial_x [(1 - n^2)u] \right]. \end{aligned} \quad (2a) \quad (2b)$$

Again, for the magnetic square well, we construct a long ferromagnetic channel subject to spin injections at two sites with injector widths W , as shown in Fig. 1. The injection sites are separated by a *well* distance L . Moreover, we set the origin for the reference frame midway between the injection sites. This construction can then be treated as a boundary value problem subject to the following set of boundary conditions:

$$\begin{aligned} \partial_t n(x = -L/2) &= 0, & \partial_t n(x = L/2) &= 0, \\ u(x = -L/2) &= \bar{u}_L, & u(x = L/2) &= \bar{u}_R, \end{aligned} \quad (3a) \quad (3b)$$

where $\bar{u}_L$ and $\bar{u}_R$ are the left and right injections/fluid velocities [14].

In the weak-injection limit (i.e., $0 < |\bar{u}| \ll \min(1, \alpha L)$), we can derive a linear dissipative exchange flow (l_{DEF}) [14, 30] that corresponds to the spin superfluid solution [4, 31]. In this case, the nonlinear terms

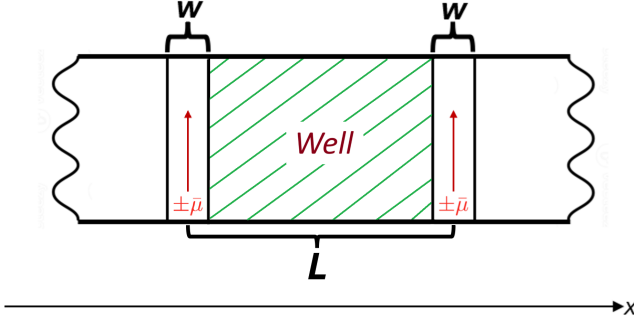


FIG. 1. Schematic illustration of the central region of the one-dimensional ferromagnetic channel (i.e., *magnetic square well* - "well"). The injection regions have identical widths W and are separated by a distance L measured from center-to-center. Moreover, the static magnetization tilts out of the plane of the thin film and in the same direction as the polarized torque.

in Eqs. (2) are negligible, therefore, we assume $\partial_t n = 0$, satisfying Eq. (3a) and allowing us to define $\Omega = \partial_t \phi$ and reducing the dispersive equations to the following set:

$$\alpha \Omega = -\partial_x u, \quad (4a)$$

$$\Omega = -n \quad (4b)$$

Next, we derive the solution for u by simple integration

$$u = -\alpha \Omega x + C, \quad (5)$$

where C is the constant of integration. Moreover, using the boundary conditions Eq. (3b), we can uniquely derive the constant of integration and an expression for the precessional frequency; the results of which are as follows:

$$u(x) = \bar{u}_L \left(\frac{1}{2} - \frac{x}{L} \right) + \bar{u}_R \left(\frac{1}{2} + \frac{x}{L} \right), \quad (6a)$$

$$\Omega = -n = \frac{\bar{u}_L - \bar{u}_R}{\alpha L} \quad (6b)$$

Equation (6a) is general. However, in Ref [22], we explore and report results for cases where the injection magnitudes have the same and opposite signs with respect to each other. With that said, for both injection schemes (same and opposite signed), the analytical solutions have the same form as a linear DEF or spin superfluid.

A. Numerical implementation

Our investigation and confirmation of the above solutions is done by numerical integration of the Landau-Lifshitz equation expressed in a pseudospectral form (PS-LL) [32], where the exchange term is replaced by the spectral kernel so that $\Delta \mathbf{m} \rightarrow \mathcal{F}^{-1} \{ \omega(k) \hat{\mathbf{m}}(k) \}$. Moreover, in this PS-LL representation, $\mathcal{F}^{-1} \{ \cdot \}$ is the inverse Fourier transform; $\omega(k) = 2 - 2 \cos(ka)$ is the dispersion relation of magnons, a is the lattice constant, and $\hat{\mathbf{m}}(k)$ represents the Fourier transform of $\mathbf{m}$. This method, which is

presented here in dimensionless units, has been demonstrated to provide a better description of the hydrodynamic solutions under strong injection regime where solitons are nucleated and stabilized [33]; this is primary focus of our study here.

It is important to note that for the sake of simplicity, the PS-LL model presents the spin injections as a spin-transfer torque terms [34–36], where all the physical parameters of that torque (i.e., current, spin polarization, and spin asymmetry) are collapsed into the single constant $\bar{\mu}$. Moreover, the spin injections causes an out-of-plane tilt in the magnetization, making it proportional to $\bar{n}$ (i.e. the fluid/spin density at the injection site). This proportionality also depends on the specifics of the channel, including geometry (i.e. injection width and channel length).

Our simulations uses the magnetic parameters of permalloy, including a saturation magnetization value of $M_s = 790$ kA/m, exchange length of $\lambda_{\text{ex}} = 5$ nm, and Gilbert damping value of $\alpha = 0.01$ as well as the gyromagnetic ratio of $\gamma = 28$ GHz/T and a vacuum permeability $\mu_0 = 4\pi \times 10^{-7}$ N/A². Moreover, the results we report are for a magnetic square well with a channel of length 1,600 nm, injection width $w = 5.5$ nm, and well distance $L = 120$ nm.

