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Hirota Bilinear Method for Soliton Solutions of the Qiao-Li Equation

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Abstract

This paper investigates a new integrable nonlinear wave equation proposed by Professor Zhijun Qiao and Professor Shengtai Li, whose multisoliton solutions and interaction dynamics remain insufficiently explored. To fill this research gap, we systematically apply the Hirota bilinear method for the first time, successfully constructing explicit expressions for the single soliton, double soliton, triple soliton and general N soliton solutions of the Qiao Li equation. Numerical simulations are employed to analyze the distinctive properties of two soliton collisions, characterizing the observed nonlinear phenomenon of valley peak transformation involving mutual conversion between antisolitons and solitons. This study confirms the rich soliton structure of this nonlinear integrable equation as a member of the integrable systems family while revealing its unique wave dynamics that distinguish it from other integrable models such as the Camassa Holm equation.

Keywords: Nonlinear integrable equations, Hirota's method, Soliton solutions, Collisions

1 Introduction

Nonlinear integrable systems are central to modern mathematical physics, with models such as the KdV equation attracting great interest for describing exact solutions including solitons. In 1993, Camassa and Holm^[1] derived an integrable shallow water wave equation with peakon solutions. After the introduction of the integrable Degasperis-Procesi equation^[2] by Degasperis and Procesi, Professor Zhi-Jun Qiao and Professor Sheng-Tai Li further extended this equation to a new integrable hierarchy (hereafter the Qiao-Li equation) via the classification theory of integrable systems, and investigated its parametric and stationary solutions. However, compared with the Camassa-Holm equation, the multi-soliton solutions and interaction properties of this nonlinear integrable equation remain unexplored, leaving its rich physical implications an open research topic.

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While current research on the Qiao-Li equation focuses on integrability proofs, specific solutions, and numerical analysis, a systematic investigation of its multi-soliton solutions using the Hirota bilinear method is lacking^[3,4]. The Hirota method provides a powerful means for constructing exact solutions to integrable equations^[5–7], particularly multi-soliton solutions. Moreover, it enables explicit expressions and analytical insights into nonlinear soliton interactions^[8,9]. Furthermore, in contrast to research on the (2+1)-dimensional KP equation, which explores complex solution structures such as periodic-soliton hybrid solutions or multi-wave mixing effects, this study establishes a comprehensive soliton solution foundation for a specific (1+1)-dimensional integrable equation^[10,11], laying a prerequisite for the subsequent exploration of other types of solutions. Meanwhile, it reflects the diversity of nonlinear wave phenomena across different dimensions and equations, as well as the differences in the application focus of the Hirota method.

This paper systematically investigates the multi-soliton solutions of the Qiao-Li equation using the Hirota bilinear method. We explicitly derive the single, double, and triple soliton solutions, and subsequently present the general N -soliton form. The core of this work involves a detailed numerical analysis of the double-soliton collision dynamics, specifically elucidating the nonlinear mechanism governing the mutual transformation between anti-peakons and peakons. This study demonstrates the unique nonlinear wave properties of the Qiao-Li equation and addresses the existing gap in the systematic research on its soliton solutions.

2 The Hirota Method for Solving the Qiao-Li Equation

The Hirota method, proposed by the Japanese mathematician Ryogo Hirota in the 1970s, is an algebraic approach for directly constructing exact solutions to nonlinear integrable equations. Its core idea is to introduce the bilinear derivative operator (Hirota D-operator), transform complex nonlinear equations into relatively simpler bilinear equations, and thereby systematically obtain solutions such as solitons, breathers, and rogue waves. This paper focuses on the application of the Hirota method to the Qiao-Li equation, emphasizing the use of the Hirota transformation in deriving exact solutions for soliton equations. Through in-depth investigation of the fundamental properties of single, double, triple, and general N -soliton solutions, as well as systematic analysis of their dynamic behaviors based on the analytical solution process, we aim to uncover the intrinsic mechanisms underlying nonlinear phenomena. This work is intended to provide new theoretical perspectives and solution methodologies for related fields such as nonlinear physics and mathematical physics.

The soliton solutions for the studied Qiao-Li equation^[12] will be derived and analyzed using the aforementioned Hirota method.

$$u_t = \partial_x^5 \left(\frac{(u^{-1/3})_{xx} - 2(u^{-1/6})_x^2}{u} \right). \quad (1)$$

To simplify the solving process of Eq. (1), the original equation contains terms with $u^{-1/3}$ and $u^{-1/6}$. We perform a power transformation $u = v^m$ to simplify the equation by introducing

a new dependent variable $v(x, t)$ (where x is the spatial variable and t is the temporal variable). This establishes a mapping relationship between the original variable $u(x, t)$ and the new variable $v(x, t)$:

$$u = v^{-6}. \quad (2)$$

The application of inverse operations and power transformations^[13,14] to Eq. (2) leads to the key relations $v = u^{-1/6}$, $u^{-1/3} = v^2$, $u^{-1/6} = v$. Differentiation of v and u with respect to the spatial variable x or time variable t based on the preceding relations provides the partial derivatives:

$$(u^{-1/3})_{xx} = (v^2)_{xx} = 2vv_{xx} + 2v_x^2, \quad (3)$$

$$(u^{-1/6})_x^2 = v_x^2, \quad (4)$$

$$u_t = -6v^{-7}v_t. \quad (5)$$

Substituting Eqs. (3), (4) and (5) into the nonlinear integrable Eq. (1) after the derivative calculations performs the variable replacement and simplifies the equation's form, establishing the foundation for deriving soliton solutions.

