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Research Article

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Controlling the Condition Number of Multiquadric RBF Matrices via Poisson Disk Sampling

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Abstract

Radial Basis Function (RBF) interpolation is a powerful tool for approximating functions from scattered data. However, the resulting interpolation matrix, especially when using the multiquadric (MQ) function, is often ill-conditioned, affecting numerical stability. This work investigates the relationship between the geometric distribution of interpolation centers and the condition number of the MQ matrix. Essentially, we impose a Poisson disk sampling constraint, which ensures a minimum distance $\boldsymbol{h}$ between points. Using asymptotic spectral analysis and matrix perturbation theory, we derive approximate expressions for the eigenvalues and the condition number $\boldsymbol{\kappa(A)}$ as functions of the number of points $\boldsymbol{N}$, the shape parameter $\boldsymbol{c}$, and the characteristic distance $\boldsymbol{r_0}$. We show that the condition number can be minimized, reaching a constant value independent of $\boldsymbol{N}$, when the average distance between points satisfies a specific functional relationship with $\boldsymbol{c}$ and $\boldsymbol{N}$. Numerical experiments in $\mathbb{R}^{\boldsymbol{2}}$ confirm the theoretical predictions, demonstrating that an adaptive choice of the Poisson disk minimum distance effectively controls the growth of ill-conditioning, unlike fixed or arbitrary choices.

Keywords: Radial basis functions; Multiquadric; Condition number; Poisson disk sampling; Spectral analysis; Meshless methods

1 Introduction

Scattered data interpolation in multiple dimensions is a fundamental problem in various areas of scientific computing, including interpolation, machine learning, computer graphics sampling, geostatistics, and solutions of partial differential equations (PDEs). Among the various techniques developed for this purpose, radial basis functions (RBFs) stand out for their ability to construct smooth and highly accurate approximations from irregularly distributed datasets.

Among the various radial functions proposed in the literature, the multiquadric function plays a prominent role. Introduced by Hardy [1, 2] in the context of topographic surface interpolation, this function has demonstrated excellent performance in reconstructing surfaces from scattered geodetic data. Subsequently, important theoretical studies established the mathematical foundations of RBF interpolation, including results on the existence and uniqueness of the interpolation problem solution [3]. The multiquadric (MQ) function offers significant advantages in solving problems with strong discontinuities, particularly when used in the context of PDE solutions. As highlighted by [4, 5], one of the main advantages of the multiquadric function in meshless methods is its geometric flexibility. Since the MQ does not depend on a rigid point structure, it can efficiently handle irregular geometries and discontinuities in the domain.

One advantage of the multiquadric function is the possibility of varying its shape parameter locally, as proposed by [6], who demonstrated that consistent cloud refinement combined with variable shape parameters improves the accuracy of solutions to the Poisson equation and fluid dynamics problems.

Despite the excellent approximation properties of multiquadric functions, their practical application presents important numerical challenges. One of the best-known problems is the ill-conditioning of the interpolation matrix, which can arise especially when interpolation points or centers become very close to each other or when the shape parameter takes high values. This behavior has been extensively investigated in the numerical analysis literature and is a fundamental aspect of efficient RBF-based method implementation [7, 8].

The multiquadric, specifically, is a popular RBF due to its spectral accuracy [9, 10]. However, according to [11], matrix conditioning can become problematic when collocation nodes are close together or when the number of points N grows in a fixed domain of size L.

The article [12] proposed the use of the MQ function as the radial basis function and reviews the multiquadric method, which is an exact interpolation scheme for scattered data. The interpolating function $s(\mathbf{x})$ is a linear combination of shifts of this function, centered at each data point. Therefore, given a function $f : D \subset \mathbb{R}^d \to \mathbb{R}$, known only through sample values $f(\mathbf{x}_i)$ at a set of scattered points

$$\Omega = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset D,$$

we seek to construct an interpolating function $s(\mathbf{x})$ such that

$$s(\mathbf{x}_i) = f(\mathbf{x}_i), \qquad i = 1, \ldots, N.$$

In multiquadric radial basis function interpolation, the approximation is given by

$$s(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j \phi(\|\mathbf{x} - \mathbf{x}_j\|),$$

where

$$\phi(r) = \sqrt{r^2 + c^2}, \qquad r \geq 0, \quad c > 0,$$

and the coefficients λ_j are determined by imposing the interpolation conditions

$$s(\mathbf{x}_i) = \sum_{j=1}^{N} \lambda_j \phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = f(\mathbf{x}_i), \qquad i = 1, \ldots, N. \tag{1}$$

Here $\|\mathbf{x}_i - \mathbf{x}_j\|$ is the Euclidean distance between evaluation points $\mathbf{x}_i$ and centers $\mathbf{x}_j$. The constant c is called the shape parameter, which controls the smoothness of the function; $c > 0$ ensures that the MQ function is always non-singular at the origin. The scalars λ_j are the weight coefficients to be determined. Applying the interpolation conditions yields the $N \times N$ linear system:

$$\begin{bmatrix} \phi(\|\mathbf{x}_1 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_1 - \mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_1 - \mathbf{x}_N\|) \\ \phi(\|\mathbf{x}_2 - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_2 - \mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_2 - \mathbf{x}_N\|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(\|\mathbf{x}_N - \mathbf{x}_1\|) & \phi(\|\mathbf{x}_N - \mathbf{x}_2\|) & \cdots & \phi(\|\mathbf{x}_N - \mathbf{x}_N\|) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f(\mathbf{x}_1) \\ f(\mathbf{x}_2) \\ \vdots \\ f(\mathbf{x}_N) \end{bmatrix}. \tag{2}$$

In matrix notation, $A\Lambda = F$. Once the linear system (2) is solved and the weight vector $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T$ is obtained, the function (1) can be evaluated at any point.