B. Frequency Comb

Again, a *frequency comb* (also referred to as *spectral comb*) is a spectrum made of discrete and regularly spaced spectral lines where the comb is the result of a low-frequency signal interacting non-linearly with a higher-frequency signal [23, 24]. It is most often discussed in the context of optics where a frequency comb can be generated by a laser source, including periodic modulation in amplitude and/or phase of a continuous wave/laser; *Four Wave Mixing* in non-linear media where four waves interact and exchange energies which then results in a new set of frequencies, and *Mode-locked laser* where a train of ultrashort laser pulses are produced by the synchronization of the phases of multiple modes within the laser cavity [25, 26]. These are just some examples of experimental models for producing frequency combs; however, other methods are actively being explored [26–28].

One of the important properties of frequency combs is that in the frequency domain representation a "perfect" frequency comb in effect has the properties of a Dirac-delta like spikes along some frequency or spatial range (where under ideal conditions, these spatial distributions of frequency spikes would have a domain of all space). In other words, in their simplest forms, one can think of frequency combs as a series of Dirac-delta like functions spaced according to:

$$f_n = f_0 + n f_r \quad (7)$$

where n is an integer, f_r is the comb tooth/modulation frequency, and f_0 is the carrier frequency offset (CFO),

where the most common (and in my opinion the best) definition of CFOs is an oscillatory mismatch between the transmitter and receiver frequencies and is less than the f_r (more on this later) [23, 24]. Moreover, these systems are sometimes referred to as *Dirac comb* and are represented by the following function:

$$\Gamma_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \quad (8)$$

The sum in the above is over all integer k values and where T is some given period, t is a real variable, and δ is the Dirac-delta function. This function results in a comb-like distribution where the δ s (i.e., discrete values) are referred to as *comb's tooth*. Moreover, because the Dirac comb function is periodic, it has the following Fourier series representation [26]:

$$\Gamma_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi n \frac{t}{T}} \quad (9)$$

Again, frequency combs are usually experimentally created using optical tools, in particular lasers. However, there have been many experimental designs that have also resulted in frequency combs. It is becoming more evident that any electronic device that can generate a series of pulses may also generate a frequency comb [26, 37, 38].

In the following sections, we report on our observation of frequency combs as a result of our *magnetic square well* scheme, including how results vary as a consequence of varying available parameter as well as perturbation.

III. HARMONICS VS FREQUENCY COMB

It is easy to confuse harmonics with frequency comb as they have many commonalities. However, there are significant differences between the two that are important both for application and theory.

The most basic definition of harmonic frequencies is that they are integer multiples of a fundamental frequency (also known as the *first harmonic*) and where the sum of all the frequencies form a harmonic series. They are sinusoidal waves where all the harmonics are periodic at the fundamental frequency. Moreover, harmonics typically exhibit lower amplitudes than the fundamental frequency [39, 40].

Frequency combs, on the other hand, are *not* another version of harmonics where the comb is just an integer multiple of the fundamental frequency. The identifying and distinguishing element of a frequency comb is that they are integer multiples of the *carrier frequency offset* f_0 [28, 41].

The general definition of *carrier frequency offset* (CFO) is the frequency difference between a carrier signal and the receiver; in other words, CFOs are oscillator mismatches. More specifically, CFOs are the difference between a transmitter's carrier frequency and the

frequency at the receiver. In addition to CFOs resulting from frequency mismatches between oscillators (i.e., transmitter/receiver), they can also result from phase slips and Doppler effects, where the transmitter and/or receiver is moving. All the integer multiples of this carrier frequency offset is then defined to be the *frequency comb* [41].

One important commonality between harmonics and frequency combs is that in general both phenomenon result from some non-linearity within the system. In the magnetic square well scheme, the non-linearity is due to the non-local interaction between the nucleated soliton and/or the harmonic perturbation that we inject at varying points within the channel, including inside and outside the well - see Figure 4.

A. Single Soliton

We tested a single injection with a magnitude that varied from the linear to the non-linear regime (e.g., From $\bar{\mu} = 0.1$ to $\bar{\mu} = 10$, where again the linear limit is $\bar{\mu} \approx 0.4$). We also varied the available parameter. Next, we harmonically perturbed the system with varying frequencies at varying points along the channel. More specifically, we varied the injection magnitude from $\bar{\mu} = 0.1$ to $\bar{\mu} = 10$, where the base current injection is $\mathbf{J} = 10^{11} \text{ A/m}^2$. We also allowed the system to evolve in time for 20 ns, which we observed as sufficient for stabilization. The channel length is again 1,600 nm, and although we tested a wide range of perturbation frequencies, we present here results for perturbation frequencies of 5 and 15 MHz - see Figure 3.

