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Covariance-Aware Compromise Allocation in Multivariate Stratified sampling under Nonlinear Constraints

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Abstract

This study proposes a compromise allocation (CAA) framework for multivariate stratified sampling with simultaneous budget and survey-time constraints that incorporates the covariance of the sample units. The allocation problem is then considered as a multi-constrained nonlinear programming problem with explicit consideration of the covariance interactions between the study variables in the objective function. Optimality conditions are derived with the Karush–Kuhn–Tucker approach and Quadratic and Logarithmic time structures are investigated. The proposed framework is tested using a numerical study with five strata and three correlated characteristics and compared with proportional allocation, Neyman allocation, and variance-based compromise allocation (VBCA). The results demonstrate that the allocation of samples can be significantly affected by covariance information and that it can enhance multivariate precision. The quadratic CAA model yielded the optimal possible allocation given the specified resource constraints, while the proportional and Neyman allocations required more surveys than the budget allowed. A sensitivity analysis also showed that the availability of budget has a greater impact on precision than the time spent on the survey. The proposed framework is a tool that is applicable in multivariate surveys with realistic resource constraints.

Keyword: *Multivariate stratified sampling; Compromise allocation; Covariance-aware allocation; Nonlinear programming; Optimal sample allocation.*

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1.0 Introduction

Stratified sampling design is among the most common sampling methods because it is very efficient and representative. The issue of determining an optimal distribution of sample sizes among different strata, in order to produce a minimum variance of an estimator at some given cost of a survey, or to produce a minimum cost at a given level of precision, is long-established in the literature on sampling, starting with (Tschuprow, 1923) and (Neyman, 1934).

Single optimal allocation is not usually an easy task in multivariate studies where an allocation that is optimal in one variable may not always be optimal in the optimality conditions of other variables. The concept of compromise allocation was introduced by (Yates, 1960) and aims to have a balanced allocation by maximising a combined objective function which is a representation of multiple variables. Later scholars expanded or adjusted this criterion to construct more effective compromise allocation methods as observed by (Aoyama, 1963), (Folks, 1965), (Kokan, 1967), (Chatterjee, 1968), (Ahsan, 1977), (M. J. Ahsan & Khan, 1982), (Melaku, 1987), (Jahan, 1994) and (Ansari, 2018), among others.

In spite of these strides, there are still a number of different lines of research to further investigate optimal sample allocation in multivariate stratified sampling. The other stream is for considering non-ideal conditions in data collection, (Haq et al., 2020) considered non-response directly in the optimization model, with solutions found through Flexible Fuzzy Goal Programming and Binary Goal Programming; (Jackson et al., 2021) adopted a combination of stratified sampling and linear cost function, using the Lagrange multiplier method to achieve a robust small-area mean estimation framework that takes into account and reduces survey costs to a target precision level. The second stream covers the problem of allocation under uncertainty, which is re-formulated as a stochastic or fuzzy multi-objective program. With the same type of problem, (Ali H. Ahmadini et al., 2021) transformed the nonlinear programming allocation model into a multi-objective problem by introducing a gamma-distributed nonlinear cost function for representing the labor costs, while (Raghav et al., 2023) extended this to the case of an intuitionistic fuzzy programming problem under pessimistic and optimistic scenarios, reporting improved sample assignments over a deterministic formulation. A third stream is on realistic, non-linear depictions of survey budgets. (Ahmad, Ansari, et al., 2022) also used the dynamic programming approach to minimize the weighted sum of variances under a quadratic travel cost constraint in a two-stage design, and (Ahmad, Rabbani, et al., 2022) also incorporated the quadratic travel costs and found that the results were better than classical allocation. In the two-stage design, (AHMAD et al., 2023) found optimum sample size and variance results for the nonlinear time functions, while (Singh et al., 2024) applied the same technique to the estimation of the population mean with the generalized exponential estimators under a linear cost structure. These studies taken together illustrate a progression toward increasingly operationally realistic allocation models (which allow for nonlinear cost and/or time behaviour, as well as non-response and uncertainty), but each address one of these complexities at a time, and as will be discussed below, none simultaneously considers the covariance structure of the study variables and nonlinear operational constraints.

