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Gegenbauer Polynomial Convolution for Bi-Univalent Functions with Applications to Image Enhancement

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Abstract

In this paper, we define a new subclass of bi-univalent functions by using convolution and Gegenbauer polynomials. We obtain the initial co-efficient bounds and the Fekete-Szegő inequalities for functions belonging to this subclass. Incorporating the coefficient bounds, a Gegenbauer Polynomial Convolution Enhancement Algorithm (GCEA) is developed by employing the derived coefficient bounds as directional convolution kernels. The proposed algorithm is applied to digital image enhancement, where it effectively improves edge detection, preserves structural features, and enhances image contrast. Experimental evaluation using PSNR, SSIM, and PCC quality measures demonstrates that the proposed approach produces enhanced images while maintaining high structural fidelity.

Keywords and Phrases: Bi-univalent functions, Coefficient estimates, Convolution, Gegenbauer polynomial, Subordination, Image enhancement, Gegenbauer polynomial convolution enhancement algorithm (GCEA).

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1 Introduction

Orthogonal polynomials form an important family of special functions characterized by their orthogonality with respect to a given weight function on a specified interval. Their mathematical properties have made them indispensable in approximation theory, numerical analysis, spectral methods, mathematical physics, and several engineering applications. Classical families, including the Legendre, Chebyshev, Hermite, and Gegenbauer polynomials, have been extensively investigated because of their analytical structure and computational efficiency. Over the years, these polynomial systems have provided effective tools for solving differential equations, constructing approximation schemes, and developing numerical algorithms, thereby continuing to play a significant role in both theoretical and applied mathematics.[12]

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials defined on the interval $[-1, 1]$ with respect to a weight function $w(x)$. The sequence is said to be orthogonal if

$$\int_{-1}^1 p_n(x)p_m(x)w(x)dx = 0, \quad n \neq m. \quad (1.1)$$

In addition to orthogonality, these polynomial systems often satisfy recurrence relations and possess explicit coefficient representations, making them computationally efficient and suitable for a wide range of theoretical

and practical applications. [8].

Among the most widely studied orthogonal polynomials are the Legendre, Chebyshev, Hermite, and Laguerre polynomials. These polynomial families naturally arise in the solutions of differential equations and are extensively employed in spectral methods, approximation theory, and mathematical modelling. A particularly important generalization of Legendre polynomials is provided by the Gegenbauer polynomials.

The Gegenbauer polynomials $G_n^\alpha(x)$ are defined by,

$$G_n^\alpha(x) = \frac{1}{2^n n!} (1-x^2)^{-\alpha} \frac{d^n}{dx^n} (x^2-1)^n, \quad (1.2)$$

where α is a real parameter. These polynomials are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$(1-x^2)^{\alpha-\frac{1}{2}}. \quad (1.3)$$

Because of the rich analytical structure, Gegenbauer polynomials have numerous applications in quantum mechanics, celestial mechanics, spectral analysis, and numerical methods for solving differential equations [19, 16].

Motivated by the growing importance of orthogonal polynomials on the field, the present work investigates a new subclass of bi-univalent functions associated with Gegenbauer polynomials. In particular, we derive estimates for the initial Taylor–Maclaurin coefficients and establish Fekete–Szegő inequalities for the newly defined class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$. Before presenting our main results, we recall several definitions and preliminary concepts that will be used throughout the paper.

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.4)$$

which are analytic in the open unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$ [9].

For two analytic functions f and g in U , the function f is said to be subordinate to g , written as

$$f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, analytic in U , satisfying $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z)).$$

If g is univalent in U , then the above condition is equivalent to $f(0) = g(0)$ and $f(U) \subseteq g(U)$ [9].

Let S denote the subclass of A consisting of univalent functions in U . By the Koebe one-quarter theorem [9], every function $f \in S$ possesses an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z (z \in U),$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (1.5)$$

A function $f \in A$ is called bi-univalent in U if both f and its inverse f^{-1} are univalent in U . The class of all such functions is denoted by Σ . Coefficient problems for bi-univalent functions have attracted considerable attention in recent years because of their intrinsic theoretical significance and their connections with geometric properties of analytic functions [10, 13, 14, 20].

Another important tool in the study of univalent and bi-univalent functions is convolution (Hadamard product). Convolution has been extensively used to characterise subclasses of analytic functions, constructs new function families, and investigate geometric properties such as starlikeness, convexity, and close-to-convexity. Through suitable kernel functions, convolution operators provide a flexible framework for generating analytic functions with prescribed geometric behaviour.

For functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

their convolution is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \quad (1.6)$$

Recently, Amourah *et al.* considered subclasses of bi-univalent functions associated with Gegenbauer polynomials through the generating function

$$H_{\alpha}(x, z) = \frac{1}{(1 - 2xz + z^2)^{\alpha}}, \quad (1.7)$$

where $x \in [-1, 1]$ and $z \in U$. Since $H_{\alpha}(x, z)$ is analytic in U , it admits the expansion

$$H_{\alpha}(x, z) = \sum_{n=0}^{\infty} G_n^{\alpha}(x) z^n, \quad (1.8)$$

where $G_n^{\alpha}(x)$ denotes the Gegenbauer polynomial of degree n [3].

For $\alpha = 0$, the generating function is given by

$$H_0(x, z) = 1 - \log(1 - 2xz + z^2) = \sum_{n=0}^{\infty} G_n^0(x) z^n. \quad (1.9)$$

Furthermore, the normalization $\alpha > -\frac{1}{2}$ is generally assumed to preserve the orthogonality properties of the Gegenbauer polynomials [8, 16].