We started our analysis for the potential of distributed spectra or frequency combs for a single soliton without perturbation. In general, for the unperturbed single soliton, we observed some secondary effects in the background, which may be due to errors in our solutions and/or some self-interaction (especially for the metastable condition) [22]. However, the result of our analysis for a single unperturbed soliton (in both the linear and nonlinear limits) is a spectrum that does not reproduce the properties of harmonics or frequency combs. Instead, these textures have very small amplitudes with no apparent coherence in their distributions. In other words, the nucleated solitons may pump waves into the background of the channel but without the apparent properties of harmonics or frequency combs. In other words in the weak injection regime, we observe linear DEF solutions equivalent to spin superfluids. However, in the strong injection regime, we observe DEFs characterized by nonlinearity in both density and fluid velocity, which then results in a stationary soliton (i.e., *contact soliton*) nucleated at the injection site. This contact soliton then smoothly transitions into a nonlinear DEF across the channel- see Figure 2.

It is important to note that in the strong injection regime (i.e., $\mu \geq 1$), higher frequency perturbation tend

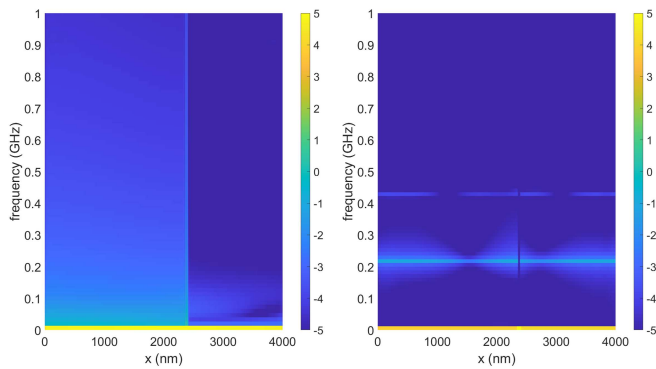


FIG. 2. These two color plots represent the solution of a single soliton with no perturbations. The left image is for injection of $\bar{\mu} = 0.1$ (i.e., linear) and the image on the right is for injection of $\bar{\mu} = 10$ (i.e., non-linear). Moreover, the image on the left is for a time step of 0.03 and the right is for a time step of 0.05, meaning the resolutions frequencies are 33 MHz and 20 MHz respectively; therefore, the observed small amplitude frequencies for both spectrums are within the limits of the frequency resolution, meaning the results are inconclusive.

to decay more rapidly. Therefore, and as a general matter across our simulations, lower frequency perturbations produced better dynamical results as can be seen in their spectral analysis - see Figure 3 for comparison.

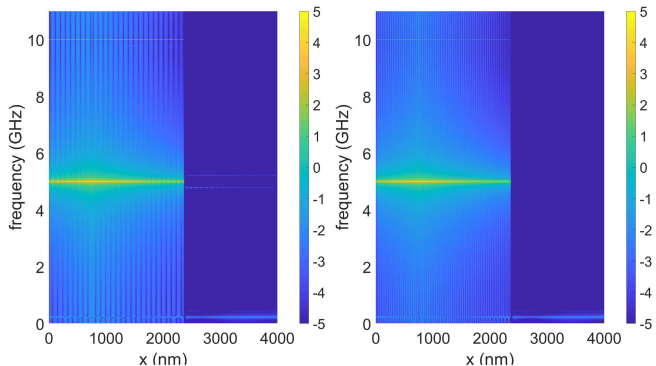


FIG. 3. The above images are the result of simulations with identical parameters - except for the perturbation frequencies. The left most image is the result of a single nucleated soliton perturbed harmonically with a frequency of 5 MHz and the second is perturbed with a frequency of 15 MHz.

Despite the failures of the perturbed and unperturbed single soliton to consistently generate frequency combs, a second nucleated soliton with proximity to the first soliton allowing for non-local *DEF-interaction* between them (i.e., the *magnetic square well*) and then perturbing that system harmonically as described above resulted in a spectrum that we concluded with a degree of confidence to be frequency combs. For an in depth description of this interaction as well as the behavior of this system in the moderate injection regime refer to Ref. [22]. In the following sections, we review the frequency comb results for the magnetic square well.

IV. FREQUENCY COMBS AND THE MAGNETIC SQUARE WELL

Again, the most effective scheme we constructed for generating frequency combs using pure torque is the magnetic square well, where we inject spin-polarized currents (i.e., $\pm\bar{\mu}$) at two sites with a separation distance that allows for non-local interactions (i.e., *DEF-interactions*). In other words, this scheme takes advantage of the algebraic decay of the spin density as a result of solving the equation of motion in the hydrodynamic representation as a boundary value problem [14]. Moreover, we varied the magnitude of injection, relative signs of the injections, and then perturbed the system harmonically inside and outside the well with varying perturbation frequencies - see Figure 4 for an illustration of this scheme.

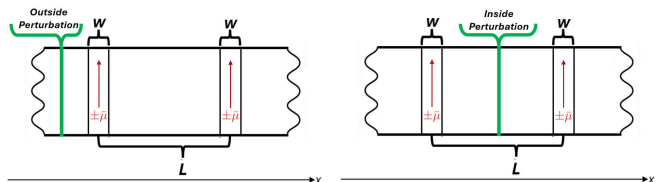


FIG. 4. This is the magnetic square well scheme, where the "well" is defined to be the region between the two injection points and with perturbation locations indicated in green. We perturbed the system outside (left image) and inside (right image) harmonically, and where L is the well distance and W is the injector width.