While considerable advances have been made in the development of compromise allocation procedures, most of these are still based on balancing the marginal variability in the characteristics of the individual studies rather than considering the possible covariance among the characteristics. A different line of work has introduced covariance in the form of global matrix functionals, such as the generalized variance for the variance–covariance matrix of the estimators (Arvantis, 1971), which was extended in multi-objective and stochastic programming formulations (Diaz-Garcia, 2008). Such methods reduce the entire covariance structure to a single scalar criterion, but do not explicitly involve the relative importance of the various characteristics or the various covariances between specific characteristics within the objective function; they also have not been developed in conjunction with nonlinear, operationally realistic time constraints. Such relationships between covariances are not considered and/or are only partly considered in the sample allocation, which may be optimal regarding the marginal variances; they may also be optimal regarding an aggregate matrix functional, but not necessarily optimal regarding a weighted characteristic-specific multivariate objective. Therefore, an allocation framework that explicitly and flexibly integrates the covariance in the allocation is desired, and at the same time supports realistic field operations with the nonlinear survey-time requirements.

This study addresses these limitations by developing a covariance-aware compromise allocation framework under nonlinear cost and time constraints. The allocation problem is formulated as a multi-constrained nonlinear programming model in which covariance-adjusted precision, survey cost, and operational time requirements are simultaneously considered. Several nonlinear time specifications, including quadratic and logarithmic forms, are incorporated to reflect realistic survey operations. The resulting optimisation problem is solved using the Lagrange

multiplier framework and evaluated against classical proportional allocation, Neyman allocation, and variance-based compromise allocation benchmarks. By integrating covariance information with explicit operational constraints, the proposed framework provides a more comprehensive approach to multivariate stratified sample allocation and yields allocations that are both statistically efficient and operationally feasible.

2.0 Problem formulation

Let finite population partitioned into L non-overlapping strata and p characteristics on each population unit.

Let

N_h : population size in h^{th} stratum,

n_h : sample size to be determine in h^{th} stratum,

$N = \sum_{h=1}^L N_h$: total population size,

y_{hj} : observation on the i^{th} unit in h^{th} stratum for characteristics j ,

$S_{h,jj}$: variance of characteristics j within stratum h ,

$S_{h,jk}$: covariance between characteristics j and k within stratum h ,

c_h : sampling cost per unit in stratum h ,

$t_h(n_h)$: operational time function for stratum h ,

C : total available survey budget,

T : total available operational time,

w_h : stratum weight.

The allocation vector is defined as: $n = (n_1, n_2, \dots, n_L)$.

2.1 Covariance structure of stratified estimators.

For characteristics j and k , the covariance between the stratified sample mean estimators is given by:

$$\text{cov}(\bar{y}_{st,j}, \bar{y}_{st,k}) = \sum_{h=1}^L w_h^2 (1/n_h - 1/N_h) S_{h,jk}$$

The covariance matrix of the multivariate stratified estimators is therefore:

$$\sum(n) = \text{cov}[(\bar{y}_{st,j}, \bar{y}_{st,k})]_{p \times p}$$

2.2 Covariance-aware objective function

Let: $a = (a_1, a_2, \dots, a_p)$ represent the vector of characteristic importance weights satisfying;

$$a > 0, \quad \sum_{j=1}^p a_j = 1.$$

To account for both individual variability and inter-characteristic dependence, the proposed covariance-aware objective function is defined as:

$$\min \Phi(n) = a^T \sum(n) a$$

Expanding the quadratic form yields;

$$a^T \sum (n) a = \sum_{j=1}^p a_j^2 \text{var}(\bar{y}_{st,j}) + 2 \sum_{j < k}^p a_j a_k \text{cov}(\bar{y}_{st,j}, \bar{y}_{st,k})$$

The above formulation extends classical weighted variance compromise allocation by explicitly incorporating covariance interactions among study characteristics.