Gegenbauer polynomials can also be defined by the following recurrence relations:

$$G_n^{\alpha}(x) = \frac{1}{n} [2x(n + \alpha - 1)G_{n-1}^{\alpha}(x) - (n + 2\alpha - 2)G_{n-2}^{\alpha}(x)] \quad (1.10)$$

with the initial values

$$G_0^{\alpha}(x) = 1, \quad G_1^{\alpha}(x) = 2\alpha x, \quad G_2^{\alpha}(x) = 2\alpha(1 + \alpha)x^2 - \alpha \quad \text{and} \quad G_3^{\alpha}(x) = \frac{4\alpha(\alpha + 1)(\alpha + 2)x^3}{3} - 2\alpha(\alpha + 1)x. \quad (1.11)$$

Several well-known polynomial families arise as special cases of the Gegenbauer polynomials. In particular,

1. For $\alpha = 1$, the Gegenbauer polynomials reduce to the Chebyshev polynomials of the second kind.
2. For $\alpha = \frac{1}{2}$, they reduce to the Legendre polynomials.

The increasing interest in orthogonal polynomial based approaches has led many researchers to investigate subclasses of bi-univalent functions associated with Gegenbauer, Chebyshev, Horadam, and other orthogonal polynomial systems [2, 3, 6, 21]. Inspired by these developments, we introduce a new subclass of bi-univalent functions defined through convolution operators and Gegenbauer polynomial generating functions. The coefficient estimates and Fekete–Szegő inequalities obtained for this class generalise several existing results in the literature and provide a framework that is subsequently applied to image enhancement through convolution-based processing techniques. Furthermore, unlike earlier studies that focused solely on coefficient estimation problems, the present work establishes a connection between Geometric Function Theory and digital image processing. By utilising the coefficient bounds derived from the proposed bi-univalent function class, we develop a Gegenbauer Polynomial Convolution Enhancement Algorithm (GCEA) for image enhancement. This interdisciplinary approach demonstrates the practical applicability of geometric function theoretic results in image analysis and computer vision.

Motivated by the analytical properties of Gegenbauer polynomials and their successful applications in the study of analytic and bi-univalent functions, we introduce a new subclass of bi-univalent functions through the convolution of analytic functions with generating functions associated with Gegenbauer polynomials. The proposed class serves as the theoretical foundation for deriving coefficient estimates, establishing Fekete–Szegő inequalities, and developing the subsequent image enhancement algorithm.

Accordingly, we define the subclass of bi-univalent functions $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$ using the Gegenbauer polynomials expansions.

Definition 1.1. Let $\eta \geq 1$, $x \in (-1/2, 1]$ and α is a nonzero real constant. A function $f(z) \in \Sigma$ is said to be in the class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$, if the following conditions are satisfied :

$$\frac{(1 - \eta)(f * g)(z) + \eta(f * h)(z)}{z} \prec H_{\alpha}(z, x) := \frac{1}{(1 - 2xz + z^2)^{\alpha}}, (z \in U) \quad (1.12)$$

and

$$\frac{(1 - \eta)(F * g)(w) + \eta(F * h)(w)}{w} \prec H_{\alpha}(w, x) := \frac{1}{(1 - 2xw + w^2)^{\alpha}}, (w \in U). \quad (1.13)$$

where $F = f^{-1}$ is defined by (1.5).

By specializing on the values of η, α, g, h , we obtain subclasses of bi-univalent functions that were studied earlier in literature [5, 7, 17, 18].

Remark 1.2. For $\alpha = 1$, the class $B_{\Sigma}^{\eta, 1}(f, g, h; x) = B_{\Sigma}^{\eta}(f, g, h; x)$ consisting of the functions $f \in \Sigma$ satisfying the condition,

$$\frac{(1 - \eta)(f * g)(z) + \eta(f * h)(z)}{z} \prec H_{\alpha}(z, x) := \frac{1}{1 - 2xz + z^2}, (z \in U)$$

and

$$\frac{(1 - \eta)(F * g)(w) + \eta(F * h)(w)}{w} \prec H_{\alpha}(w, x) := \frac{1}{1 - 2xw + w^2}, (w \in U).$$

where $F = f^{-1}$ is defined by (1.5). This class was introduced and studied by Sivasankari et al [18].

Remark 1.3. For $g(z) = \frac{z}{1 - z}$, $h(z) = \frac{z}{(1 - z)^2}$, we obtain the class $B_{\Sigma}^{\eta, \alpha}\left(f, \frac{z}{1 - z}, \frac{z}{(1 - z)^2}; x\right) = B_{\Sigma}(\eta, x, \alpha)$, consisting of functions $f \in \Sigma$ satisfying the condition,

$$(1 - \eta)\frac{f(z)}{z} + \eta f'(z) \prec \frac{1}{(1 - 2xz + z^2)^{\alpha}}, (z \in U)$$

and

$$(1-\eta)\frac{F(w)}{w} + \eta F'(w) \prec \frac{1}{(1-2xw+w^2)^\alpha}, (w \in U).$$

where the function $F = f^{-1}$ is defined by (1.5). (See example 1 in [3])

Remark 1.4. For $g(z) = \frac{z}{1-z}$, $h(z) = \frac{z}{(1-z)^2}$, we obtain the class $B_\Sigma^\eta\left(f, \frac{z}{1-z}, \frac{z}{(1-z)^2}; x\right) = B_\Sigma(\eta, x)$, consisting of functions $f \in \Sigma$ satisfying the condition,

$$(1-\eta)\frac{f(z)}{z} + \eta f'(z) \prec \frac{1}{1-2xz+z^2}, (z \in U)$$

and

$$(1-\eta)\frac{F(w)}{w} + \eta F'(w) \prec \frac{1}{1-2xw+w^2}, (w \in U).$$

where the function $F = f^{-1}$ is defined by (1.5). This class was introduced and studied by Bulut et al. [5]

Remark 1.5. For $h(z) = \frac{z}{(1-z)^2}$, $\eta = 1$, we obtain the class $B_\Sigma^{1,\alpha}\left(f, g, \frac{z}{(1-z)^2}; x\right) = B_\Sigma(x, \alpha)$, consisting of functions $f \in \Sigma$ satisfying the condition :

$$f'(z) \prec \frac{1}{(1-2xz+z^2)^\alpha}, (z \in U)$$

and

$$F'(w) \prec \frac{1}{(1-2xw+w^2)^\alpha}, (w \in U).$$

where the function $F = f^{-1}$ is defined by (1.5) (See example 3 in [3]).