Substituting $\phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = \sqrt{r_{ij}^2 + c^2}$, with $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$, the matrix A is written as:

$$A = \begin{bmatrix} \sqrt{r_{11}^2 + c^2} & \sqrt{r_{12}^2 + c^2} & \cdots & \sqrt{r_{1N}^2 + c^2} \\ \sqrt{r_{21}^2 + c^2} & \sqrt{r_{22}^2 + c^2} & \cdots & \sqrt{r_{2N}^2 + c^2} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{r_{N1}^2 + c^2} & \sqrt{r_{N2}^2 + c^2} & \cdots & \sqrt{r_{NN}^2 + c^2} \end{bmatrix}. \tag{3}$$

Matrix A is radially symmetric since $\phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = \phi(\|\mathbf{x}_j - \mathbf{x}_i\|)$; dense, since $c > 0$; has computational cost $\mathcal{O}(N^2)$; and according to [3], matrix A is non-singular provided $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ are distinct and the parameter $c > 0$. In general, many authors highlight the advantage of guaranteed invertibility of matrix A, but dedicate much of their research effort to dealing with the numerical difficulties arising from its ill-conditioning.

Authors such as [13, 14] point out that a small fill distance, i.e., very close points, leads to a more accurate approximation of the true function. However, this makes the columns of matrix A almost linearly dependent, since the basis functions centered at these points are very similar, making the condition number arbitrarily high. [14] established theoretical results relating interpolation error and the condition number of RBF matrices, showing that such values depend on the geometry of centers, the fill distance, and the minimum separation.

Subsequent studies on MQ-RBF showed that increasing the shape parameter can improve approximation accuracy, but at the cost of a severe increase in the condition number, especially when centers are non-uniformly distributed.

In this work, we discuss the origin of the ill-conditioning problem in multiquadric function interpolation, with particular emphasis on the relationship between matrix conditioning and the distance between interpolation points. The distribution and density of the point cloud directly determine the quality, accuracy, and smoothness of the multiquadric RBF surface. Without a point cloud, there are no centers for the radial basis functions, and therefore interpolation simply does not exist.

There are point generation techniques such as pseudo-random generation (Fig. a), Delaunay triangulation (Fig. b), and Poisson disk sampling (Fig. c).

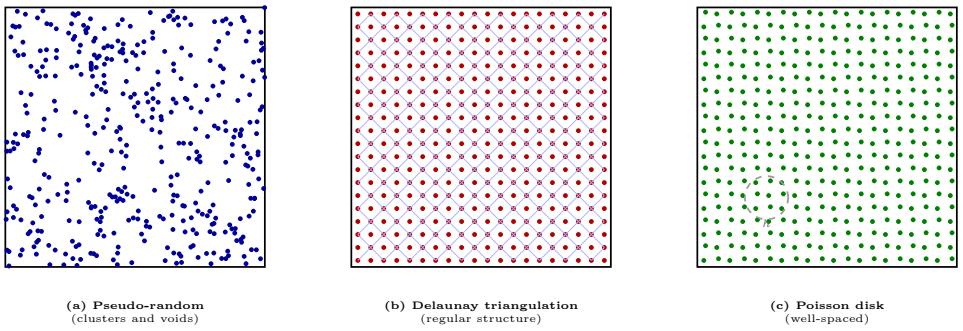


Figure 1 Comparison of different point cloud generation techniques in $[0,1]^2$: (a) pseudo-random distribution, exhibiting clusters and voids; (b) Delaunay triangulation, producing a regular structure; (c) Poisson disk distribution with minimum radius h, combining randomness and uniform spacing.

The first technique generates poor clouds because each coordinate of each new point is chosen randomly, disregarding existing points, automatically generating dense and sparse regions. The second technique generates point clouds from Delaunay triangulation. Once a mesh with nearly equilateral triangles (good quality) is guaranteed, vertices are extracted and the mesh structure is discarded, obtaining a good quality point cloud. However, mesh generation is indeed one of the most complex, time-consuming, and critical pre-processing steps in simulations with irregular geometries.

Poisson Disk distribution is the most advantageous technique when one needs spatial sampling that is random and uniform without being clustered or structured, overcoming the recurring problem of random point generators as well as mesh generators. This method is called Poisson disk packing, and it naturally and easily fills a space, avoiding both the clusters and voids of randomness and the regularity of a mesh.

The Poisson disk distribution of [15] does not need to generate a mesh structure to obtain nodes; furthermore, while Delaunay triangulation algorithms have complexity $\mathcal{O}(N \log N)$, Poisson disk distribution is $\mathcal{O}(N)$. Its main algorithm is efficient

and widely used in computer graphics and games to create natural, high-quality distributions.