Substituting the covariance expression gives:

$$\Phi(n) = \sum_{j=1}^p \sum_{k=1}^p a_j a_k \sum_{h=1}^L w_h^2 (1/n_h - 1/N_h) S_{h,jk}$$

$$\text{Let: } B_h = w_h^2 \sum_{j=1}^p \sum_{k=1}^p a_j a_k S_{h,jk}$$

$$\text{where } \sum_{j=1}^p \sum_{k=1}^p a_j a_k S_{h,jk} = a^T \sum(n) a$$

Now, the objective function simplifies to:

$$\Phi(n) = \sum_{h=1}^L B_h (1/n_h - 1/N_h)$$

Since $1/N_h$ is constant when differentiating the above function with respect to n_h , the objective function reduces to:

$$\min \Phi(n) = \sum_{h=1}^L \frac{B_h}{n_h} \quad (1)$$

The quantity B_h represents the combined covariance-weighted multivariate variability contribution of stratum h.

2.3 Operational constraints.

- Linear constraint

The total survey cost constraint is defined as: $\sum_{h=1}^L c_h n_h \leq C$

- Quadratic operational time constraint

To represent escalating operational burden arising from interviewer fatigue, transportation complexity, and coordination overhead. The quadratic operational time function is defined as:

$$t_h(n_h) = \alpha_h n_h + \beta_h n_h^2 \quad \text{where } \alpha_h, \beta_h > 0.$$

The corresponding time constraint becomes:

$$\sum_{h=1}^L (\alpha_h n_h + \beta_h n_h^2) \leq T$$

- Logarithmic operational time function

To represent learning effects and operational efficiency gains during survey implementation, the logarithmic operational time function is defined as:

$$t_h(n_h) = \alpha_h n_h + b_h \ln(n_h)$$

Where $\alpha_h > 0$.

The corresponding time constraint becomes:

$$\sum_{h=1}^L (\alpha_h n_h + b_h \ln(n_h)) \leq T$$

- Feasibility constraint

The allocation variables satisfy:

$$2 \leq n_h \leq N_h \text{ where } h = 1, 2, \dots, L$$

These constraints ensure positive feasible allocation within each stratum.

2.4 Karush-Kuhn-Tucker Optimization framework

The covariance-aware multivariate allocation problem is solved using nonlinear programming techniques under KKT optimality conditions.

The optimization problem is expressed as:

$$\min \Phi(n) = \sum_{h=1}^L \frac{B_h}{n_h}$$

Subject to:

$$\sum_{h=1}^L c_h n_h \leq C$$

$$\sum_{h=1}^L t_h(n_h) \leq T$$

$$2 \leq n_h \leq N_h$$

2.5 Lagrangian function

Introduce the non-negative Lagrange multipliers:

- $\lambda \geq 0$: multiplier associated with the cost constraint.
- $\gamma \geq 0$: multiplier associated with the time constraint.

The Lagrangian function is therefore defined as:

$$L(n, \lambda, \gamma) = \sum_{h=1}^L \frac{B_h}{n_h} + \lambda \left(\sum_{h=1}^L c_h n_h - C \right) + \gamma \left(\sum_{h=1}^L t_h(n_h) - T \right)$$

2.6 Karush-Kuhn-tucker conditions

Under appropriate regularity assumptions, the optimal allocation satisfies the following conditions:

- Stationarity condition

For each stratum h: $\frac{\delta L}{\delta n_h} = 0$

Thus; $-\frac{B_h}{n_h^2} + \lambda c_h + \gamma t_h^1(n_h) = 0$

Rearranging gives the general covariance-aware optimality equation:

$$\frac{B_h}{n_h^2} = \lambda c_h + \gamma t_h^1(n_h) \tag{2}$$

- Primal feasibility

The optimal allocation must satisfy:

$$\sum_{h=1}^L c_h n_h \leq C$$

$$\sum_{h=1}^L t_h(n_h) \leq T$$

and $2 \leq n_h \leq N_h$

- Dual feasibility

The Lagrange multipliers satisfy: $\lambda \geq 0, \gamma \geq 0$.