Remark 1.6. For $h(z) = \frac{z}{(1-z)^2}$, $\eta = 1$, we obtain the class $B_\Sigma^{1,\alpha}\left(f, g, \frac{z}{(1-z)^2}; x\right) = B_\Sigma(x)$, consisting of functions $f \in \Sigma$ satisfying the condition :

$$f'(z) \prec \frac{1}{1-2xz+z^2}, (z \in U)$$

and

$$F'(w) \prec \frac{1}{1-2xw+w^2}, (w \in U).$$

where the function $F = f^{-1}$ is defined by (1.5) (See [5]).

Remark 1.7. For $g(z) = \frac{z}{(1-z)^{n+1}}$, $h(z) = \frac{z}{(1-z)^{n+2}}$, we get the class

$B_\Sigma^\eta\left(f, \frac{z}{(1-z)^{n+1}}, \frac{z}{(1-z)^{n+2}}; x\right) = B_\Sigma^{\eta,n}(x)$, consisting of functions $f \in \Sigma$ satisfying the condition :

$$\frac{(1-\eta)D^n f(z) + \eta D^{n+1} f(z)}{z} \prec \frac{1}{(1-2xz+z^2)^\alpha}, (z \in U)$$

and

$$\frac{(1-\eta)D^n F(w) + \eta D^{n+1} F(w)}{w} \prec \frac{1}{(1-2xw+w^2)^\alpha}, (w \in U).$$

where D^n is the Ruscheweyh derivative operator [17] defined by $D^n(z) = \frac{z(z^{n-1}f(z))^{(n)}}{n!}$ and the function $F = f^{-1}$ as in (1.5) .

Remark 1.8. For $g(z) = \phi(a, c; z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k$, and $h(z) = \phi(a+1, c; z) = z + \sum_{k=2}^{\infty} \frac{(a+1)_{k-1}}{(c)_{k-1}} z^k$, we obtain the class $B_{\Sigma}^{\eta}(f, \phi(a, c; z), \phi(a+1, c; z); x) = B_{\Sigma}^{\eta}(a, c, x)$, consisting of functions $f \in \Sigma$ satisfying the condition :

$$\frac{(1-\eta)L(a, c)f(z) + \eta L(a+1, c)f(z)}{z} \prec \frac{1}{(1-2xz+z^2)^{\alpha}} (z \in U) \quad (c \neq 0, -1, -2, \dots)$$

and

$$\frac{(1-\eta)L(a, c)F(w) + \eta L(a+1, c)F(w)}{w} \prec \frac{1}{(1-2xw+w^2)^{\alpha}}, (w \in U) \quad (c \neq 0, -1, -2, \dots)$$

where the function $F = f^{-1}$ as in (1.5), $L(a, c)f(z)$ is the Carlson-Shaffer operator [7] defined by

$$L(a, c)f(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} a_k z^k \text{ and } (a)_k \text{ is the Pochhammer symbol given by } (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

In order to prove our main results, we shall need the following lemma.

Let P denote the class of functions $p(z)$ of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots, \quad (1.14)$$

which are analytic in the open unit disk U .

Lemma 1.9. [15] If the function $p \in P$ is given by the series (1.14), then the following sharp estimate holds:

$$|c_k| \leq 2, k = 1, 2, \dots \quad (1.15)$$

Lemma 1.10. [11] If the function $p \in P$ is given by the series (1.14), then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2 Coefficient bounds for the class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x,)$

Theorem 2.1. For $\eta \geq 1$ and $x \in (-1/2, 1]$, let the function $f(z) \in \Sigma$ given by (1.4) be in the class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x, \alpha)$. Then

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{(2[(1-\eta)g_3 + \eta h_3] - [(1-\eta)g_2 + \eta h_2]^2)2\alpha^2 x^2 + \alpha(1-2x^2)[(1-\eta)g_2 + \eta h_2]^2}}, \quad (2.1)$$

$$|a_3| \leq \frac{4\alpha^2 x^2}{[(1-\eta)g_2 + \eta h_2]^2} + \frac{2|\alpha x|}{[(1-\eta)g_3 + \eta h_3]} \quad (2.2)$$

Proof. Let $f \in \Sigma$ given by (1.4) be in the class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$.

From (1.5) and (1.13), we have

$$\frac{(1-\eta)(f * g)(z) + \eta(f * h)(z)}{z} = 1 + G_1^{\alpha}(x)p(z) + G_2^{\alpha}(x)p^2(z) + \dots \quad (2.3)$$

and for its inverse map, $F = f^{-1}$,

$$\frac{(1-\eta)(F * g)(w) + \eta(F * h)(w)}{w} = 1 + G_1^{\alpha}(x)q(w) + G_2^{\alpha}(x)q^2(w) + \dots \quad (2.4)$$

for some analytic function

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \dots \quad (z \in U) \quad (2.5)$$

and

$$q(w) = d_1 w + d_2 w^2 + d_3 w^3 + \dots \quad (w \in U), \quad (2.6)$$

such that $p(0) = q(0) = 0$, $|p(z)| < 1$ ($z \in U$) and $|q(w)| < 1$ ($w \in U$).