The Poisson disk point distribution is not a probability distribution in the traditional statistical sense, **but maintains the idea of spatial randomness**. The Poisson disk distribution consists of a spatial pattern of points generated by a stochastic process under a minimum distance constraint. Mathematically, such a distribution can be defined as follows:

Let $D \subset \mathbb{R}^d$ be a domain (e.g., a square or cube). A set of points $\Omega = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ in D is a Poisson Disk sample with radius h if:

(i) **Minimum point spacing**: For any pair of distinct points $\mathbf{x}_i$ and $\mathbf{x}_j$ in the sample, the Euclidean distance satisfies:

$$\|\mathbf{x}_i - \mathbf{x}_j\| \geq h$$

where $h > 0$ is the exclusion radius. This ensures points are well-spaced, avoiding clusters.

(ii) **Maximum domain coverage**: The set of sampled points covers the domain D such that:

$$D \subseteq \bigcup_{i=1}^{N} \overline{B}(\mathbf{x}_i, h)$$

where $\overline{B}(\mathbf{x}_i, h)$ is the closed n-dimensional ball centered at $\mathbf{x}_i$ with radius h. This ensures that every point in D is at most a distance h from some sampled point, guaranteeing complete domain coverage. These aspects of Poisson disk distribution can be visualized in Figure 2.

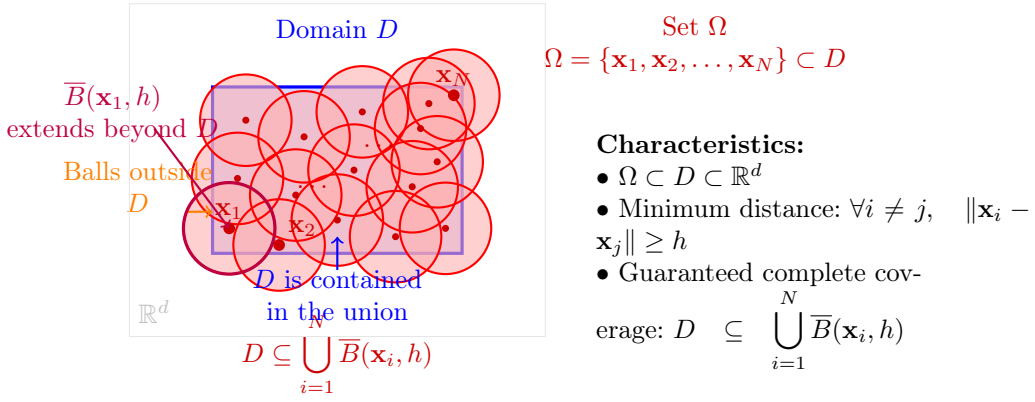


Figure 2 Schematic representation of Poisson disk distribution. Domain D is covered by the union of closed balls of radius h centered at points $\mathbf{x}_i$, guaranteeing the coverage property, while the minimum distance h between any two centers is respected.

In other words, it is a sphere packing (or circles, in $\mathbb{R}^2$) where the centers of the spheres are the sample points. Unlike a regular grid, point positions have a random component, but unlike a random sample, it prevents points from clustering.

The most common method for generating such samples is Poisson rejection sampling, proposed by [16] and later refined by [15] with an efficient algorithm.

In many practical applications, a point distribution without clustering is desirable, modeled by the Poisson Disk distribution. In game object placement, to distribute trees, rocks, and bushes realistically on terrain, avoiding overlap. In geostatistics, to define sampling locations on terrain avoiding clustering bias.

Therefore, this work seeks to answer the following question: **Given the Poisson Disk restriction described in (i) and (ii) for the set $\Omega \subset [0, L]^d \subset \mathbb{R}^d$, what is the optimal configuration of points with h for matrix (3) that minimizes $\kappa(A)$?**

Problem Formulation

Let $\Omega = \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N\} \subset [0, L]^d$ be a set of points. Matrix $A \in \mathbb{R}^{N\times N}$ is symmetric and defined by the multiquadric radial basis function:

$$\phi(\|\mathbf{x}_i - \mathbf{x}_j\|) = \sqrt{\|\mathbf{x}_i - \mathbf{x}_j\|^2 + c^2}, \quad c > 0.$$

The condition number in the 2-norm is given by:

$$\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)},$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the largest and smallest singular values of A, respectively.

Thus, we wish to determine:

$$\min_{\mathbf{x}_1,\ldots,\mathbf{x}_N \in [0,L]^d} \kappa(A)$$

subject to the Poisson Disk constraint:

$$\|\mathbf{x}_i - \mathbf{x}_j\| \geq h \quad \forall i \neq j.$$

It is known that $\kappa(A) \to \infty$ if $\min_{i\neq j} \|\mathbf{x}_i - \mathbf{x}_j\| \to 0$ or if $N \to \infty$ in a fixed domain L.