- Complementary slackness

The complementary slackness conditions are:

$$\lambda \left(\sum_{h=1}^L c_h n_h - C \right) = 0$$

and

$$\gamma \left(\sum_{h=1}^L t_h(n_h) - T \right) = 0$$

220

2.7 Quadratic operational time Model.

$$t_h(n_h) = \alpha_h n_h + \beta_h n_h^2$$

Differentiating with respect to n_h gives;

$$t_h^1(n_h) = \alpha_h + 2\beta_h n_h$$

Substituting $t_h^1(n_h)$ in equation 2 above we have;

$$\frac{B_h}{n_h^2} = \lambda c_h + \gamma(\alpha_h + 2\beta_h n_h) \quad (3)$$

227

2.8 Logarithmic operational time model.

$$t_h(n_h) = \alpha_h n_h + b_h \ln(n_h)$$

Differentiating with respect to n_h gives;

$$t_h^1(n_h) = \alpha_h + \frac{b_h}{n_h}$$

Substituting $t_h^1(n_h)$ in equation 2 above we have;

$$\frac{B_h}{n_h^2} = \lambda c_h + \gamma \left(\alpha_h + \frac{b_h}{n_h} \right) \quad (4)$$

The resulting nonlinear equations (3) and (4) are solved numerically to obtain the optimal allocation vector subject to the operational constraints.

2.9 Theoretical properties of the proposed framework.

- Existence of optimal solution.

Consider the optimization problem:

239
$$\min \Phi(n) = \sum_{h=1}^L \frac{B_h}{n_h}$$

240 Subject to:

241
$$\sum_{h=1}^L c_h n_h \leq C$$

242
$$\sum_{h=1}^L t_h(n_h) \leq T$$

243
$$2 \leq n_h \leq N_h$$

244 The feasible region is nonempty provided there exists at least one allocation vector satisfying all constraints.

245 Since: $2 \leq n_h \leq N_h$

246 For all strata, the feasible set is bounded. Furthermore, the inequality constraints define close subset of the feasible
247 domain. Hence, the feasible region is compact (close & bounded set).

248 Now, the objective function:

249
$$\Phi(n) = \sum_{h=1}^L \frac{B_h}{n_h}$$

250 Is continuous for; $n_h > 0$, because all feasible allocations satisfy:

251
$$n_h \geq 2,$$

252 the objective function remains continuous over the feasible region. Therefore, by the weierstrass extreme value
253 theorem, the proposed optimization problem admits at least one optimal solution.

- 254 • Convexity of the objective function.

255 Consider the component function:

256
$$f(n_h) = \frac{B_h}{n_h} \text{ where } n_h > 0.$$

257 The first derivative is:
$$f^1(n_h) = -\frac{B_h}{n_h^2}$$

258 The second derivative is:
$$f^{11}(n_h) = \frac{2B_h}{n_h^3},$$

259 Since $B_h > 0$ and $n_h > 0$, it follows that $f^{11}(n_h) > 0$. Hence, $f(n_h)$ is strictly convex.

260 Because finite sum of strictly convex function remains strictly convex, the objective function:

261
$$\Phi(n) = \sum_{h=1}^L \frac{B_h}{n_h} \text{ is strictly convex over the feasible domain.}$$

- 262 • Convexity of the constraints.
- 263 i. Cost constraint:

264
$$\sum_{h=1}^L c_h n_h \leq C$$

265 Is linear and therefore convex.

- 266 ii. Quadratic time constraint:

267
$$t_h(n_h) = \alpha_h n_h + \beta_h n_h^2,$$

268 With $\beta_h > 0$, the second derivative;

269
$$t_h^{11}(n_h) = 2\beta_h > 0.$$

270 Hence, the quadratic operational time function is convex. Consequently, the feasible region associated with the
271 quadratic time model remains convex.

272 iii. Logarithmic time constraint:

273
$$t_h(n_h) = \alpha_h n_h + b_h \ln(n_h) \text{ with } \alpha_h > 0.$$

274 The second derivative will be;

275
$$t_h^{11}(n_h) = -\frac{b_h}{n_h^2} < 0.$$

276 Hence, the logarithmic operational time function is non convex. As a result, the feasible region associated
277 with it may exhibit nonconvex behaviour. Accordingly, the quadratic and logarithmic operational
278 framework require separate theoretical interpretation.