$$-6v^{-7}v_t = \partial_x^5 \left(\frac{2vv_{xx} + 2v_x^2 - 2v_x^2}{v^{-6}} \right) = 2\partial_x^5(v^7v_{xx}). \quad (6)$$

Simplifying Eq. (6) yields the final relation:

$$v_t = -\frac{1}{3}v^7\partial_x^5(v^7v_{xx}). \quad (7)$$

To ultimately obtain the standard bilinear form as shown in Eq. (8), we adopt the bilinearization approach used for the Sawada-Kotera-type equations. We introduce a function $F = F(x, t)$ such that $v(x, t) = [2\partial_x^2 \ln F(x, t)]^{1/7}$, and subsequently apply the Hirota bilinear operator.

$$D_t^m D_x^n F \cdot F = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n F(x, t) F(x', t') \Big|_{x'=x, t'=t}.$$

Substituting the above expression into Eq. (7) and simplifying yields the bilinear equation:

$$(D_x^6 + D_x D_t) F \cdot F = 0. \quad (8)$$

where $v(x, t) = [2\partial_x^2 \ln F(x, t)]^{1/7}$, F is the tau function in the Hirota method, which satisfies the subsequent bilinear Eq. (8).

2.1 The Single-Soliton Solution of the Qiao-Li Equation

The form of the Qiao-Li equation suggests a single-soliton solution with the double exponential expansion^[15]:

$$F = 1 + e^\eta, \quad \eta = kx - \omega t + \delta. \quad (9)$$

Among them, k and ω control the soliton's wave speed and angular frequency, and δ is the phase.

Substituting the tau function Eq. (9) $F = 1 + e^\eta$ with $\eta = kx - \omega t + \delta$ into the bilinear equation Eq. (8) yields:

$$\begin{aligned} (D_x^6 + D_x D_t)F \cdot F &= D_x^6(1 + e^\eta) \cdot (1 + e^\eta) + D_x D_t(1 + e^\eta) \cdot (1 + e^\eta) \\ &= [D_x^6 1 \cdot 1 + D_x^6 1 \cdot e^\eta + D_x^6 e^\eta \cdot 1 + D_x^6 e^\eta \cdot e^\eta] \\ &\quad + [D_x D_t 1 \cdot 1 + D_x D_t 1 \cdot e^\eta + D_x D_t e^\eta \cdot 1 + D_x D_t e^\eta \cdot e^\eta]. \end{aligned} \quad (10)$$

We then compute each group of bilinear terms in Eq. (10) term by term:

$$\begin{aligned} D_x^6 1 \cdot 1 &= 0, \quad D_x D_t 1 \cdot 1 = 0, \\ D_x^m D_t^n 1 \cdot e^\eta &= (-k)^m \omega^n e^\eta, \quad D_x^m D_t^n e^\eta \cdot 1 = k^m (-\omega)^n e^\eta, \\ D_x^6 1 \cdot e^\eta &= (-k)^6 e^\eta = k^6 e^\eta, \\ D_x^6 e^\eta \cdot 1 &= k^6 e^\eta, \quad D_x D_t 1 \cdot e^\eta = (-k) \cdot \omega \\ e^\eta &= -k\omega e^\eta, \quad D_x D_t e^\eta \cdot 1 = k \cdot (-\omega) e^\eta = -k\omega e^\eta, \\ D_x^6 e^\eta \cdot e^\eta &= 0, \quad D_x D_t e^\eta \cdot e^\eta = 0. \end{aligned}$$

Substituting all the above terms into Eq. (8), we obtain:

$$\begin{aligned} (D_x^6 + D_x D_t)F \cdot F &= [0 + k^6 e^\eta + k^6 e^\eta + 0] + [0 - k\omega e^\eta - k\omega e^\eta + 0] \\ &= 2k^6 e^\eta - 2k\omega e^\eta. \end{aligned} \quad (11)$$

From the constraint of the bilinear equation $(D_x^6 + D_x D_t)F \cdot F = 0$ and the fact that $e^\eta \neq 0$, canceling the common factor $2e^\eta$ on both sides gives:

$$k^6 - k\omega = 0.$$

Further simplifying the above expression yields the dispersion relation:

$$\omega = k^5. \quad (12)$$

Next, substituting the tau function form Eq. (9) and the dispersion relation Eq. (12) into the expression for v , $v = \left(2 \frac{\partial^2}{\partial x^2} \ln F\right)^{1/7}$, yields:

$$\ln F = \ln(1 + e^\eta),$$

$$\frac{\partial}{\partial x} \ln F = \frac{ke^\eta}{1 + e^\eta},$$

$$\frac{\partial^2}{\partial x^2} \ln F = \frac{k^2 e^\eta}{(1 + e^\eta)^2}.$$

Thus

$$\begin{aligned} 2 \frac{\partial^2}{\partial x^2} \ln F &= \frac{2k^2 e^\eta}{(1 + e^\eta)^2}, \\ v &= \left(\frac{2k^2 e^\eta}{(1 + e^\eta)^2} \right)^{1/7}. \end{aligned} \quad (13)$$

Substituting the expression for the new variable $v = \left(\frac{2k^2 e^\eta}{(1 + e^\eta)^2} \right)^{1/7}$ with $\eta = kx - \omega t + \delta$, together with the dispersion relation $\omega = k^5$ from Eq. (12), into the mapping $u = v^{-6}$ given by Eq. (2), we obtain the single-soliton solution of the Qiao-Li equation after simplification.

$$u(x, t) = \left(\frac{2k^2 e^\eta}{(1 + e^\eta)^2} \right)^{-6/7}, \quad \eta = kx - k^5 t + \delta. \quad (14)$$

Further simplification of the above expression yields the single-soliton solution in exponential form.

$$u(x, t) = \frac{(1 + e^\eta)^{12/7}}{(2k^2)^{6/7} e^{6\eta/7}}, \quad \eta = kx - k^5 t + \delta. \quad (15)$$

The parameters k and δ govern the soliton's wave speed and phase.