For symmetric matrices, singular values coincide with the absolute values of eigenvalues, i.e., $\sigma_i = |\lambda_i|$. Therefore,

$$\kappa_2(A) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}. \tag{4}$$

Note that in the case of the multiquadric function, matrix A is symmetric but not necessarily positive definite. The MQ is conditionally positive definite of order 1 [3],

i.e., A is positive definite on the subspace of vectors whose components sum to zero. Thus, A may have eigenvalues of different signs. Therefore, with A non-singular, its spectral condition number is given by (8).

2 Spectral Analysis of the Interpolation Matrix

The multiquadric basis function expanded in Taylor series around $r = 0$ is:

$$\phi(r) = \sqrt{r^2 + c^2} = c + \frac{r^2}{2c} - \frac{r^4}{8c^3} + \frac{r^6}{16c^5} - \frac{5r^8}{128c^7} + \mathcal{O}(r^{10}). \tag{5}$$

This expansion is valid when $r \ll c$. For Poisson disk distributions in a fixed domain L, the minimum distance h is of order $L/\sqrt{N}$. We assume $h \ll c$ to guarantee approximation accuracy. The truncation error is $\mathcal{O}(r^4/c^3)$, which will be neglected in the following analysis.

Matrix A can be written as:

$$A = cJ + \frac{1}{2c}R^{(2)} - \frac{1}{8c^3}R^{(4)} + \cdots, \tag{6}$$

where J is the matrix whose elements $J_{ij} = 1$ for all i, j and

$$R_{ij}^{(k)} = \|\mathbf{x}_i - \mathbf{x}_j\|^k \quad \text{for } i \neq j, \qquad R_{ii}^{(k)} = 0. \tag{7}$$

Spectral analysis of a matrix is fundamental for understanding the stability and numerical conditioning of interpolation problems. The objective of this section is to approximate the eigenvalues of matrix A, generated by the multiquadric function, to understand how they behave as a function of the number of points N, the characteristic distance r_0 between them, and the shape parameter c.

2.1 Simplification for Quasi-Uniform Distributions

The analysis becomes tractable if we assume points are quasi-uniformly distributed in the domain. In this case, we can characterize the distribution by a typical distance r_0. The crucial approximation is that for most pairs of points $i \neq j$, distances are approximately equal to r_0. This allows us to write:

$$R_{ij}^{(2)} \approx r_0^2, \quad R_{ij}^{(4)} \approx r_0^4, \quad i \neq j.$$

Thus, matrices $R^{(2)}$ and $R^{(4)}$ can be approximated by a structure depending on matrix J and the identity I:

$$R^{(2)} \approx r_0^2(J - I), \quad R^{(4)} \approx r_0^4(J - I).$$

Substituting these approximations into the expression for A, we obtain:

$$A \approx cJ + \frac{1}{2c}r_0^2(J - I) - \frac{1}{8c^3}r_0^4(J - I).$$

Rearranging terms to group contributions from I and J, we arrive at the fundamental form:

$$A \approx \left[c - \frac{r_0^2}{2c} + \frac{r_0^4}{8c^3}\right] I + \left[c + \frac{r_0^2}{2c} - \frac{r_0^4}{8c^3}\right] J.$$

This is a linear combination of matrices I and J, i.e., $A = \alpha I + \beta J$, where:

$$\alpha = c - \frac{r_0^2}{2c} + \frac{r_0^4}{8c^3}, \quad \beta = c + \frac{r_0^2}{2c} - \frac{r_0^4}{8c^3}.$$

2.2 Eigenvalues and Eigenvectors of $A = \alpha I + \beta J$

Lemma 2.1. *For matrix $J \in \mathbb{R}^{N\times N}$ with $J_{ij} = 1$ for all i, j, the eigenvalues and eigenvectors are:*

- *$\lambda_1 = N$, with eigenvector $v_1 = (1, 1, \ldots, 1)^T$;*
- *$\lambda_i = 0$ for $i = 2, \ldots, N$, with eigenvectors w satisfying $\sum_{j=1}^{N} w_j = 0$ (orthogonal to v_1).*

Proof. The proof follows directly from properties of J: $Jv_1 = Nv_1$ and, for any w with zero sum, $Jw = 0$. □

Proposition 2.1. *For matrix $A = \alpha I + \beta J$, the eigenvalues are:*

- *$\lambda_1 = \alpha + \beta N$, with eigenvector $v_1 = (1, \ldots, 1)^T$;*
- *$\lambda_i = \alpha$ for $i = 2, \ldots, N$, with eigenvectors w orthogonal to v_1.*

Proof. Applying A to v_1: $Av_1 = \alpha v_1 + \beta Jv_1 = \alpha v_1 + \beta Nv_1 = (\alpha + \beta N)v_1$. For w with zero sum: $Aw = \alpha w + \beta Jw = \alpha w$. □

In the context of our spectral analysis of the interpolation matrix, we have the specific values of α and β:

$$\alpha = c - \frac{r_0^2}{2c} + \frac{r_0^4}{8c^3}, \quad \beta = c + \frac{r_0^2}{2c} - \frac{r_0^4}{8c^3}.$$

Substituting:

- For eigenvector v_1 (constant mode):

$$\lambda_1 = \alpha + \beta N = c(N + 1) + \frac{N - 1}{2c} r_0^2 - \frac{N - 1}{8c^3} r_0^4.$$