It is well-known that if $|p(z)| < 1$ and $|q(w)| < 1$, then

$$|c_k| \leq 1 \quad \text{and} \quad |d_k| \leq 1 \quad \text{for all} \quad k \in \mathbb{N}. \quad (2.7)$$

From (2.3), (2.4), (2.5) and (2.6), we have

$$\frac{(1-\eta)(f * g)(z) + \eta(f * h)(z)}{z} = 1 + G_1^\alpha(x)c_1(z) + [G_1^\alpha(x)c_2 + G_2^\alpha(x)c_1^2]z^2 + \dots \quad (2.8)$$

$$\frac{(1-\eta)(F * g)(w) + \eta(F * h)(w)}{w} = 1 + G_1^\alpha(x)d_1(w) + [G_1^\alpha(x)d_2 + G_2^\alpha(x)d_1^2]w^2 + \dots \quad (2.9)$$

Equating the coefficients in (2.8) and (2.9), we get

$$a_2[(1-\eta)g_2 + \eta h_2] = G_1^\alpha(x)c_1, \quad (2.10)$$

$$a_3[(1-\eta)g_3 + \eta h_3] = G_1^\alpha(x)c_2 + G_2^\alpha(x)c_1^2, \quad (2.11)$$

$$-a_2[(1-\eta)g_2 + \eta h_2] = G_1^\alpha(x)d_1, \quad (2.12)$$

$$(2a_2^2 - a_3)[(1-\eta)g_3 + \eta h_3] = G_1^\alpha(x)d_2 + G_2^\alpha(x)d_1^2. \quad (2.13)$$

From (2.10) and (2.12), we obtain

$$c_1 = -d_1 \quad (2.14)$$

and

$$2[(1-\eta)g_2 + \eta h_2]^2 a_2^2 = [G_1^\alpha(x)]^2 (c_1^2 + d_1^2). \quad (2.15)$$

Also, by using (2.11) and (2.13) we obtain

$$(2a_2^2)[(1-\eta)g_3 + \eta h_3] = G_1^\alpha(x)(c_2 + d_2) + G_2^\alpha(x)(c_1^2 + d_1^2). \quad (2.16)$$

By using (2.15) in (2.16), we get

$$\left([(1-\eta)g_3 + \eta h_3] - \frac{G_2^\alpha(x)}{[G_1^\alpha(x)]^2} [(1-\eta)g_2 + \eta h_2]^2 \right) 2a_2^2 = G_1^\alpha(x)(c_2 + d_2). \quad (2.17)$$

From (1.11), (2.7) and (2.17), we have

$$|a_2| \leq \frac{2|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{2[(1-\eta)g_3 + \eta h_3] - [(1-\eta)g_2 + \eta h_2]^2} 2\alpha^2 x^2 + \alpha(1-2x^2)[(1-\eta)g_2 + \eta h_2]^2}. \quad (2.18)$$

Next, by subtracting (2.13) from (2.11), we have

$$2a_3[(1-\eta)g_3 + \eta h_3] - 2a_2^2[(1-\eta)g_3 + \eta h_3] = G_1^\alpha(x)(c_2 - d_2) + G_2^\alpha(x)(c_1^2 - d_1^2). \quad (2.19)$$

Further, in view of (2.14), we obtain

$$a_3 = a_2^2 + \frac{G_1^\alpha(x)(c_2 - d_2)}{2[(1-\eta)g_3 + \eta h_3]}. \quad (2.20)$$

Hence using (2.15) and applying (1.11), we get

$$|a_3| \leq \frac{4\alpha^2 x^2}{[(1-\eta)g_2 + \eta h_2]^2} + \frac{2|\alpha|x}{[(1-\eta)g_3 + \eta h_3]} \quad (2.21)$$

This completes the proof. $\square$

3 Fekete-Szegő inequality for function in $B_{\Sigma}^{\eta,\alpha}(f, g, h; x)$

Theorem 3.1. For $\eta \geq 1$ and $x \in (-1/2, 1]$, let the function $f(z) \in \Sigma$ given by (??) be in the class $B_{\Sigma}^{\eta,\alpha}(f, g, h; x, \alpha)$ and for some $\mu \in \mathbb{R}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\alpha|x}{[(1-\eta)g_3 + \eta h_3]}, & \text{for } |\mu - 1| \leq \frac{|(2[(1-\eta)g_3 + \eta h_3] - [(1-\eta)g_2 + \eta h_2]^2)2\alpha^2 x^2 + \alpha(1-2x^2)[(1-\eta)g_2 + \eta h_2]^2|}{4\alpha^2 x^2[(1-\eta)g_3 + \eta h_3]} \\ \frac{8|(1-\mu)|\alpha^3 x^3}{|(2[(1-\eta)g_3 + \eta h_3] - [(1-\eta)g_2 + \eta h_2]^2)2\alpha^2 x^2 + \alpha(1-2x^2)[(1-\eta)g_2 + \eta h_2]^2|}, & \text{for } |\mu - 1| \geq \frac{|(2[(1-\eta)g_3 + \eta h_3] - [(1-\eta)g_2 + \eta h_2]^2)2\alpha^2 x^2 + \alpha(1-2x^2)[(1-\eta)g_2 + \eta h_2]^2|}{4\alpha^2 x^2[(1-\eta)g_3 + \eta h_3]} \end{cases} \quad (3.1)$$

Proof. Let $f(z) \in \Sigma$ given by (1.4) be in the class $B_{\Sigma}^{\eta,\alpha}(f, g, h; x, \alpha)$.