To better visualize the soliton waveform and facilitate analysis of physical properties like amplitude, width, and propagation, the expression can be transformed using the fundamental identity relating exponential and hyperbolic cosine functions. The exponential form of solution Eq. (15) is first converted using the identity $e^\eta + e^{-\eta} = 2 \cosh \eta$ into a hyperbolic cosine form and then transformed via the identity $1/\operatorname{sech} \eta = \cosh \eta$ into the final hyperbolic secant form.

$$u(x, t) = \frac{2^{6/7}}{k^{12/7}} \operatorname{sech}^2 \left(\frac{kx - k^5 t + \delta}{2} \right). \quad (16)$$

The parameters k and δ determine the velocity and phase of the soliton.

2.2 The Two-Soliton Solution of the Qiao-Li Equation

Based on the Hirota bilinear equation $(D_x^6 + D_x D_t) F \cdot F = 0$ and the dependent variable transformations $u = v^{-6}$, $v(x, t) = [2\partial_x^2 \ln F(x, t)]^{1/7}$, we construct the two-soliton solution by introducing the standard two-soliton tau function:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2}. \quad (17)$$

Where the phase variables are defined as $\eta_i = k_i x - \omega_i t + \delta_i$, $i = 1, 2$. Here k_i represents the wave number, δ_i represents the arbitrary phase constant, and A_{12} is the undetermined phase factor describing the nonlinear interaction between two solitons.

To determine the interaction coefficient A_{12} , we use the definition of the Hirota bilinear operator $D_t^m D_x^n F \cdot F = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n F(x, t) F(x', t') \Big|_{x'=x, t'=t}$. For exponential functions $f = e^{\eta_i}$ and $g = e^{\eta_j}$, the core bilinear operator identities are:

$$D_x^n e^{\eta_i} \cdot e^{\eta_j} = (k_i - k_j)^n e^{\eta_i + \eta_j}, \quad D_t D_x e^{\eta_i} \cdot e^{\eta_j} = (\omega_i - \omega_j)(k_i - k_j) e^{\eta_i + \eta_j}. \quad (18)$$

Substitute the tau function Eq. (17) into the bilinear equation Eq. (8). The constant terms and single exponential terms e^{η_1}, e^{η_2} are automatically satisfied by the dispersion relation $\omega_i = k_i^5$. The only undetermined coefficient A_{12} comes from the coefficient of the cross term $e^{\eta_1 + \eta_2}$.

$$[(A_{12} - 1)(\omega_1 k_2 + k_1 \omega_2 + 6k_1^3 k_2^3) + 4(A_{12} + 1)k_1^2 k_2^2 (k_1^2 + k_2^2)] e^{\eta_1 + \eta_2} = 0. \quad (19)$$

Substitute the dispersion relation $\omega_i = k_i^5$ into Eq. (19), and solve equation Eq. (19) for the phase factor A_{12} :

$$A_{12} = \frac{(k_1^3 - k_2^3)(k_1 - k_2)^3 - (k_1 - k_2)^6}{(k_1 + k_2)^6 - (k_1^3 + k_2^3)(k_1 + k_2)^3}. \quad (20)$$

After algebraic expansion and simplification, the exact expression of A_{12} is obtained:

$$A_{12} = \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4}. \quad (21)$$

Substituting Eq. (21) into Eq. (17), the tau function of the two-soliton solution is determined as:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4} e^{\eta_1 + \eta_2}, \quad (22)$$

where the phase variables are rewritten as $\eta_i = k_i x - k_i^5 t + \delta_i$, $i = 1, 2$.

Substitute the tau function F and the phase factor A_{12} into the variable transformation $v = [2\partial_x^2 \ln F]^{1/7}$, and then use $u = v^{-6}$. The exact two-soliton solution of the Qiao-Li equation (1) is derived as:

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln \left[1 + e^{\eta_1} + e^{\eta_2} + \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4} e^{\eta_1 + \eta_2} \right] \right)^{-6/7}. \quad (23)$$

Where the phase variables are simplified to $\eta_i = k_i x - k_i^5 t + \delta_i$, $i = 1, 2$. This solution describes the elastic collision of two solitons. Parameters k_i, δ_i govern their velocities and amplitudes, while the coefficient A_{12} characterizes their nonlinear interaction.

To better visualize the soliton waveform and facilitate analysis of physical properties like amplitude, width, and propagation, the expression can be transformed using the fundamental identity relating exponential and hyperbolic cosine functions. The exponential form of the two-soliton solution Eq. (23) is first converted using the identity $e^\eta + e^{-\eta} = 2 \cosh \eta$ into a hyperbolic cosine form and then transformed via the identity $1/\operatorname{sech} \eta = \cosh \eta$ into the final hyperbolic function form.

$$u(x, t) = \frac{2^{6/7}}{(k_1 k_2)^{6/7}} \left[\frac{\cosh \left(\frac{\eta_1 - \eta_2}{2} \right) + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \cosh \left(\frac{\eta_1 + \eta_2}{2} \right)}{\cosh \left(\frac{\eta_1 - \eta_2}{2} \right) + \cosh \left(\frac{\eta_1 + \eta_2}{2} \right)} \right]^{-12/7}. \quad (24)$$

Where the phase variables are simplified to $\eta_i = k_i x - k_i^5 t + \delta_i$, $i = 1, 2$. This solution describes the elastic collision of the two-soliton solution. Parameters k_i, δ_i govern their velocities and amplitudes, while the coefficient A_{12} characterizes their nonlinear interaction.