- For the $N - 1$ eigenvectors with zero sum (oscillatory modes):

$$\lambda_i = \alpha = c - \frac{1}{2c} r_0^2 + \frac{1}{8c^3} r_0^4, \quad i = 2, \ldots, N.$$

Therefore, the approximate eigenvalues of the interpolation matrix A are:

$$\boxed{\begin{aligned} \lambda_1 &\approx c(N + 1) + \frac{N - 1}{2c} r_0^2 - \frac{N - 1}{8c^3} r_0^4, \\ \lambda_i &\approx c - \frac{1}{2c} r_0^2 + \frac{1}{8c^3} r_0^4, \quad i = 2, \ldots, N. \end{aligned}}$$

This spectral structure reveals crucial information:

- **Eigenvector v_1 (constant mode)**: Represents a constant function over all points. The eigenvalue λ_1 is dominant and grows linearly with the number of points N.
- **Zero-sum eigenvectors (oscillatory modes)**: Represent functions that oscillate so that their average over all points is zero. These $N-1$ modes are equally amplified by a factor α, which is approximately constant and positive for c sufficiently large relative to r_0.
- **Condition Number**: The condition number of matrix A is given by the ratio $\kappa(A) = \lambda_1/\lambda_i$. This analysis shows that for sufficiently large c such that $\alpha > 0$:

$$\kappa(A) \approx \frac{c(N+1) + \frac{N-1}{2c} r_0^2}{c - \frac{r_0^2}{2c}} \approx N + 1,$$

  considering only the leading terms. The condition number grows linearly with N.
- **Practical Implications**: To ensure numerical stability, it is necessary to balance c and point density.

2.3 Approximate Condition Number

$$\kappa(A) \approx \frac{c(N+1) + \frac{N-1}{2c} r_0^2 - \frac{N-1}{8c^3} r_0^4}{c - \frac{1}{2c} r_0^2 + \frac{1}{8c^3} r_0^4} \tag{8}$$

3 Condition Function Analysis and Minimization

Consider the function $f(r_0) = \kappa(A)$ given by (8). To minimize $\kappa(A)$ subject to $\|\mathbf{x}_i - \mathbf{x}_j\| \geq r_{\min}$, the optimal configuration is an arrangement where all points are equally spaced with distance $r_{\min}$ (a regular pattern, such as a triangular or hexagonal lattice in 2D).

To simplify the analysis, set $x = r_0^2$. The derivative of f with respect to x is:

$$f'(x) = \frac{N\left(1 - \frac{x}{2c^2}\right)}{\left(c - \frac{x}{2c} + \frac{x^2}{8c^3}\right)^2}.$$

The critical point occurs when $1 - \frac{x}{2c^2} = 0$, i.e., $x = 2c^2$, which implies:

$$r_0^* = c\sqrt{2}. \tag{9}$$

The second derivative confirms this is a minimum. Substituting $r_0 = c\sqrt{2}$:

$$\alpha = \frac{c}{2}, \quad \beta = \frac{3c}{2}, \quad \lambda_1 = \frac{c(3N+1)}{2}, \quad \lambda_i = \frac{c}{2}.$$

Therefore:

$$f(c\sqrt{2}) = \frac{\frac{c(3N+1)}{2}}{\frac{c}{2}} = 3N + 1.$$

3.1 Minimum Condition Number Value

- **Unconstrained Case:** If $r_{\min} \leq c\sqrt{2}$, the minimum is attained at $r_0 = c\sqrt{2}$:

$$\kappa_{\min} = 3N + 1. \tag{10}$$

- **Constrained Case:** If $r_{\min} > c\sqrt{2}$, the minimum is attained at the boundary $r_0 = r_{\min}$:

$$\kappa_{\min} = \frac{c(N+1) + \frac{N-1}{2c} r_{\min}^2 - \frac{N-1}{8c^3} r_{\min}^4}{c - \frac{1}{2c} r_{\min}^2 + \frac{1}{8c^3} r_{\min}^4}. \tag{11}$$

4 Spectral Analysis for Unstructured Points

Generally, for scattered points $\mathbf{x}_1, \mathbf{x}_2, \cdots \mathbf{x}_N$ in $\mathbb{R}^d$, matrix A is of order $N \times N$ with form

$$A_{ij} = \sqrt{r_{ij}^2 + c^2}, \tag{12}$$

with $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|_2$. Using the first-order Taylor expansion $\sqrt{r^2 + c^2} \approx c + \frac{r^2}{2c}$, we can write $A \approx A_0 + \Delta A$, where $(A_0)_{ij} = c$ for all i, j and $(\Delta A)_{ij} = \frac{r_{ij}^2}{2c}$ for $i \neq j$, and $(\Delta A)_{ii} = 0$.