The magnetic square well even without perturbation exhibits textures that appears to have the properties of frequency combs. In general, nucleated solitons exhibit resonant frequencies that are a function of the magnitude of injection - see publication [22]. Moreover, this unperturbed well can also be divided into the categories of linear and nonlinear as well as same and opposite signed injections.

In the linear limit, the square well as can be seen from the color plots in Figures 5 and 6 exhibits a spectral distributions that appear to have the properties of frequency combs. Moreover, this spectral distribution was observed for both same and opposite signed injections. However, an important difference between the two results, which is also a consistent difference across all results, is that the frequency of the pumped waves is higher for the opposite signed injections than it is for the same signed injections.

In the following sections, we discuss the results of perturbed square well in the non-linear regime for both same- and opposite-signed injections, where their properties are much more apparent in terms of generating frequency combs.

A. Same Signed Well - With Perturbation

Again, even without perturbation, the magnetic square well exhibits a spectrum with the properties of a fre-

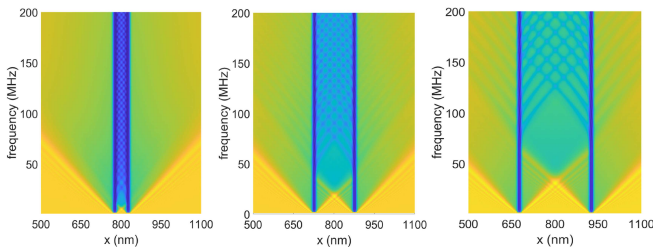


FIG. 5. These are *same signed* square well solutions, with no perturbation, within the linear limit (i.e. $\bar{\mu} = [0.1 \ 0.1]$), and with varying well sizes. As can be seen from their spectrum, they exhibit textures inside and outside the wells.

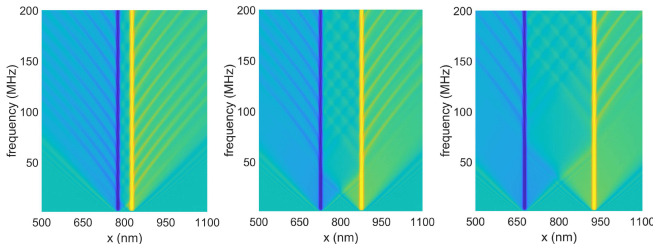


FIG. 6. These are *opposite signed* square well solutions also with no perturbation, within the linear limit (i.e. $\bar{\mu} = [-0.1 \ 0.1]$), and again with varying well sizes. They also exhibit textures inside and outside the wells.

quency comb. This means that this dynamical result is intrinsic to the property of the magnetic square well. Moreover, and as can be seen in panels two to four of Figures 7 and 8 where the system is harmonically perturbed at a location inside and outside the well, the generation of frequency combs becomes more obvious. The frequency distribution of the system is now a function of the magnitude of the frequency of the perturbation. Furthermore, lower frequency perturbation results in a greater distribution of frequency combs; this is also evident in the Fourier transform, where the frequency spikes in Fourier space (and where the blue curve represents discrete values while the red curve represents an average over time) are more abundant - see Figures 7 to 10. In other words, lower frequency harmonic perturbation of the square well results in a more textured frequency combs. This stands to reason because as previously stated higher frequency perturbations are likely to damp out more quickly.

One of the important observed differences between the perturbations inside and outside the well is that for the same set of parameters, perturbation inside the well tends to result in a larger number of frequency comb distributions. We believe this is likely due to the fact that there is significant more reflection, absorption and radiation that the perturbed incoming wave experiences as a result of being in the middle of the two solitons, which in effect operate as the walls for the well. When the perturbation is outside the well, reflection, absorption and radiation occur at only one of the solitons. This is because outside the well the channel has a spin-free boundary condition

at the edges. This means that the channel edges are in effect a sink for the incoming waves; therefore, no interaction takes place thus reducing the nonlinearity of the system and in turn reducing the number of observed frequency combs as visualized in our analysis. This result is further highlighted in the condition where the injection signs are opposite to each other; we explore this result in the next section.

B. Opposite Signed Well - With Perturbation

In general, the observed properties of the magnetic square well are similar for both same- and opposite-signed injections, in particular when the systems are perturbed. However, the one significant difference consistently observed for all permutation of the parameters is that the synchronized frequencies/oscillation measured is higher for systems of same-signed injections. We also observe that this frequency has an inverse relationship with the injection magnitude, where larger injections resulted in smaller frequencies. This is likely due to the exchange force increasing in magnitude with increased injection and thus in effect "sucking up" all the energies of the system and preventing that energy from being pumped into the channel in the form of outgoing DEFs. Moreover, opposite-signed injections resulted in a larger distribution of frequency combs as can be seen from the Figures 9 and 10. This is likely due to the additional nonlinearity introduced by the differing torque signs.

Moreover, in our simulations, the harmonic perturbation frequencies for all the magnetic square well schemes that resulted in frequency combs were a constant increment above and below or integer multiple of the synchronization frequencies. As an example, for opposite-signed injections where the measured synchronized frequency was found to be approximately 0.175 MHz, we perturbed the system inside and outside the well by frequencies of 0.175, 1.75, and 3.5 MHz (see figures 9 and 10). This process was the same for the same-signed injections, where the difference was in the perturbation frequency magnitudes - i.e., synchronized frequency of approximately 1.05 MHz and perturbation frequencies of 0.105, 1.05, and 5.25 MHz (see figures 7 and 8).