2.3 The Three-Soliton Solution of the Qiao-Li Equation

On the basis of the single- and two-soliton solutions constructed above, the three-soliton solution of the Qiao-Li equation can be established by extending the τ function to the standard three-exponential form in the Hirota bilinear method. For the bilinear equation $(D_x^6 + D_x D_t)F \cdot F = 0$, we assume the τ function for the three-soliton solution takes the form:

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + Be^{\eta_1+\eta_2+\eta_3}, \quad (25)$$

where the phase variables are $\eta_i = k_i x - \omega_i t + \delta_i$, $\omega_i = k_i^5$, $i = 1, 2, 3$, and A_{ij} are the two-soliton interaction coefficients already determined in Section 2.2:

$$A_{ij} = \frac{(k_i - k_j)^4}{(k_i + k_j)^4}, \quad 1 \leq i < j \leq 3. \quad (26)$$

The coefficient B is the unknown three-wave interaction parameter to be determined by substituting Eq. (25) into the bilinear equation (8). Substitute the tau function (25) into $(D_x^6 + D_x D_t)F \cdot F = 0$. According to the properties of the Hirota bilinear operator,

$$D_x^n e^{\eta_i} \cdot e^{\eta_j} = (k_i - k_j)^n e^{\eta_i + \eta_j}, \quad D_t D_x e^{\eta_i} \cdot e^{\eta_j} = (\omega_i - \omega_j)(k_i - k_j) e^{\eta_i + \eta_j}.$$

The constant terms, linear exponential terms e^{η_i} , and pairwise terms $e^{\eta_i + \eta_j}$ are automatically satisfied by the dispersion relation $\omega_i = k_i^5$ and the two-soliton condition for A_{ij} . The only nontrivial condition comes from the coefficient of the term $e^{\eta_1 + \eta_2 + \eta_3}$, which must vanish identically.

Using the dispersion relation $\omega_i = k_i^5$, we substitute

$$(k_i - k_j)^6 + (\omega_i - k_j)(k_i - k_j) = (k_i - k_j)^6 - (k_i^5 - k_j^5)(k_i - k_j) = 0,$$

This identity is consistent with the conditions for single- and two-soliton solutions. For the three-soliton case, extracting the coefficient of the term $e^{\eta_1 + \eta_2 + \eta_3}$ yields the simplified equation:

$$\begin{aligned} & [(k_1 + k_2 - k_3)^6 - (k_1^5 + k_2^5 - k_3^5)(k_1 + k_2 - k_3)] A_{12} e^{\eta_1 + \eta_2 + \eta_3} \\ & + [(k_1 - k_2 + k_3)^6 - (k_1^5 - k_2^5 + k_3^5)(k_1 - k_2 + k_3)] A_{13} e^{\eta_1 + \eta_2 + \eta_3} \\ & + [(-k_1 + k_2 + k_3)^6 - (-k_1^5 + k_2^5 + k_3^5)(-k_1 + k_2 + k_3)] A_{23} e^{\eta_1 + \eta_2 + \eta_3} \\ & - B [(k_1 + k_2 + k_3)^6 - (k_1^5 + k_2^5 + k_3^5)(k_1 + k_2 + k_3)] e^{\eta_1 + \eta_2 + \eta_3} = 0. \end{aligned}$$

For the bilinear equation $(D_x^6 + D_x D_t)F \cdot F = 0$, which admits pure multi-soliton solutions with standard interaction coefficients, the three-soliton condition is satisfied if and only if

$$B = A_{12}A_{13}A_{23}.$$

Substituting the known Eq. (25), we obtain

$$B = \frac{(k_1 - k_2)^4 (k_1 - k_3)^4 (k_2 - k_3)^4}{(k_1 + k_2)^4 (k_1 + k_3)^4 (k_2 + k_3)^4}. \quad (27)$$

Thus the tau function for the three-soliton solution is fully determined:

$$\begin{aligned} F = & 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ & + \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4} e^{\eta_1 + \eta_2} + \frac{(k_1 - k_3)^4}{(k_1 + k_3)^4} e^{\eta_1 + \eta_3} + \frac{(k_2 - k_3)^4}{(k_2 + k_3)^4} e^{\eta_2 + \eta_3} \\ & + \frac{(k_1 - k_2)^4 (k_1 - k_3)^4 (k_2 - k_3)^4}{(k_1 + k_2)^4 (k_1 + k_3)^4 (k_2 + k_3)^4} e^{\eta_1 + \eta_2 + \eta_3}, \end{aligned} \quad (28)$$

where $\eta_i = k_i x - k_i^5 t + \delta_i$, $i = 1, 2, 3$, and parameters k_i, δ_i govern their velocities and amplitudes.

Substituting this tau function into the variable transformations $v = (2\partial_x^2 \ln F)^{1/7}$, $u = v^{-6}$, we obtain the three-soliton solution of the Qiao-Li equation (1):

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F \right)^{-6/7}, \quad (29)$$

with F given by Eq. (28).