279 • Slater's condition

280 For the convex quadratic operational framework, strong duality holds provided Slater's condition is satisfied. That
281 is, there exist at least one strictly feasible allocation vector:

282
$$n = (n_1, n_2, \dots, n_L)$$

283 Such that

284
$$\sum_{h=1}^L c_h n_h < C$$

285
$$\sum_{h=1}^L t_h(n_h) < T$$

286 Under this condition, the KKT conditions characterized the unique globally optimal allocation.

287 3.0 Numerical Optimization Procedure.

288 The covariance-aware optimization models were implemented into LINGO using nonlinear programming routines
289 capable of handling constrained convex and nonconvex optimization structures.

290 For each optimization experiment:

- 291 • Feasibility conditions were verified,
- 292 • KKT stationarity residuals were monitored,
- 293 • Complementary slackness conditions were evaluated,
- 294 • Convergence tolerance were enforced.

295 For the logarithmic operational model, multiple initialization procedures were employed to reduce sensitivity to
296 local stationary solutions.

297

298 Convergence diagnostics:

299 I. Feasibility residual
300 Verify:

301
$$\sum_{h=1}^L c_h n_h - C \leq \theta$$

302
$$\sum_{h=1}^L t_h(n_h) - T \leq \theta$$

303 Where, θ is a numerical tolerance.

304

- II. Stationarity residual
 For each stratum: $R_h = -\frac{B_h}{n_h^2} + \lambda c_h + \gamma t_h^1(n_h)$
 at convergence $|R_h| < \theta$.
 III. Complementary slackness residual

Verify:

$$\lambda \left(\sum_{h=1}^L c_h n_h - C \right) \approx 0$$

and

$$\gamma \left(\sum_{h=1}^L t_h(n_h) - T \right) \approx 0$$

3.1 Numerical illustration

To demonstrate the applicability of the proposed covariance-aware allocation framework, a hypothetical multivariate stratified population consisting of five strata and three study variables was considered. Population sizes, covariance matrices, cost coefficients and time coefficients were specified to reflect realistic survey conditions. The proposed model and benchmark allocation methods were then solved under identical budget and time constraints, and their performances were compared using a common covariance-aware evaluation criterion.

The numerical parameters were selected to reflect heterogeneous survey environments characterized by varying optimal burden, sampling costs, and inter-characteristics dependence across. The parameter settings were designed to evaluate the robustness and sensitivity of the proposed covariance-aware allocation framework under diverse operational conditions.

Now, we compute the covariance coefficients:

Using $B_h = w_h^2 a^T \Sigma_h a$ where a is weight vector and Σ_h is the covariance matrix.

Let $a = \begin{pmatrix} 0.333 \\ 0.333 \\ 0.333 \end{pmatrix}$ $w_h = (0.15, 0.20, 0.25, 0.18, 0.22)$

and $w_h^2 = (0.0225, 0.04, 0.0625, 0.0324, 0.0484)$

Stratum covariance matrices.

Stratum 1: $\Sigma_1 = \begin{pmatrix} 16 & 6 & 4 \\ 6 & 14 & 5 \\ 4 & 5 & 12 \end{pmatrix}$ Stratum 2: $\Sigma_2 = \begin{pmatrix} 20 & 10 & 8 \\ 10 & 18 & 9 \\ 8 & 9 & 15 \end{pmatrix}$

Stratum 3: $\Sigma_3 = \begin{pmatrix} 18 & 4 & 3 \\ 4 & 16 & 2 \\ 3 & 2 & 14 \end{pmatrix}$ Stratum 4: $\Sigma_4 = \begin{pmatrix} 25 & 15 & 12 \\ 15 & 22 & 10 \\ 12 & 10 & 20 \end{pmatrix}$

Stratum 5: $\Sigma_5 = \begin{pmatrix} 21 & 8 & 7 \\ 8 & 19 & 6 \\ 7 & 6 & 17 \end{pmatrix}$

Substituting in $B_h = a^T \Sigma_h a$ we have;

Stratum1: $B_1 = 0.1800$ Stratum2: $B_2 = 0.4756$ Stratum3: $B_3 = 0.4583$

Stratum4: $B_4 = 0.5076$ Stratum5: $B_5 = 0.5324$

Table 1: Population and operational parameters

Stratum	N_h	c_h	α_h	β_h	V_h	B_h	b_h
1	150	8	2.0	0.010	12	0.1800	10
2	200	10	2.3	0.015	17	0.4756	12
3	250	12	3.0	0.020	15	0.4583	15
4	180	9	2.2	0.012	20	0.5076	11
5	220	11	2.8	0.018	18	0.5324	13

Table 2: Comparison of the models when cost limit = 1800 units and Time limit = 650 units.