Matrix A_0 has one eigenvalue Nc (constant eigenvector) and $N-1$ zero eigenvalues. Using first-order perturbation theory, we obtain:

$$\lambda_{\max} = Nc + \frac{1}{Nc} \sum_{i<j} r_{ij}^2, \qquad \lambda_{\min} \approx -\frac{1}{cN} \sum_{i<j} r_{ij}^2. \tag{13}$$

The condition number is then:

$$\kappa(A) = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right| = 1 + \frac{N^2 c^2}{\sum_{i<j} r_{ij}^2}. \tag{14}$$

For large N, if distances have mean μ_r and variance σ_r^2, then $\sum_{i<j} r_{ij}^2 \approx \frac{N(N-1)}{2}(\mu_r^2 + \sigma_r^2)$, and

$$\kappa(A) \approx 1 + \frac{2c^2}{\mu_r^2 + \sigma_r^2}.$$

4.1 Minimum Condition Number Calculation

Note that N is a positive integer. The following analysis uses derivatives with respect to N, treating it as a continuous variable. This is done to study the sensitivity of the condition number κ to variations in problem size and should be interpreted as an asymptotic analysis for large N, where finite differences can be approximated by

derivatives. The function $\kappa(N)$, originally defined for integers, is extended to a smooth function $\tilde{\kappa}(x)$ with $x \in \mathbb{R}^+$, of which $\kappa(N)$ is the restriction. The optimality conditions derived below refer to this continuous extension and are valid in the large N limit.

Assuming the condition number as a function κ dependent on N, where

$$\kappa(N) = 1 + \frac{2Nc^2}{(N-1)\sigma_{\text{eff}}^2}, \tag{15}$$

with $c > 0$, $\sigma_{\text{eff}}^2 = \mu_r^2 + \sigma_r^2 > 0$ and $N > 1$. Suppose c and σ_{eff}^2 depend on N differentiably. The critical point condition $\frac{d\kappa}{dN} = 0$ leads, after algebraic manipulation, to the differential equation:

$$\frac{2}{c} \cdot \frac{dc}{dN} - \frac{1}{\sigma_{\text{eff}}^2} \cdot \frac{d\sigma_{\text{eff}}^2}{dN} = \frac{1}{N(N-1)}. \tag{16}$$

Integrating, we obtain the functional relation at the optimal point:

$$\frac{c^2}{\sigma_{\text{eff}}^2} = K \cdot \frac{N-1}{N}, \tag{17}$$

or equivalently:

$$\sigma_{\text{eff}} = \frac{c}{\sqrt{K}} \cdot \sqrt{\frac{N}{N-1}}. \tag{18}$$

where $K = e^C > 0$ is an integration constant.

Constant K can be chosen to ensure the minimum distance $r_{\min} = \sqrt{\sigma_{\text{eff}}^2}$ respects the Poisson disk constraint. For practical applications, $K = 1$ is suggested, yielding $r_{\min}(N) = c\sqrt{(N-1)/N}$, which tends to c as $N \to \infty$.

Substituting (17) into (15), we obtain the condition number value at the critical point:

$$\kappa(N) = 1 + \frac{2N}{N-1} \cdot \left[K \cdot \frac{N-1}{N} \right] = 1 + 2K. \tag{19}$$

This is a fundamental result: for any N satisfying relation (17), the condition number κ is constant, independent of N.

The second derivative test confirms this critical point is a local minimum, since $\frac{d^2\kappa}{dN^2}\Big|_{\text{critical point}} = \frac{4K}{N(N-1)^2} > 0$.

The interpretation of results allows us to conclude:

1. The first-order condition leads to the functional relation (17).
2. When this relation is satisfied, $\kappa = 1 + 2K$, constant.
3. The second derivative test confirms it is a local minimum.
4. The theoretical lower bound occurs when $K \to 0^+$, implying $\kappa \to 1^+$.
5. Any deviation from the optimal relation produces larger κ values.

5 Numerical Experiments

5.1 Objective

The objective of this section is to numerically investigate the influence of the choice of the minimum separation parameter $r_{\min}$, associated with center distribution, on the conditioning of the multiquadric matrix used in radial basis function interpolation methods.

Consider the multiquadric matrix

$$A_{ij} = \sqrt{\|x_i - x_j\|^2 + c^2}, \tag{20}$$

where $c > 0$ is a fixed parameter and centers $\{x_i\}_{i=1}^{N} \subset [0,1]^2$ are generated via Poisson disk-type distributions characterized by a minimum distance $r_{\min}$ between points. The interpolation matrix A has dimension $N \times N$ where N is the number of points (centers) in set Ω.

The main focus is to evaluate how different choices of $r_{\min}$ affect the condition number 4, as a function of the number of centers N. The condition number was computed using Python routines on a computer with double precision (64 bits).

5.2 Experimental Setup

The numerical experiments compare three distinct strategies for choosing the parameter $r_{\min}$:

1. **Non-optimized distribution**, with a fixed value

$$r_{\min} = 0.05, \tag{21}$$

   independent of the number of centers N.

2. **Optimization based on equation 9 for quasi-uniform points**

$$r_{\min} = c\sqrt{2}. \tag{22}$$

3. **Theoretical N-dependent optimization for unstructured points**, obtained from equation 18, given by

$$r_{\min}(N) = \frac{c}{\sqrt{K}}\sqrt{\frac{N}{N-1}}, \tag{23}$$

   where $K = e^{-2.9}$ is a constant chosen to minimize the average condition number over all computational simulations.