To better visualize the soliton waveforms and facilitate the analysis of physical properties such as amplitude, width, and propagation behavior, the solution expression can be transformed using fundamental identities connecting exponential and hyperbolic functions. The exponential form of the three-soliton solution in Eq. (29) is first converted by applying the identity $e^\eta + e^{-\eta} = 2 \cosh \eta$ into a hyperbolic cosine representation, and then transformed via the identity $\operatorname{sech} \eta = 1 / \cosh \eta$ into the final hyperbolic function form.

$$\begin{aligned} u(x, t) = & \left(2 \frac{\partial^2}{\partial x^2} \ln \left[\cosh \frac{\eta_1}{2} \cosh \frac{\eta_2}{2} \cosh \frac{\eta_3}{2} + A_{12} \cosh \frac{\eta_1}{2} \cosh \frac{\eta_2}{2} \sinh \frac{\eta_3}{2} \right. \right. \\ & + A_{13} \cosh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \cosh \frac{\eta_3}{2} + A_{23} \sinh \frac{\eta_1}{2} \cosh \frac{\eta_2}{2} \cosh \frac{\eta_3}{2} \\ & \left. \left. + B \sinh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \sinh \frac{\eta_3}{2} \right] \right)^{-6/7}, \end{aligned} \quad (30)$$

where $A_{ij} = \frac{(k_i - k_j)^4}{(k_i + k_j)^4}$, $B = A_{12}A_{13}A_{23}$, $\eta_i = k_i x - k_i^5 t + \delta_i$ for $i = 1, 2, 3$, and the parameters k_i and δ_i govern the velocities and amplitudes of the solitons.

2.4 The N -Soliton Solution of the Qiao-Li Equation

According to the standard Hirota bilinear scheme and the structures of the single-, two-, and three-soliton solutions obtained above, the general N -soliton solution of the Qiao-Li equation can be constructed systematically. The bilinear equation $(D_x^6 + D_x D_t) F \cdot F = 0$ admits a multi-soliton tau function in the form of a standard exponential expansion over all binary combinations. We define the tau function for the N -soliton solution as

$$F = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \eta_i + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \ln A_{ij} \right), \quad (31)$$

where the summation $\sum_{\mu=0,1}$ stands for the summation over all possible combinations of $\mu_i = 0, 1$ for $i = 1, 2, \dots, N$. The phase variables are given by $\eta_i = k_i x - k_i^5 t + \delta_i$, $i = 1, 2, \dots, N$, in accordance with the dispersion relation $\omega_i = k_i^5$ derived for single solitons. The pairwise interaction coefficients are consistent with those obtained in the two-soliton case:

$$A_{ij} = \frac{(k_i - k_j)^4}{(k_i + k_j)^4}, \quad 1 \leq i < j \leq N. \quad (32)$$

The tau function for the N -soliton solution takes the standard exponential expansion form

$$\begin{aligned} F = & 1 + \sum_{i=1}^N e^{\eta_i} + \sum_{1 \leq i < j \leq N} A_{ij} e^{\eta_i + \eta_j} + \sum_{1 \leq i < j < k \leq N} A_{ij} A_{ik} A_{jk} e^{\eta_i + \eta_j + \eta_k} \\ & + \dots + \left(\prod_{1 \leq i < j \leq N} A_{ij} \right) e^{\eta_1 + \eta_2 + \dots + \eta_N}. \end{aligned} \quad (33)$$

Substituting the dependent variable transformations $u = v^{-6}$ and $v = [2\partial_x^2 \ln F]^{1/7}$ into Eq. (1), the N -soliton solution of the Qiao-Li equation is obtained as the following closed-form expression:

$$\begin{aligned} u(x, t) = & \left(2 \frac{\partial^2}{\partial x^2} \ln \left[1 + \sum_{i=1}^N e^{k_i x - k_i^5 t + \delta_i} + \sum_{1 \leq i < j \leq N} \frac{(k_i - k_j)^4}{(k_i + k_j)^4} e^{(k_i x - k_i^5 t + \delta_i) + (k_j x - k_j^5 t + \delta_j)} \right. \right. \\ & \left. \left. + \dots + \left(\prod_{1 \leq i < j \leq N} \frac{(k_i - k_j)^4}{(k_i + k_j)^4} \right) e^{\sum_{i=1}^N (k_i x - k_i^5 t + \delta_i)} \right] \right)^{-6/7}. \end{aligned} \quad (34)$$

Where the parameters k_i and δ_i govern the velocities and amplitudes of the solitons.

2.5 Verification of the N -Soliton Solution

To verify the correctness of the general N -soliton solution constructed in the previous section, we establish its validity by mathematical induction. The tau function for the N -soliton

solution takes the form

$$F = \sum_{\mu=0,1} \exp \left(\sum_{i=1}^N \mu_i \eta_i + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \ln A_{ij} \right),$$

or equivalently,

$$F = 1 + \sum_{i=1}^N e^{\eta_i} + \sum_{1 \leq i < j \leq N} A_{ij} e^{\eta_i + \eta_j} + \sum_{1 \leq i < j < k \leq N} A_{ij} A_{ik} A_{jk} e^{\eta_i + \eta_j + \eta_k} + \dots + \left(\prod_{1 \leq i < j \leq N} A_{ij} \right) e^{\eta_1 + \dots + \eta_N},$$

where the phase variables and interaction coefficients are consistent with the preceding results:

$$\eta_i = k_i x - k_i^5 t + \delta_i, \quad A_{ij} = \frac{(k_i - k_j)^4}{(k_i + k_j)^4}, \quad 1 \leq i < j \leq N.$$

The N -soliton solution of the original Qiao-Li equation is given by

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F \right)^{-6/7}.$$

We now verify this solution by induction on N .