From (1.5) and (1.13), we have

$$\frac{(1-\eta)(f * g)(z) + \eta(f * h)(z)}{z} = 1 + G_1^{\alpha}(x)p(z) + G_2^{\alpha}(x)p^2(z) + \dots \quad (3.2)$$

and for its inverse map, $F = f^{-1}$,

$$\frac{(1-\eta)(F * g)(w) + \eta(F * h)(w)}{w} = 1 + G_1^{\alpha}(x)q(w) + G_2^{\alpha}(x)q^2(w) + \dots \quad (3.3)$$

Now, by using (2.17) and (2.20) for some $\mu \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu) \left[\frac{[G_1^{\alpha}(x)]^3(c_2 + d_2)}{2[(1-\eta)g_3 + \eta h_3][G_1^{\alpha}(x)]^2 - 2G_2^{\alpha}(x)[(1-\eta)g_2 + \eta h_2]^2} \right] + \frac{G_1^{\alpha}(x)(c_2 - d_2)}{2[(1-\eta)g_3 + \eta h_3]} \\ &= G_1^{\alpha}(x) \left[\left(h(\mu) + \frac{1}{[(1-\eta)g_3 + \eta h_3]} \right) c_2 + \left(h(\mu) - \frac{1}{[(1-\eta)g_3 + \eta h_3]} \right) d_2 \right], \end{aligned}$$

where

$$h(\mu) = \frac{[G_1^{\alpha}(x)]^2(1-\mu)}{2 \left([(1-\eta)g_3 + \eta h_3][G_1^{\alpha}(x)]^2 - G_2^{\alpha}(x)[(1-\eta)g_2 + \eta h_2]^2 \right)}.$$

Hence, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2|\alpha|x}{[(1-\eta)g_3 + \eta h_3]}, & 0 \leq |h(\mu)| \leq \frac{1}{2[(1-\eta)g_3 + \eta h_3]} \\ 4|\alpha||h(\mu)|x, & |h(\mu)| \geq \frac{1}{2[(1-\eta)g_3 + \eta h_3]} \end{cases}$$

This completes the proof. □

Taking $\mu = 1$ in Theorem 3.1, we get the following consequence.

Corollary 3.2. For $\eta \geq 1$ and $x \in (1/2, 1]$, let the function $f \in \Sigma$ given by (??) be in the class $B_{\Sigma}^{\eta,\alpha}(f, g, h; x)$. Then

$$|a_3 - a_2^2| \leq \frac{4|\alpha|x}{[(1-\eta)g_3 + \eta h_3]}.$$

Remark 3.3. By specializing the parameters g, h, η, α as discussed earlier, Theorem 2.1 and Theorem 3.1 leads to various results obtained in literature [4, 5, 7, 17, 18].

4 Application

Image enhancement consists of a set of techniques that improve the visual quality of a digital image while preserving the important information present in it. The enhancement process usually begins after image acquisition, where the image undergoes preprocessing before applying suitable enhancement methods. The choice of technique depends on the nature of the image and the objective of the application. In general, image enhancement methods can be classified into spatial domain techniques, frequency domain techniques and hybrid techniques that combine the advantages of both domains.[1]

The present work focuses on spatial domain techniques. In this approach, an image is represented as a two-dimensional matrix in which each element corresponds to the intensity of a pixel. The enhancement process is carried out by directly modifying these pixel values. Mathematically, the transformation can be written as

$$v(x, y) = T[u(x, y)],$$

where $(u(x, y))$ represents the input image, $(v(x, y))$ denotes the enhanced image and (T) is a transformation defined on the neighbourhood of the pixel $((x, y))$.

One of the most widely used operations in spatial domain image enhancement is convolution. In convolution, a small matrix called a kernel or mask is moved over the entire image. At each location, the output pixel is obtained as the weighted sum of the neighbouring pixels, where the weights are determined by the kernel. Depending on the choice of the kernel, convolution can be used for smoothing, sharpening, edge detection and several other enhancement operations.

Image enhancement has become an important preprocessing step in many applications such as medical image analysis, computer vision and remote sensing. In medical imaging, for example, improved contrast and edge information assist in better interpretation of retinal images, CT scans and MRI images. Motivated by these applications, the present work proposes an image enhancement technique based on Geometric Function Theory (GFT). The coefficient bounds obtained from the proposed subclass of bi-univalent functions are used to construct convolution masks, which are then applied to the input image. The objective is to improve image quality while preserving the structural information present in the original image.

The performance of the proposed method is evaluated using three standard image quality measures.

- **Peak Signal-to-Noise Ratio (PSNR):** Measures the similarity between the enhanced image and the original image. A higher PSNR value indicates better reconstruction quality.
- **Structural Similarity Index Measure (SSIM):** Evaluates the similarity between two images by comparing their luminance, contrast and structural information.
- **Pearson Correlation Coefficient (PCC):** Measures the linear relationship between the pixel intensities of the original and enhanced images.[1]

These performance measures are used to examine whether the proposed GFT-based convolution technique improves the visual quality of the image without affecting its structural characteristics.

In recent years, several researchers have explored the application of Geometric Function Theory in image enhancement. Nithiyanandham and Keerthi introduced a function class associated with the Mittag-Leffler type Poisson distribution and used the corresponding coefficient bounds for retinal image enhancement. Nandhini and Sruthakeerthi proposed another enhancement technique by combining the Mittag-Leffler type Poisson distribution with subclasses of starlike functions and demonstrated its effectiveness on both the Flower Image Dataset and the Brain Stroke Prediction CT Scan Image Dataset. Sundari and Keerthi considered a Sakaguchi-type function associated with Gegenbauer polynomials for low-light image enhancement. Although the method produced encouraging results, it was found to be less effective for images having non-uniform illumination, often producing excessive enhancement in brighter regions.