V. TRAIN OF MAGNETIC SQUARE WELLS

Another method used to confirm the results of our magnetic square well was to extend it to a train of magnetic square wells as can be seen in Figure 11. In effect, we injected multiple supersonic torques (i.e., $\bar{\mu} = [10 \ 10 \ 10 \ 10]$) along a one-dimensional ferromagnetic channel and compared the perturbed and unperturbed dynamics - see Figures 12 and 14 respectively. Moreover, the Fourier representation of these perturbed and unperturbed train of magnetic square wells also confirm the results - see Figures 13 and 15, where again the blue

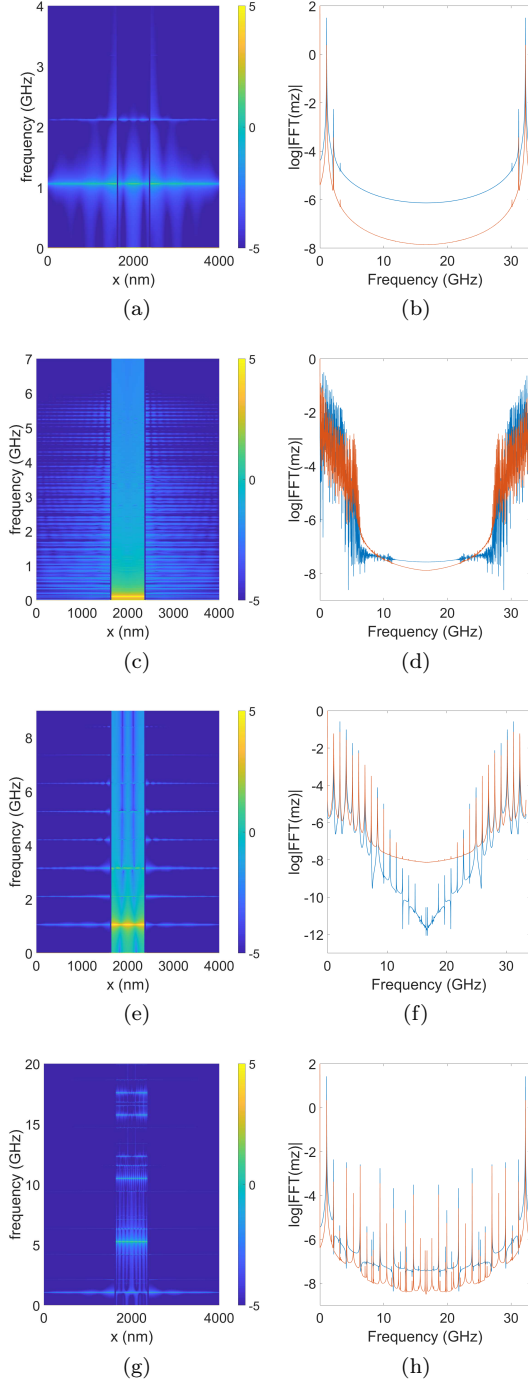


FIG. 7. These series of images and their respective Fourier space representations (where the blue curve represents discrete values while the red curve represents an average over time) are for the same signed square well. Panel one and its Fourier representation does not include any perturbation while panels two to four include harmonic perturbation *inside the well* of increasing frequency. It is also apparent that the greatest distribution of frequency combs results from low frequency perturbation. The images are for parameters of $\bar{\mu} = [10 \ 10]$, well size of 120 nm, and a channel length of 1,600 nm. Moreover, the synchronized frequency is approximately 1.05 MHz and perturbation frequencies (f_p) are as follow: (a)/(b): $f_p = 0$; (c)/(d): $f_p = 0.105$ MHz; (e)/(f): $f_p = 1.05$ MHz, and (g)/(h): $f_p = 5.25$ MHz