In all cases, the MQ shape parameter is kept fixed at $c = 0.15$. For each value of N, independent center distributions are generated according to each strategy, the corresponding MQ matrix is constructed, and the spectral condition number $\kappa(A)$ is computed.

5.3 Theoretical Reference Values

Spectral analysis of the multiquadric matrix leads to the rational function

$$f(r_0) = \frac{c(N+1) + \frac{N-1}{2c} r_0^2 - \frac{N-1}{8c^3} r_0^4}{c - \frac{1}{2c} r_0^2 + \frac{1}{8c^3} r_0^4}, \tag{24}$$

which provides a theoretical approximation for the ratio between the extreme eigenvalues of the matrix.

Substituting $r_0 = c\sqrt{2}$ into this expression yields exactly

$$f(c\sqrt{2}) = 3N + 1. \tag{25}$$

This result predicts linear growth of the condition number for the choice $r_{\min} = c\sqrt{2}$.

On the other hand, the N-dependent choice for $r_{\min}$ leads to a uniform upper bound for the condition number, given by

$$\kappa(A) \leq 1 + 2K = 1.110046440112814. \tag{26}$$

Both theoretical values are included in the numerical plots as reference curves.

5.4 Results Discussion

The numerical results are presented in Tables 1 and 2. For the non-optimized strategy ($r_{\min} = 0.005$), $\kappa(A)$ grows dramatically from 6.98×10^1 to 6.33×10^{17} as N increases from 5 to 1955, evidencing severe ill-conditioning. The strategy with $r_{\min} = c\sqrt{2}$ yields values between 30.57 and 452.00, with a mean of 285.47, significantly lower than the theoretical prediction $3N + 1$ (which would reach 5866 for $N = 1955$). Remarkably, the N-dependent optimized strategy maintains $\kappa(A)$ in an exceptionally low range between 1.00 and 10.48 (mean 4.14) for the entire range of N analyzed, confirming the theoretical prediction that it is possible to keep the condition number constant and close to $1+2K \approx 1.11$, regardless of the number of centers. These results can be verified in Figure 3. Computing $\kappa(A)$ for each matrix has cost $\mathcal{O}(N^3)$ due to the required spectral decomposition.

6 Conclusion

This paper investigated the relationship between the geometric distribution of interpolation centers and the condition number $\kappa(A)$ of the multiquadric radial basis function (MQ) matrix under Poisson disk constraints. The main contributions can be summarized as follows.

Theoretical results.

Using asymptotic spectral analysis and first-order matrix perturbation theory, we derived explicit approximations for the eigenvalues of the MQ matrix. For quasi-uniform point distributions characterized by a typical distance r_0, the condition

Table 1 Comparison between numerical and theoretical values for strategies $r_{\min} = c\sqrt{2}$ and optimized $r_{\min}(N)$ for $\kappa(A)$ where A is of order $N \times N$.

N	$r_{\min} = c\sqrt{2}$		Optimized $r_{\min}(N)$	
	Numerical	Theoretical $(3N+1)$	Numerical	Theoretical $(1+2K)$
5	30.57	16	1.45	1.11
55	153.69	166	4.47	1.11
105	233.76	316	8.86	1.11
155	366.14	466	3.46	1.11
205	227.32	616	3.37	1.11
305	344.39	916	1.00	1.11
405	225.63	1216	1.58	1.11
505	298.08	1516	1.59	1.11
605	249.24	1816	9.20	1.11
705	452.00	2116	3.06	1.11
805	289.81	2416	3.38	1.11
905	186.66	2716	3.51	1.11
1005	192.21	3016	9.14	1.11
1105	409.81	3316	3.73	1.11
1205	176.04	3616	1.50	1.11
1305	333.04	3916	2.98	1.11
1405	215.03	4216	2.99	1.11
1505	268.45	4516	10.48	1.11
1605	402.49	4816	2.92	1.11
1705	327.89	5116	2.80	1.11
1805	269.97	5416	9.47	1.11
1905	260.16	5716	4.10	1.11
1955	279.44	5866	3.04	1.11

Table 2 Descriptive statistics of condition numbers for each strategy.

Statistic	$r_{\min} = 0.005$ (fixed)	$r_{\min} = c\sqrt{2}$	Optimized $r_{\min}(N)$
Mean	2.57×10^{16}	285.47	4.14
Median	5.15×10^{15}	279.44	3.38
Standard Deviation	2.84×10^{17}	83.17	2.86
Minimum	6.98×10^{1}	30.57	1.00
Maximum	5.44×10^{15}	452.00	10.48

number is approximately given by

$$\kappa(A) \approx \frac{c(N+1) + \frac{N-1}{2c} r_0^2 - \frac{N-1}{8c^3} r_0^4}{c - \frac{1}{2c} r_0^2 + \frac{1}{8c^3} r_0^4}.$$

Minimizing this expression with respect to r_0 yields the optimal separation distance $r_0^* = c\sqrt{2}$, which leads to $\kappa_{\min} = 3N + 1$.