Model	Stratum					Total Sample	CAA metric	Cost	Time
	1	2	3	4	5				
Proportional	27	36	45	32	39	179	0.0596	1833.00	565.50
Neyman	24	38	38	39	40	179	0.0584	1819.00	558.15
VBCA	35	37	32	42	36	182	0.0594	1800.00	546.69
CAA(Quadratic)	26	38	34	42	39	179	0.0587	1800.00	549.43
CAA(logarithmic)	25	37	33	41	37	173	0.0608	1738.41	650.00

Where;

- VBCA: Variance-Based Compromise Allocation,
- CAA (Quadratic): Covariance-aware allocation (quadratic),
- CAA (logarithmic): Covariance-aware allocation (logarithmic).

Table 3: KKT verification.

Constraint	Slack	Dual price
Cost	0	0.0000326
Time	100.5685	0
Lower bounds	>0	0

- KKT verification mathematically.

For CAA model: $\min f(n) = \sum_{h=1}^5 \frac{B_h}{n_h}$

Subject to: $g_1(n) = Tcost - 1800 \leq 0$

$g_2(n) = Ttime - 650 \leq 0$

The KKT conditions require:

- Primal feasibility: $g_1(n) \leq 0$
- $g_2(n) \leq 0$ Satisfied.
- Dual feasibility: $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$
Satisfied because $\lambda_1 = 0.0000326 > 0$ and $\lambda_2 = 0$.
- Complementary slackness: $\lambda_i g_i(n) = 0$

For cost: $0.00003263791 * 0 = 0$ satisfied.

For time: $0 * 100.5685 = 0$ satisfied.

3.2 Sensitivity analysis

Table 4: For CAA quadratic when budget remain 1800 units.

Time Limit	Total sample	Objective	TTime	Tcost
500	166	0.06362	500.0000	1663.039
550	179	0.0587	549.4315	1800
600	179	0.0587	549.4315	1800
650	179	0.0587	549.4315	1800
700	179	0.0587	549.4315	1800

Table 5: For CAA quadratic when Time limit = 650 units.

Budget	Total sample	Objective	Ttime	Tcost
1600	158	0.0661	478.5481	1600
1700	168	0.0622	513.6824	1700
1800	179	0.0587	549.4315	1800
1900	189	0.0557	585.7953	1900
2000	198	0.0529	622.7738	2000

4.0 Discussion of results.

The summary of the data used in this study is provided in Table 1, and the summary of the allocation results from the proportional allocation framework, Neyman allocation framework, variance-based compromise allocation (VBCA), and the proposed covariance-aware allocation (CAA) is provided in Table 2, where all five allocations are evaluated under the same fiscal and temporal constraints ($c = 1800$ units and $t = 650$ units). All allocations, including those based on the various principles, are evaluated on the same objective measure based on covariance (1), so that differences between allocations in Table 2 are due to differences in allocation quality, not differences in evaluation principles. In this discussion, a lower objective value means multivariate precision is higher, but feasibility in terms of the cost and time caps come first, as a possible allocation which is infeasible cannot be implemented despite its apparent degree of precision.

The proportional allocation method does not take into account variation and covariance structure differences between strata. With 179 sampling units, it gives an objective value of 0.0596, which is numerically the lowest, and hence is the most precise looking of all the five allocations in Table 2. The costs it will have to pay, however, are 1833.00 units, which is more than the 1800.00 unit budget, thus this allocation is not feasible within the operational limits of the study. This is not so in the Neyman allocation, which takes stratum-level variability into account, and obtains an objective value of 0.0584; it also fails to meet the cost constraint, however, which would need 1819.00 units, instead of the 1800 units. The apparent precision advantage is illusory in both cases, since it is obtained by running the survey for more than the budget permits, and hence not available given the resources actually available to the survey.