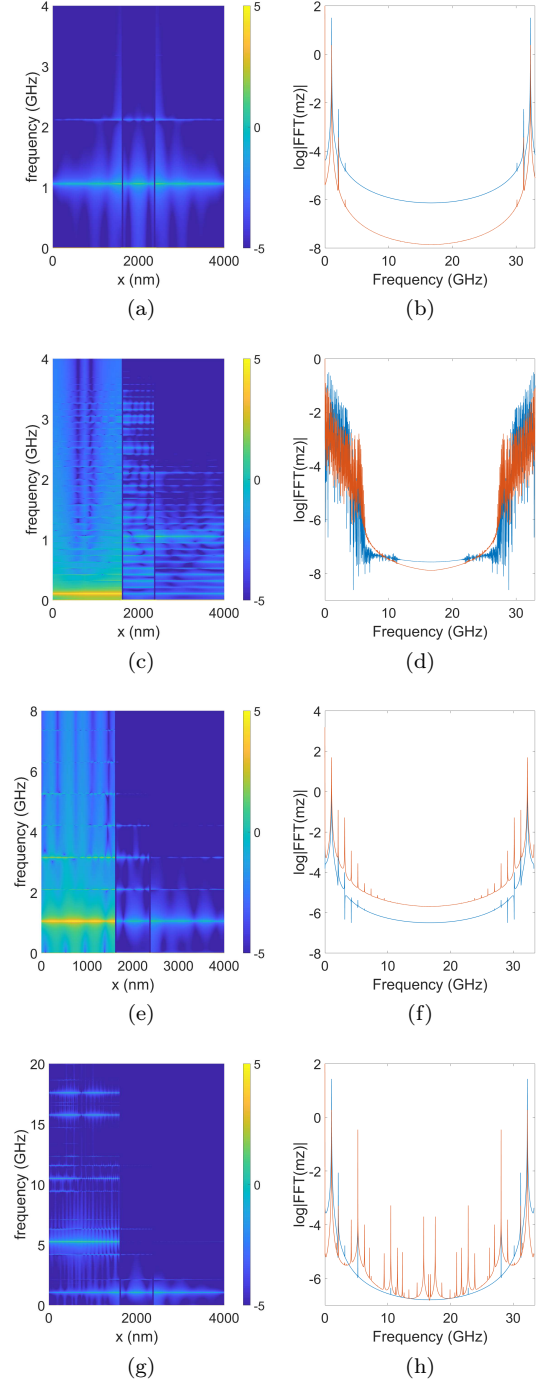


FIG. 8. These series of images and their respective Fourier space representations are for the same signed square well. Panel one and its Fourier representation does not include any perturbation while panels two to four include harmonic perturbation *outside the well* of increasing frequency. The result are again for parameters of $\bar{\mu} = [10 \ 10]$, well size of 120 nm, and a channel length of 1,600 nm. Moreover, the synchronized frequency is approximately 1.05 MHz and perturbation frequencies (f_p) are as follow: (a)/(b): $f_p = 0$; (c)/(d): $f_p = 0.105$ MHz; (e)/(f): $f_p = 1.05$ MHz, and (g)/(h): $f_p = 5.25$ MHz

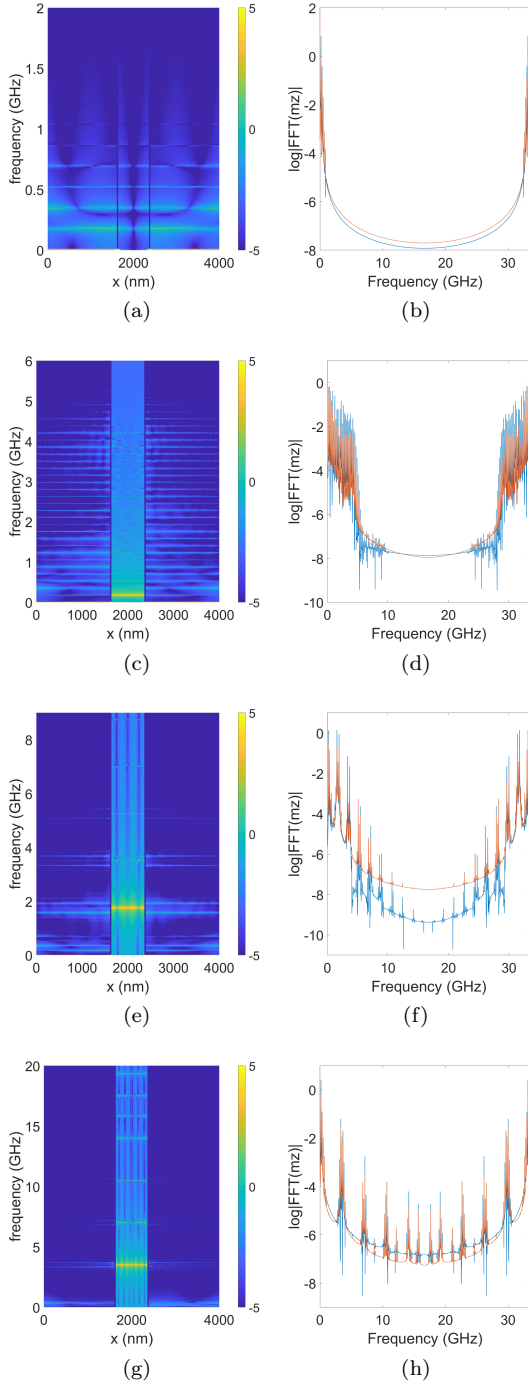


FIG. 9. These series of images and their respective Fourier space representations are for opposite signed square well. Panel one and its Fourier representation does not include any perturbation while panels two to four include harmonic perturbation *inside the well* of increasing frequency. The images are the result for parameters of $\bar{\mu} = [10 \ -10]$, well size of 120 nm, and a channel length of 1,600 nm. Moreover, the synchronized frequency is approximately 0.175 MHz and perturbation frequencies (f_p) are as follow: (a)/(b): $f_p = 0$; (c)/(d): $f_p = 0.175$ MHz; (e)/(f): $f_p = 1.75$ MHz, and (g)/(h): $f_p = 3.05$ MHz