For unstructured point distributions, the optimality condition $\frac{d\kappa}{dN} = 0$ results in the functional relation

$$\frac{c^2}{\sigma_{\text{eff}}^2} = K \frac{N-1}{N},$$

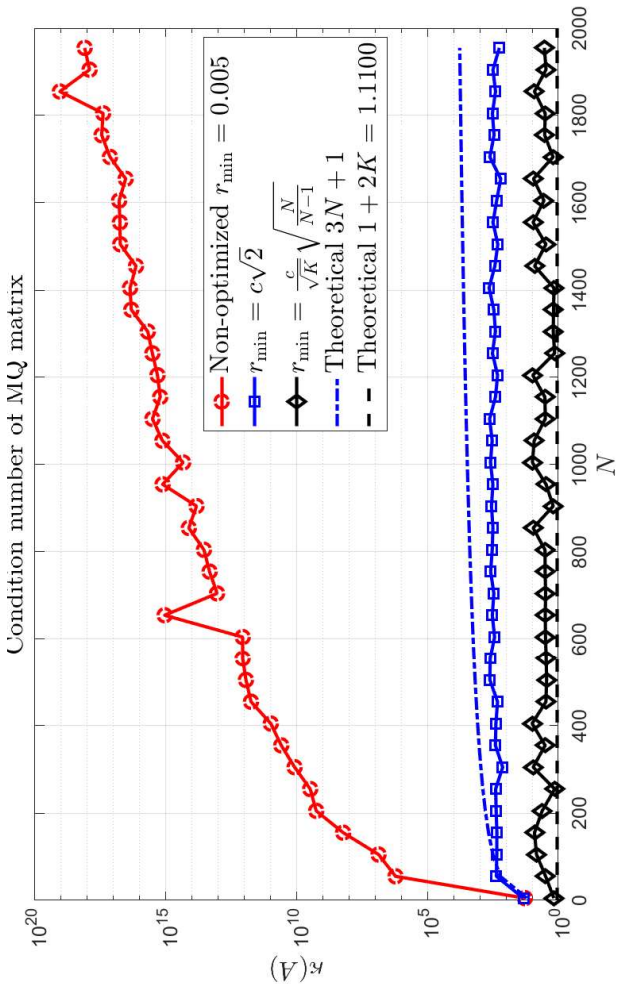


Figure 3 Comparison of condition number $\kappa(A)$ as a function of number of centers N for the three $r_{\min}$ choice strategies. The $r_{\min} = c\sqrt{2}$ strategy exhibits linear growth $\approx 3N + 1$, while the N-dependent optimized strategy keeps $\kappa(A)$ constant and close to 3, as predicted by theoretical analysis. The non-optimized case ($r_{\min} = 0.05$) grows exponentially.

which, when satisfied, makes the condition number constant and independent of N:

$$\kappa(A) = 1 + 2K.$$

Numerical validation.

Numerical experiments in $\mathbb{R}^2$ with $c = 0.15$ and $K = \exp(-2.9) \approx 0.055$ confirmed the theoretical predictions. The fixed strategy ($r_{\min} = 0.005$) produced severe ill-conditioning, with $\kappa(A)$ growing from 6.98×10^1 to 6.33×10^{17} as N increased from 5 to 1955. The heuristic strategy $r_{\min} = c\sqrt{2}$ yielded values between 30.57 and 452.00 (mean 285.47). Remarkably, the N-dependent optimized strategy kept $\kappa(A)$ in the

range 1.00 to 10.48 (mean 4.14) over the entire range of N analyzed, confirming that a constant condition number can be achieved regardless of the number of centers.

Limitations and future work.

The analysis relies on the Taylor expansion $\sqrt{r^2 + c^2} \approx c + r^2/(2c)$, which assumes $r \ll c$. For problems where the shape parameter c is small or points are widely spaced, higher-order terms may become relevant. Additionally, the Poisson disk sampling algorithm becomes computationally expensive for $N \gg 2000$, as generating a valid configuration with the required minimum distance becomes increasingly difficult. Future work includes extending the analysis to other RBFs (e.g., inverse multiquadric, Gaussian), investigating the trade-off between condition number and interpolation accuracy, and developing adaptive strategies for three-dimensional domains.

Final remarks.

The results provide a rigorous mathematical and numerical justification for using adaptive center generation strategies in RBF-based methods. The proposed N-dependent choice of $r_{\min}(N)$ guarantees uniform spectral stability, making it a robust alternative for applications such as meshless PDE solvers, geostatistical interpolation, and computer graphics, where ill-conditioning has traditionally been a limiting factor.

7 Declarations

7.1 Ethical Approval

Not applicable.

7.2 Availability of supporting data

Not applicable.

7.3 Competing interests

Not applicable.

7.4 Funding

Research supported by project Controlling the Condition Number of Multiquadric RBF Matrices via Poisson Disk Sampling at Federal University of Acre.

7.5 Authors' contributions

The author demonstrates, theoretically and numerically, that the linear growth of the condition number of the multiquadric RBF interpolation matrix with the number of centers can be eliminated by establishing an optimal functional relation between the shape parameter c, the number of points N, and the minimum sampling distance $r_{\min}$ via Poisson disk sampling. This strategy maintains the condition number constant

and close to 1, regardless of N, ensuring numerical stability for meshless methods in problems with irregular geometries.

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