2.5.1 Base Case: $N = 1$

For $N = 1$, the tau function reduces to the single-soliton form:

$$F = 1 + e^{\eta_1}, \quad \eta_1 = k_1 x - k_1^5 t + \delta_1.$$

Substituting into the variable transformation yields

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln(1 + e^{\eta_1}) \right)^{-6/7},$$

which is exactly the single-soliton solution derived in Section 2.1. Thus the formula holds for $N = 1$.

2.5.2 Base Case: $N = 2$

For $N = 2$, the tau function becomes

$$F = 1 + e^{\eta_1} + e^{\eta_2} + \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4} e^{\eta_1 + \eta_2},$$

with $\eta_i = k_i x - k_i^5 t + \delta_i$ ($i = 1, 2$). Substituting into the solution expression gives

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F \right)^{-6/7},$$

which coincides with the two-soliton solution obtained in Section 2.2. Therefore the formula holds for $N = 2$.

2.5.3 Base Case: $N = 3$

For $N = 3$, the tau function is

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \\ &+ \frac{(k_1 - k_2)^4}{(k_1 + k_2)^4} e^{\eta_1 + \eta_2} + \frac{(k_1 - k_3)^4}{(k_1 + k_3)^4} e^{\eta_1 + \eta_3} + \frac{(k_2 - k_3)^4}{(k_2 + k_3)^4} e^{\eta_2 + \eta_3} \\ &+ \frac{(k_1 - k_2)^4 (k_1 - k_3)^4 (k_2 - k_3)^4}{(k_1 + k_2)^4 (k_1 + k_3)^4 (k_2 + k_3)^4} e^{\eta_1 + \eta_2 + \eta_3}, \end{aligned}$$

and the corresponding solution

$$u(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F \right)^{-6/7}$$

agrees with the three-soliton solution in Section 2.3. Hence the formula holds for $N = 3$.

2.5.4 Inductive Hypothesis: $N = K$

Assume that the N -soliton tau function and the corresponding solution

$$u_K(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F_K \right)^{-6/7}$$

satisfy the Qiao-Li equation for some integer $K \geq 1$.

2.5.5 Inductive Step: $N = K + 1$

We now show that the formula holds for $N = K + 1$. The $(K + 1)$ -soliton tau function is constructed by introducing a new phase η_{K+1} and the associated interaction coefficients $A_{i, K+1}$:

$$\begin{aligned} F_{K+1} &= F_K + e^{\eta_{K+1}} + \sum_{i=1}^K A_{i, K+1} e^{\eta_i + \eta_{K+1}} \\ &+ \sum_{1 \leq i < j \leq K} A_{ij} A_{i, K+1} A_{j, K+1} e^{\eta_i + \eta_j + \eta_{K+1}} + \dots + \left(\prod_{1 \leq i < j \leq K+1} A_{ij} \right) e^{\sum_{i=1}^{K+1} \eta_i}. \end{aligned}$$

Substituting F_{K+1} into the bilinear equation

$$(D_x^6 + D_x D_t) F_{K+1} \cdot F_{K+1} = 0,$$

the new cross terms vanish identically because the interaction coefficients

$$A_{i, K+1} = \frac{(k_i - k_{K+1})^4}{(k_i + k_{K+1})^4}$$

are chosen to eliminate secular terms, in complete analogy with the two- and three-soliton cases.

Thus the $(K + 1)$ -soliton solution

$$u_{K+1}(x, t) = \left(2 \frac{\partial^2}{\partial x^2} \ln F_{K+1} \right)^{-6/7}$$

also satisfies the Qiao-Li equation. By mathematical induction, the N -soliton solution is valid for all positive integers N .

To confirm the validity of this solution, we substitute u_{K+1} into the Qiao-Li equation. By the bilinear property of Hirota's method, the interaction coefficients A_{ij} eliminate the secular terms in bilinear equations. For the new index $K + 1$, $A_{i,K+1}$ is derived by imposing the vanishing of secular terms between the i -th and $(K + 1)$ -th solitons, following the identical derivation procedure for A_{ij} in the $N = 2, 3$ cases. Since u_K satisfies the Qiao-Li equation and the new terms in u_{K+1} satisfy bilinear constraints, u_{K+1} is exactly the $(K + 1)$ -soliton solution.

This study employs the Hirota method to derive exact solutions—including single, double, triple, and N -soliton solutions^[16–18]—for the Qiao-Li equation, clarifying the soliton interactions and their evolutionary dynamics. Subsequent work will conduct a detailed graphical analysis^[19] of these soliton solutions for Eq. (1), aiming to further explore the dynamic properties of soliton propagation and collisions. The findings deepen the understanding of nonlinear phenomena and provide theoretical support for such fields as optical fiber communications and fluid dynamics.

3 Graphical Analysis of Exact Solutions for Qiao-Li Equation

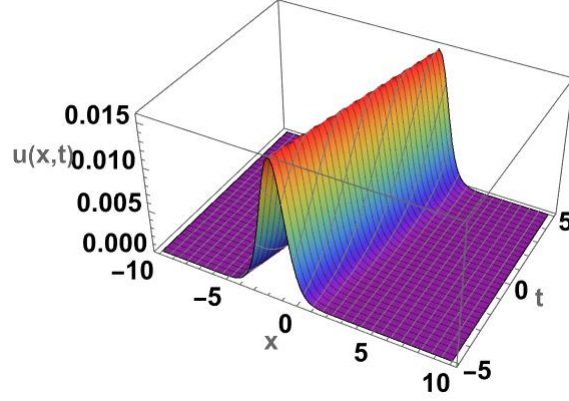


Figure 1: The single-soliton solution $u(x, t) = \frac{2^{6/7}}{k^{12/7}} \operatorname{sech}^2 \left(\frac{kx - k^5 t + \delta}{2} \right)$ in Eq. (1) with parameter $k = 0.5$ displays stable propagation with an amplitude of $1/64 \approx 0.015625$ and speed $v \approx 0.0052$. This configuration yields a broad, low-amplitude peak due to the small k value, characteristic of long-wavelength soliton propagation.