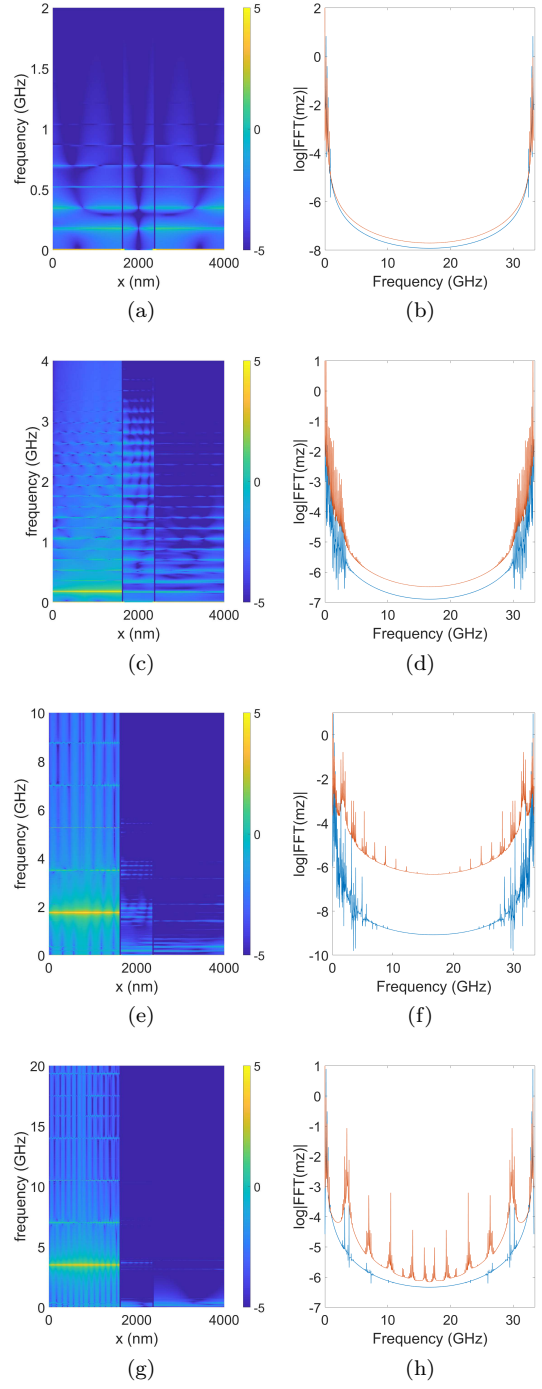


FIG. 10. These series of images and their respective Fourier space representations are for opposite signed square well. Panel one and its Fourier representation does not include any perturbation while panels two to four include harmonic perturbation *Outside the well* of increasing frequency. The images are the result for parameters of $\bar{\mu} = [10 \ -10]$, well size of 120 nm, and a channel length of 1,600 nm. Moreover, the synchronized frequency is approximately 0.175 MHz and perturbation frequencies (f_p) are as follow: (a)/(b): $f_p = 0$; (c)/(d): $f_p = 0.175$ MHz; (e)/(f): $f_p = 1.75$ MHz, and (g)/(h): $f_p = 3.05$ MHz

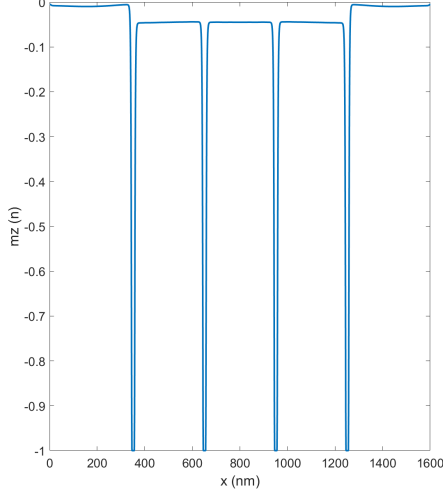


FIG. 11. This image is of a series of four evenly spaced super-sonic torque injections (i.e., $\bar{\mu} = [10 \ 10 \ 10 \ 10]$) along a one-dimensional ferromagnetic channel, in effect creating three magnetic square wells with spin-free conditions at the edges of the channel.

curve represents discrete values in Fourier space while the red curve represents an average over time. Moreover, perturbing this train of magnetic square wells again resulted in more pronounced spectra, which include the properties of a frequency comb. These results once again demonstrate that in the nonlinear limit an intrinsic property of the magnetic square well is a stable spectrum with the properties of the frequency combs.

VI. CONCLUSIONS

The frequency comb results illustrated in this paper are not aberrations or the result of a specific set of parameters finely tweaked or tuned. Instead, they are consistent fixtures of the magnetic square well scheme, regardless of the injection regime (i.e., linear vs. nonlinear). In general, the tweaking or tuning of the parameters in this scheme either enhances or reduces the observed frequency combs - but not eliminate them. In other words, the magnetic square well scheme has significant nonlinearities that result in interesting and often unexpected dynamics and textures that require further exploration.