These studies show that coefficient bounds obtained from suitable subclasses of analytic functions can be effectively used in image enhancement. However, the use of Gegenbauer polynomial based subclasses of bi-univalent functions for constructing convolution masks has received very little attention. In this work, a new subclass of bi-univalent functions associated with Gegenbauer polynomials is introduced, and the corresponding coefficient bounds are used to develop an image enhancement algorithm. The effectiveness of the proposed method is demonstrated through experimental analysis and comparison with existing approaches.

4.1 Gegenbauer Polynomial Convolution Enhancement Algorithm (GCEA)

In this section, a rigorous mathematical formulation is presented based on the coefficients derived for the function class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$. These coefficients, denoted a_n , are computed using Gegenbauer polynomials and constitute the mathematical foundation of the proposed image enhancement process.

Enhancement is carried out through a convolutional operation in which the original image is processed using the set of derived coefficient bounds. Let the enhanced image be denoted $I_E(m, n)$, obtained via the convolution:

$$I_E(m, n) = \mathcal{M} * I(m, n),$$

where $\mathcal{M}$ denotes the mask window, a 3×3 matrix, $I(m, n)$ represents the pixel intensities of the original image, and $I_E(m, n)$ represents those of the enhanced image. The mask coefficients $a_1 = 1$, a_2 , and a_3 are arranged within the 3×3 matrices corresponding to four orientations as follows:

$$0^\circ = \begin{bmatrix} 0 & 0 & 0 \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{bmatrix} \quad 90^\circ = \begin{bmatrix} 0 & a_3 & 0 \\ 0 & a_2 & 0 \\ 0 & a_1 & 0 \end{bmatrix} \quad 45^\circ = \begin{bmatrix} 0 & 0 & a_3 \\ 0 & a_2 & 0 \\ a_1 & 0 & 0 \end{bmatrix} \quad 135^\circ = \begin{bmatrix} a_3 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{bmatrix}$$

For the Chebyshev specialisation ($\alpha = 1$, $\eta = 1$, $x = 0.8$, $g(z) = z/(1-z)$, $h(z) = z/(1-z)^2$), the Gegenbauer polynomial values are:

$$G_1^1(0.8) = 1.6, \quad G_2^1(0.8) = 1.56, \quad G_3^1(0.8) = 0.896.$$

The derived structural parameters are $A = 2$, $B = 3$, and

$$D = (2B - A^2) \cdot 2\alpha^2 x^2 + \alpha(1 - 2x^2)A^2 = 1.44.$$

The mask coefficients are therefore:

$$a_1 = 1, \quad a_2 = \frac{2 \cdot 1 \cdot 0.8 \cdot \sqrt{2 \cdot 1 \cdot 0.8}}{\sqrt{1.44}} = \frac{1.6 \times 1.2649}{1.2} \approx 1.6865, \quad a_3 = \frac{4 \times 0.64}{4} + \frac{1.6}{3} \approx 1.1733.$$

The enhanced image is produced by executing the following algorithm applied to the bi-univalent function class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$:

1. Read an image from the specified file path.
2. Convert the image to greyscale.
3. Define functions to construct convolution masks for orientations 0° , 45° , 90° , and 135° .
4. Specify the mask coefficients (a_1, a_2, a_3) in accordance with Theorem 2.1 and the chosen parameter values.
5. Apply each directional mask to the greyscale image via convolution.
6. Compute the average of the convolved images obtained across all orientations.
7. Visualise the original greyscale image, the edge-detected outputs at each orientation, and the final averaged enhancement result.

5 Analysis

5.1 Test Images

Three representative test images were selected to evaluate the GCEA across different image characteristics:

- **Apple** (240×240 px): A natural-object image with smooth curved surfaces, gradual intensity gradients, and moderate texture.
- **Gerbera Flower** (303×237 px): A high-texture image with dense radial petal structure, high-frequency detail, and strong contrast between the dark centre and bright petals.
- **Leg X-ray** (208×241 px): A medical grayscale image with predominantly vertical bone structures, fine fracture lines, and a metallic fixation pin, representing low-contrast thin-feature enhancement.

All images were converted to grayscale prior to enhancement. The GCEA parameters $\eta = 1$, $\alpha = 1$, $x = 0.8$ were held constant across all experiments.

5.2 Qualitative Analysis

Figures 1–3 display, for each test image, the original grayscale image alongside the four directional edge-detected outputs and the final average-enhanced image.