The VBCA model is an improvement of both of the above two procedures in that it balances the variances of multiple characteristics simultaneously and strictly does satisfy the cost constraint (1800 units). Its allocation vector, however, does not consider the covariance interactions among variables and will allocate resources in an unequal fashion towards the strata that have the largest individual variance, not necessarily those that are most important to the overall multivariate risk. This leads to an overstated sample size (182) and sub-optimal objective value (0.0594) when compared to the proposed model, despite the fact that all the work is performed within the same budget constraints.

The proposed covariance-aware allocation paradigm explicitly includes variance and covariance data through the use of the coefficients.

$$B_h = w_h^2 \sum_{j=1}^p \sum_{k=1}^p a_j a_k S_{h,jk}$$

Intervariable interdependencies are considered, and which quantifies the overall contribution of multivariate variability for each stratum. The covariance-risk coefficients of the strata 4 and 5 are the highest ($B_4 = 0.5076$ and $B_5 = 0.5324$) while the Stratum 1 contributes significantly less ($B_1 = 0.1800$). Thus, the proposed allocation method focuses on strata 4 and 5 and focuses away from strata with relatively small contributions to the overall multivariate risk.

In the quadratic time structure this covariance-sensitive allocation allows the proposed model, CAA (Quadratic), to have a sample size of 179 units, which is the same as with proportional and Neyman allocations, and it allows it to satisfy the cost constraint exactly (1800 units), and also to use less time (549.43 units) than the available budget (650 units). Its objective value of 0.0587 is lower than both the population-proportional and the variance-only compromise approaches (0.0596 and 0.0594, respectively), indicating that a careful modeling of the covariance gives an allocation that is more accurate (when feasibility is carefully applied) than the other two compromise approaches. The story of the comparison with Neyman allocation (0.0584) is more complicated. Neyman's slightly better objective value is only achieved by the sacrifice of exceeding the cost ceiling by 19 units, based upon identical resources, so it is not a viable alternative. The model described will form the basis of the most realistic solution for the restricted budget of four alternatives explored.

The proposed framework, with the logarithmic form, operates in a different nonlinear temporal framework, and achieves a divergent position on the cost–precision continuum. It is able to achieve a smaller total sample size (173) at a much lower cost (1738.41) and thus utilises the time budget exactly (650.00) but has a higher objective value (0.0608) compared with CAA (Quadratic). The observation implies that the type of architecture used in the logarithmic case would compromise some degree of multivariate precision for reduced expenditure of resources, thus offering an advantage when the minimization of resource expenditure is desired and the maximization of precision is not, while the quadratic case would offer the advantage of precision when resources can be allocated to support it.

In concert, these results illustrate that incorporating the covariance interaction is more than a matter of simply allocating samples; it is a critical question to ask to determine if a model's apparent accuracy can be achieved in a practical setting. When the cost overrun is ignored, both proportional and Neyman allocation seem to be competitive; but when the feasibility constraint is imposed, the framework proposed in this paper is seen as being most efficient in the use of a limited budget in terms of the reduction of covariance.

Meanwhile, Table 3 illustrated that the budget constraint was active at the optimum, whereas the time constraint remained non-binding. The corresponding shadow price associated with the budget constraint was 3.26×10^{-5} , indicating that a marginal increase in the available budget would improve the objective function. In contrast, the zero-shadow price associated with the time constraint suggests that additional survey time would not affect the optimal solution under the current parameter settings as illustrated in the sensitivity analysis. In order to test the robustness of the proposed framework, sensitivity analyses with respect to the survey budget and temporal constraints were carried out, and the results are presented in Tables 4 and 5. We explore the implications of relaxing the temporal constraint while keeping the fixed budget of 1800 units in Table 4. Two different modes of operation are observed. The temporally limited constraint of 500 is binding, so that the total number of samples decreases to 166, and the value of the objective increases to 0.06362, signaling a substantive reduction in precision due to the shorter completion time. As time limit is relaxed further, the allocation stabilizes when time limit approaches around 550 units, and the objective value is kept constant at 0.0587, regardless of the increase of time limit. After this point it becomes irrelevant when the constraint of time and the constraint of cost are the only ones operating, and the achievable precision is only determined by the latter.