That said, the magnetic square well also presents a novel method for inducing frequency combs in ferromagnetic channels by the injection of pure spin-polarized currents and/or by perturbing the system harmonically at different regions along the channel (e.g., inside and outside the well). Moreover, this harmonic perturbation is found to be most effective when the perturbation frequency is low. In other words, we have theoretically demonstrated that by minimal injection of energy into this magnetic square well, we can induce useful behaviors

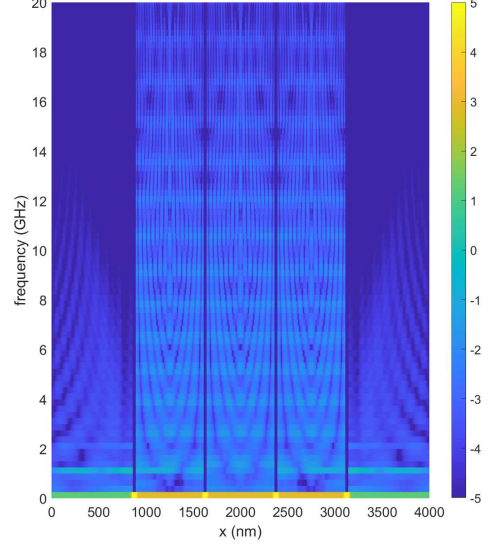


FIG. 12. This color plot represents an *unperturbed* train of three magnetic square solutions for injection magnitude of $\bar{\mu} = [10 \ 10 \ 10 \ 10]$ along a one-dimensional ferromagnetic channel. Even without perturbation the spectra inside each well are constant and evenly spaced as expected for a frequency combs distribution.

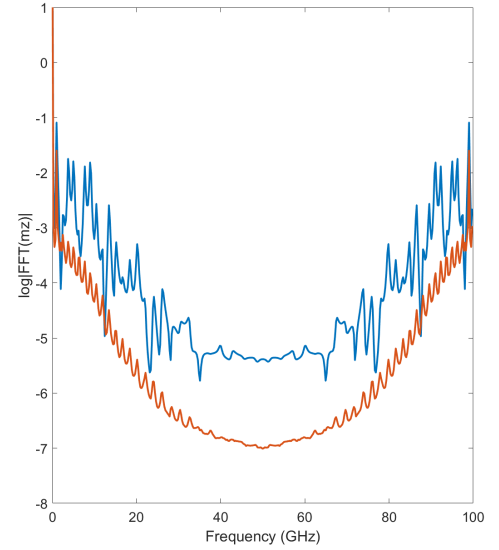


FIG. 13. This is the Fourier representation of the *unperturbed* train of three magnetic square solutions for injection magnitude of $\bar{\mu} = [10 \ 10 \ 10 \ 10]$ along a one-dimensional ferromagnetic channel. The blue curve represents discrete values in Fourier space while the red curve represents an average over time. Even without perturbation there is a clear spectra inside each well with the properties of frequency combs.

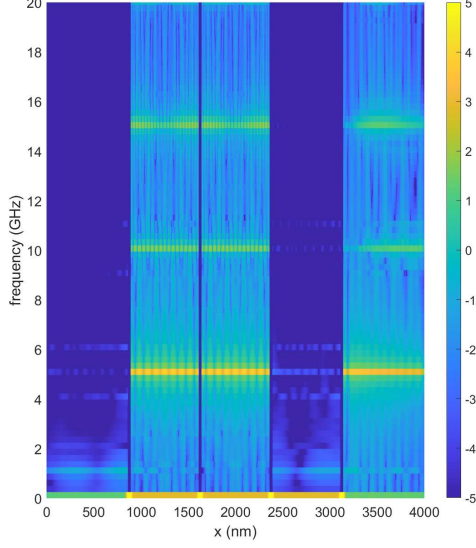


FIG. 14. This color plot represents the *perturbed* train of three magnetic square solutions, where the first two wells are perturbed harmonically with a frequency of the 5 MHz and the third is left unperturbed for comparison. The third perturbation is placed outside the well. The results for the interior and exterior of the three wells are as expected.

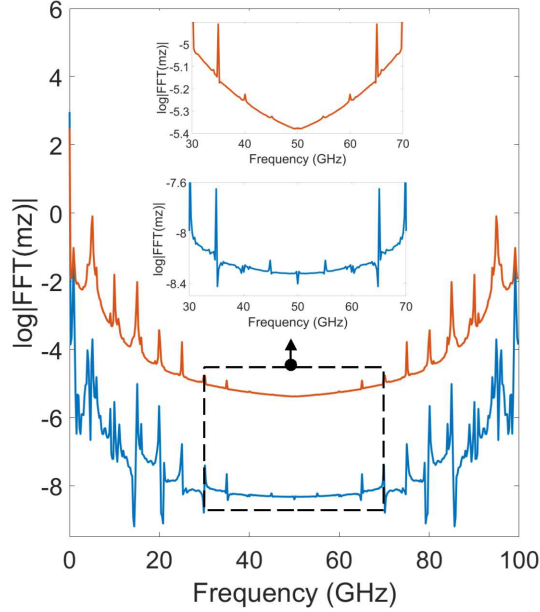


FIG. 15. This is the Fourier representation of the *perturbed* train of magnetic square well solutions, where again the first two wells are perturbed while the third is not (i.e., the Fourier transform of Figure 14). The blue curve represents discrete values in Fourier space while the red curve represents an average over time. Again, a spectral distribution with the properties of frequency combs is more apparent for the perturbed wells.

and textures that often require much more energy and a much bigger experimental setup. Finally, our results also provide a deeper insight into the fluid-like behavior of magnetism, particularly in the nonlinear limit, where magnetic solitons exist and can interact non-locally via DEFs.

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