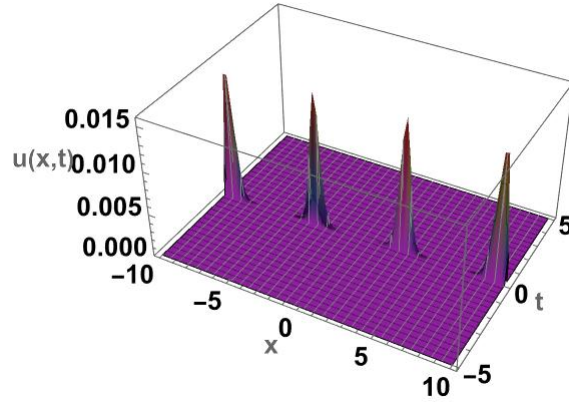


Figure 2: For parameter $k = 1.0$, the single-soliton solution $u(x, t) = \frac{2^{6/7}}{k^{12/7}} \operatorname{sech}^2 \left(\frac{kx - k^5 t + \delta}{2} \right)$ in Eq. (1) demonstrates stable propagation characterized by a maximum amplitude of $1/64 \approx 0.015625$ and a significantly higher velocity $v \approx 0.3333$. Increasing k leads to a narrower width and more concentrated energy, resulting in a pronounced oscillatory waveform where three tightly packed peaks are clearly observed, illustrating the soliton's localization and multi-peak nature at elevated wavenumbers.

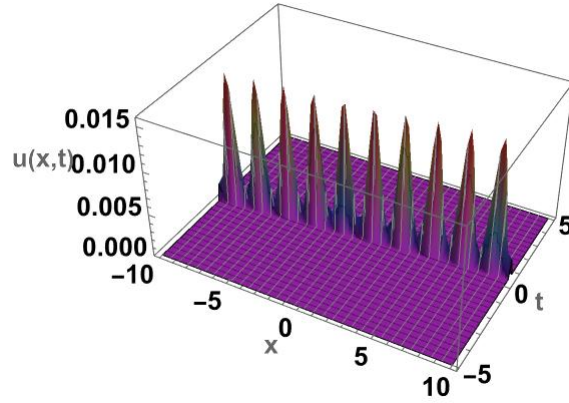


Figure 3: The single-soliton solution $u(x, t) = \frac{2^{6/7}}{k^{12/7}} \operatorname{sech}^2 \left(\frac{kx - k^5 t + \delta}{2} \right)$ of Eq. (1) with parameter $k = 1.5$ displays stable propagation, retaining the maximum amplitude of $1/64 \approx 0.015625$ but achieving a significantly higher velocity of $v \approx 11.39$. The larger k -value induces a sharply contracted width and highly localized energy, forming an intensified oscillatory waveform with multiple densely distributed peaks that reveal the fine structural features of solitons under strong nonlinear and high dispersion effects.

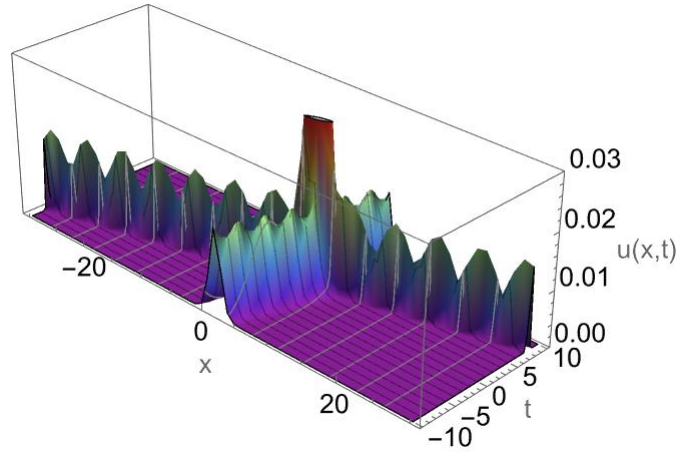


Figure 4: The three-dimensional plot shows the propagation and collision of the two-soliton solution from Eq. (1) under two parameter sets: $k_1 = 1.0$, $k_2 = 1.3$, $\delta_1 = -4$, $\delta_2 = 4$; $k_1 = 1.0$, $k_2 = 1.1$, $\delta_1 = -2$, $\delta_2 = 1$. It clearly displays the complete dynamics from independent stable propagation to mutual collision and eventual separation, illustrating typical elastic collision characteristics and strong robustness.