The complementary scenario – changing the budget and fixing the temporal limit – is analyzed in Table 5. In this case, the relationship is monotonic and continuous as opposed to threshold-driven. As the budget increases from 1600 to 2000 units, the total number of samples that are taken increases from 158 to 198, and at the same time the objective value decreases from 0.0661 to 0.0529. The corresponding rise in the number of units required to conduct the survey (478.55 to 622.77) reinforces the fact that improved precision requires more effort in the field, not just a larger nominal sample.

These two sensitivity patterns are a natural and self-explanatory contrast. While the effect of time is described as having a threshold below which it is beneficial, but not an effect beyond a threshold, the effect of budget is monotonic throughout the range studied. It would seem that, for the conditions considered in this paper, the precision obtainable with the proposed model is limited mainly by budget and not by time; that is, as the number

of years of the survey is increased, no further improvement is found in the precision of the multivariate, and each additional dollar spent in the survey delivers a measurable improvement in that precision. If survey designers are faced with a combination of cost and schedule constraints, it means they can find better precision gains by asking for more funds rather than more field time (if they have a minimum amount of time that is acceptable).

5.0 Conclusion

This research effort developed a framework for multivariate stratified sampling based on a nonlinear operational constraint that is covariance-aware and operates on a compromise allocation. A key factor is the ability of the proposed allocation methodology to directly target overall multivariate estimation precision as opposed to many other allocation methods, where the overall precision is a secondary result of optimizing for marginal variances or population proportions.

The allocation dilemma was stated as a multi-constrained nonlinear programming model with cost and temporal constraints; both quadratic and logarithmic time structures were considered, and the optimality conditions were set out using the Karush-Kuhn-Tucker model. It was also shown that the quadratic formulation leads to a convex optimization problem, which guarantees global optimality, while the logarithmic formulation is nonconvex, and only locally optimal solutions can be guaranteed in general.

The numerical results explained the implications of this identification. Both allocation methods gave seemingly competitive objective values (1833.00 and 1819.00, respectively, as opposed to 1800). However, these values were above the budget limit and, under the conditions of the experiment, were not feasible. The VBCA model did fit within the budget constraints, but it did require a larger sample size (182 units) to attain a less efficient objective value (0.0594) compared to the suggested model. The quadratic covariance-aware allocation, on the other hand, was able to meet the cost constraint exactly and had a smaller objective value (0.0587) than the other two methods, as well as achieving a better fit than the seemingly better-fitting allocations that didn't pay attention to budget constraints, suggesting that explicit consideration of the covariance, not just the variance, did indeed help produce a feasible allocation that could come closer to the precision of the seemingly better-fitting allocations that ignored the cost constraint. The logarithmic variant is the complementary alternative, which gives a modest change of the objective value (0.0608) for a significantly reduced cost (1738.41 units) and, therefore, is advantageous if the conservation of the budget is more important than the accuracy of the survey.

This conclusion was confirmed by sensitivity analysis, which showed that adding more time or time up to a certain point had a beneficial effect on precision but then continued in the same direction for successive additions of time, while adding more budget had a consistently monotonic effect on precision across the range studied. In this regard, financial resources turn out to be the key variable that can enhance the precision of multivariate estimation after having drawn up a feasible schedule, a conclusion consistent with the feasibility gap between the classical and proposed allocations as described above.

The framework can be expanded in future investigations to cover scenarios of stochastic environments, non-response, two-stage sampling designs and multi-objective optimization formulations that combine uncertainty in both the cost and time parameters.

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Author contributions

All authors contributed to the study conception and design. Material preparation, data analysis were performed by [Shuaibu Idris Adam] and [Quazzafi Rabbani]. The first draft of the manuscript was written by [Shuaibu Idris Adam] and my supervisor [Quazzafi Rabbani] commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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