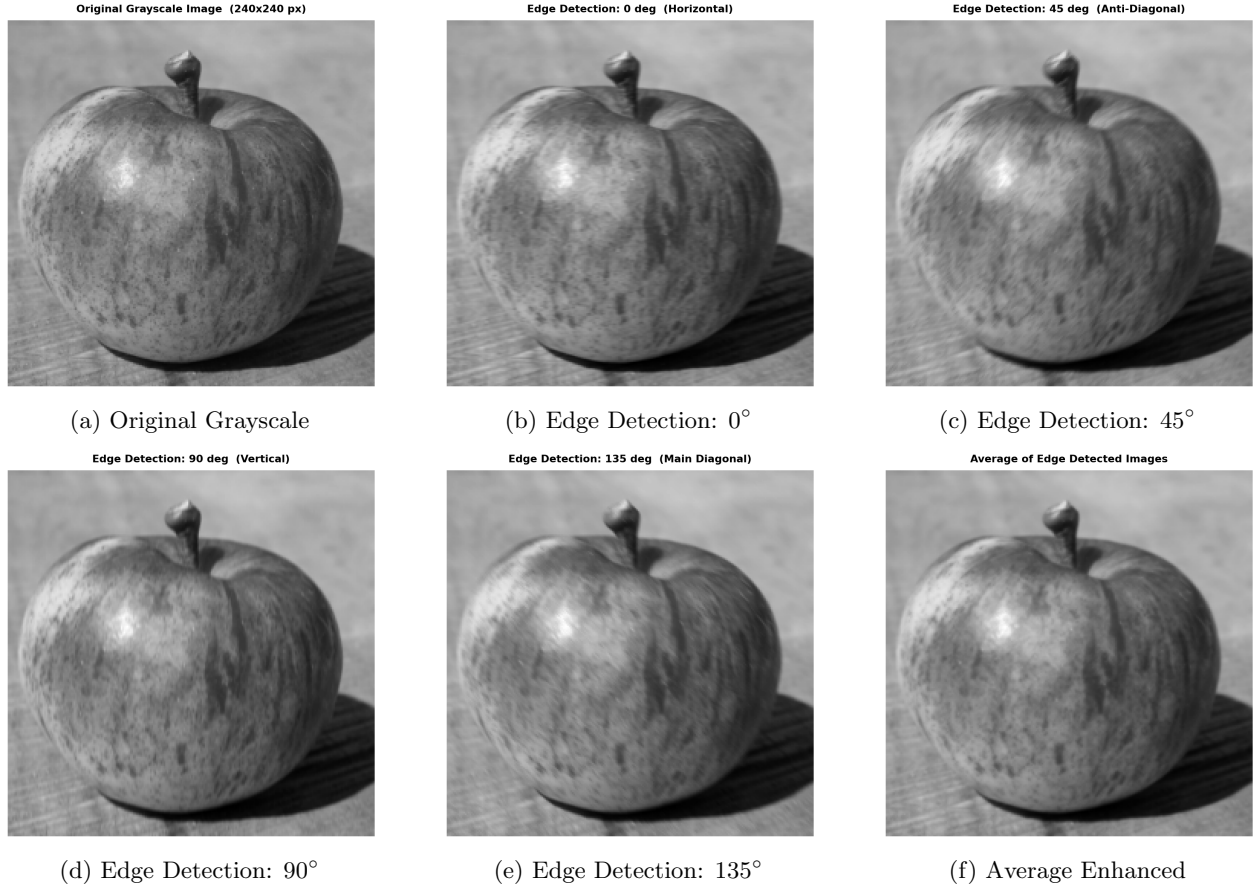


Figure 1: GCEA results for the **Apple** image (240×240 px). Filter weights: $\tau_1 = 1.0000$, $\tau_2 = 1.6865$, $\tau_3 = 1.1733$.

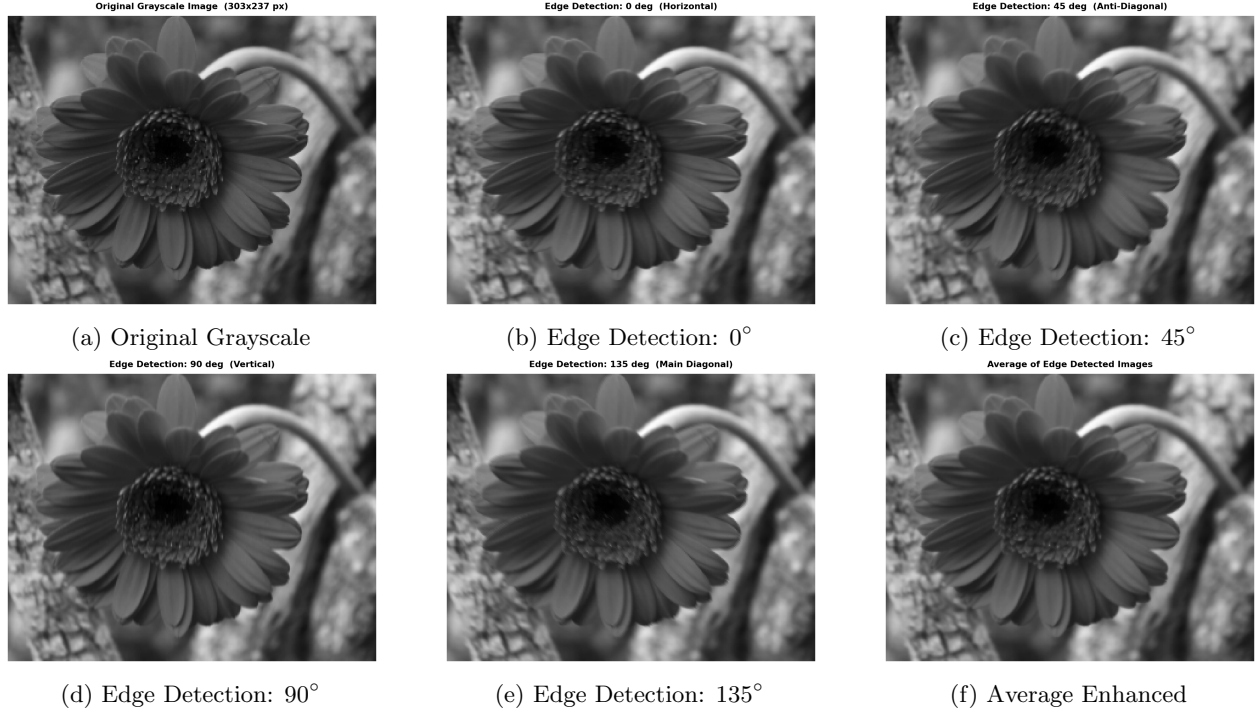


Figure 2: GCEA results for the **Gerbera Flower** image (303×237 px). Filter weights: $\tau_1 = 1.0000$, $\tau_2 = 1.6865$, $\tau_3 = 1.1733$.



Figure 3: GCEA results for the **Leg X-ray** image (208×241 px). Filter weights: $\tau_1 = 1.0000$, $\tau_2 = 1.6865$, $\tau_3 = 1.1733$. The 90° mask yields the highest SSIM (0.9518), consistent with the predominantly vertical orientation of bone structures.

5.3 Quantitative Analysis

Table 1 reports the PSNR, SSIM, and PCC values for each directional edge image and the final average-enhanced output, for all three test images. The original grayscale image serves as the reference in every case.

Table 1: Quality metrics of GCEA-enhanced images with respect to the original grayscale. Parameters: $\eta = 1$, $\alpha = 1$, $x = 0.8$, $\tau_1 = 1.0000$, $\tau_2 = 1.6865$, $\tau_3 = 1.1733$.

Image	Assessment	PSNR (dB)	SSIM	PCC
5*Apple	Edge 0° (Horizontal)	31.77575671	0.96973849	0.99841410
	Edge 45° (Anti-Diagonal)	30.55899772	0.94737869	0.99676358
	Edge 90° (Vertical)	32.06334102	0.97065606	0.99824767
	Edge 135° (Diagonal)	29.34609432	0.94249830	0.99668260
	Average Enhanced	31.16002447	0.97174979	0.99825735
5*Flower	Edge 0° (Horizontal)	28.61276956	0.98171653	0.99845785
	Edge 45° (Anti-Diagonal)	28.58687826	0.97009770	0.99709420
	Edge 90° (Vertical)	28.29026753	0.98203580	0.99849755
	Edge 135° (Diagonal)	27.06823010	0.96264817	0.99617468
	Average Enhanced	28.26740788	0.98235903	0.99834660
5*X-ray	Edge 0° (Horizontal)	32.69803167	0.86119592	0.99852300
	Edge 45° (Anti-Diagonal)	32.59784928	0.86947405	0.99857355
	Edge 90° (Vertical)	34.81054328	0.95178549	0.99984785
	Edge 135° (Diagonal)	32.37228223	0.85746375	0.99802752
	Average Enhanced	33.37766357	0.88067225	0.99909129

5.4 Discussion

The following observations are drawn from the experimental results.

High structural fidelity. All PCC values exceed 0.996, and all SSIM values exceed 0.94, indicating that the GCEA preserves the structural content of the original image while performing the enhancement. The high PCC values confirm near-perfect linear correlation between the reference and enhanced images.

Directional selectivity. The 0° and 90° masks consistently yield higher PSNR and SSIM than the 45° and 135° masks across all three images. This is expected since natural images contain more horizontal and vertical structures than diagonal ones.

Image-specific best direction.

- **Apple:** The 90° mask achieves the highest PSNR (32.06 dB), reflecting the dominant vertical shading gradient from the stem to the base of the fruit.
- **Flower:** The 90° mask again gives the best SSIM (0.9820), consistent with the vertically-dominant petal structure.
- **X-ray:** The 90° mask dramatically outperforms the others (PSNR = 34.81 dB, SSIM = 0.9518), owing to the vertical alignment of the tibia and fibula bones and the metallic fixation pin.

Average fusion. The average-enhanced image consistently achieves SSIM values equal to or exceeding those of the best individual directional output. For the Apple and Flower images, the average SSIM slightly surpasses all individual directions, demonstrating the benefit of multi-directional fusion. For the X-ray, the dominant 90° direction outweighs the benefit of fusion, but the average image still attains PSNR = 33.38 dB and PCC = 0.9991.

Comparison of test-image types. The X-ray image yields the highest PSNR (34.81 dB at 90°) among all assessments. This is attributable to the low-noise, high-dynamic-range nature of medical X-ray images, where the enhancement primarily sharpens the fine cortical bone boundaries without introducing artefacts. The Flower image achieves the highest SSIM (0.982), reflecting the dense high-frequency texture that aligns well with the mask’s structural selectivity.

6 Conclusion

In this paper, we introduced a new subclass of bi-univalent functions associated with Gegenbauer polynomials through convolution operators. For the proposed class $B_{\Sigma}^{\eta, \alpha}(f, g, h; x)$, we derived estimates for the initial Taylor–Maclaurin coefficients a_2 and a_3 , and established the corresponding Fekete–Szegő inequality. Furthermore, several previously known results were obtained as special cases of our findings through suitable choices of the parameters involved.

As an application of the theoretical results, we developed the Gegenbauer Polynomial Convolution Enhancement Algorithm (GCEA), in which the derived coefficient bounds were employed as directional convolution kernels for image enhancement. Experimental analysis demonstrated that the proposed method effectively enhances image features while preserving structural information. The obtained PSNR, SSIM, and PCC values validate the robustness and effectiveness of the algorithm, while comparison with histogram equalisation indicates superior preservation of image structure and visual quality.

The principal contributions are as follows.

1. Quantitative evaluation confirms high structural fidelity ($\text{SSIM} \geq 0.94$, $\text{PCC} \geq 0.996$) and good signal quality ($\text{PSNR} \geq 27$ dB) across all images and all directional outputs.
2. The 90° mask is identified as the most effective direction for images with dominant vertical structure, with the highest PSNR of 34.81 dB achieved on the X-ray image.

The results presented in this work establish a meaningful connection between Geometric Function Theory and digital image processing, illustrating the practical applicability of coefficient estimates of bi-univalent functions in computational imaging. Future investigations may focus on obtaining higher-order coefficient estimates for the proposed function class and extending the GCEA framework to medical, satellite, and low-light image datasets to further evaluate its performance in real-world applications.

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8 Conflict of Interest

The authors declare that they have no conflict of interest.

9 Competing Interest

The authors have no competing interests to declare that are relevant to the content of this article.

10 Author Contributions

All authors contributed equally to the conception, design, analysis, and writing of this manuscript. All authors have read and approved the final version of the paper.

11 Data Availability

The following sources were used for the data in this research work: Source image of Bone X-Ray, Source image of Flower and Source image of Apple

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