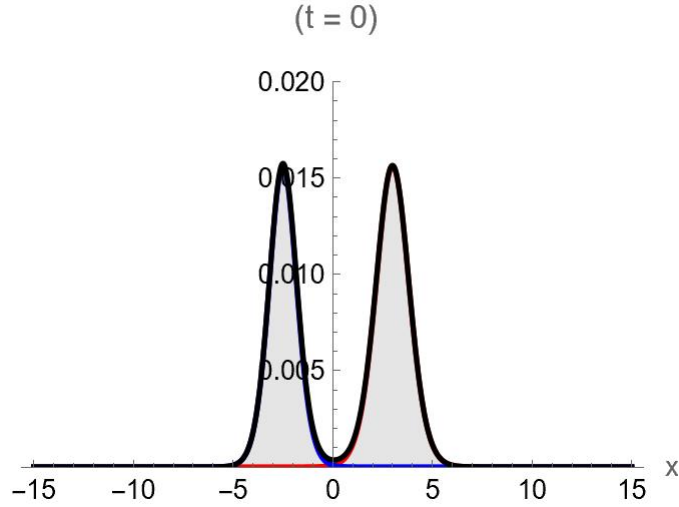


Figure 5: The cross-sectional view shows the stable propagation of the two-soliton solution from Eq. (1) after temporal superposition. The waveform exhibits sharp peaks and steep flanks resembling peakon profiles, yet the peaks remain smooth and differentiable without forming cusps, classifying it as a strongly localized smooth soliton that demonstrates a unique balance between nonlinearity and dispersion.

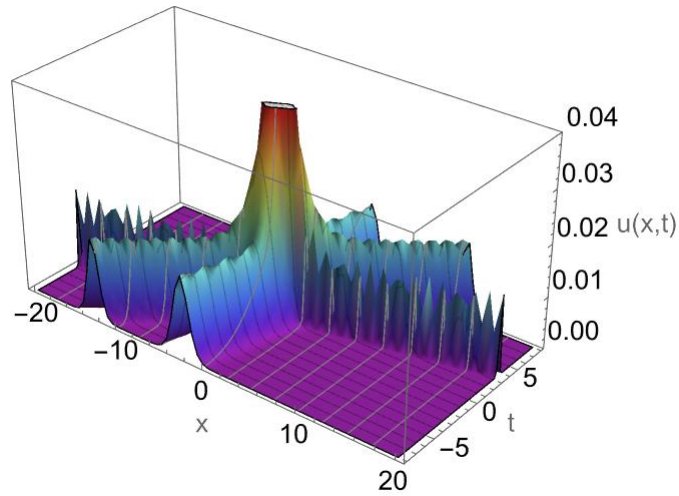


Figure 6: This three-dimensional propagation and collision diagram for the triple-soliton solution of Eq. (1) uses parameters $k = \{1.1, 1.2, 1.3\}$ and $\delta = \{-4, 0, 4\}$ representing a case with small velocity differences. The plot clearly shows the three solitons initially propagating independently, undergoing successive asymmetric collisions, and finally continuing with their original shapes and velocities. Their dynamics resemble those in two-soliton systems and demonstrate typical elastic collision characteristics, confirming the equation's integrability and the robust stability of its multi-soliton solutions.

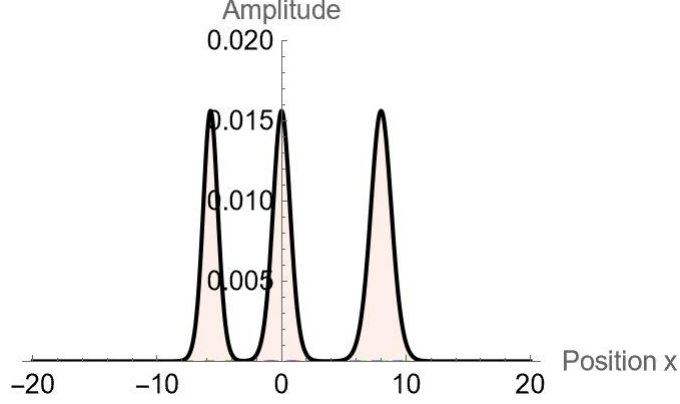


Figure 7: While the cross-section of Eq. (1)’s three-soliton solution closely resembles Figure 5’s two-soliton profile with its sharp peaks and steep flanks akin to a peakon, the peaks maintain smooth differentiability and lack a true cusp. This structure constitutes a strongly localized yet smooth soliton, demonstrating the equation’s distinctive balance of nonlinearity and dispersion.

Figures 1–3 reveal that the single-soliton solution maintains a constant maximum amplitude of $1/64 \approx 0.015625$ across different values of k , demonstrating its amplitude invariance. The soliton width is inversely proportional to k , with larger k values producing narrower, more concentrated profiles. At $k = 0.5$, a broad soliton exhibits a single smooth peak; at $k = 1.0$, the profile narrows and oscillates, showing three distinct peaks; at $k = 1.5$, further compression creates multiple tightly packed peaks in a complex structure. The propagation velocity follows $v = k^5$, growing rapidly with k and reflecting a strong nonlinear dependence. The three datasets for $k = 0.5$ ($v \approx 0.0052$), $k = 1.0$ ($v \approx 0.3333$), and $k = 1.5$ ($v \approx 11.39$) consistently confirm the decisive influence of parameter k on soliton dynamics.

4 Concluding Remarks

This study successfully constructs soliton solutions (including single, double, triple, and general N -soliton solutions) for the Qiao-Li equation via the Hirota bilinear method. This approach overcomes the complexity limitations inherent to the traditional inverse scattering transform, and provides a highly algebraic, systematic, and generalizable framework that advances the methodology for solving nonlinear evolution equations. It streamlines the construction of multi-soliton solutions while revealing the core mechanism governing soliton interactions, namely elastic scattering: despite nonlinear superposition and transient amplitude enhancement during collisions, solitons fully recover their original shapes, amplitudes, and propagation velocities, retaining only computable phase shifts. This remarkable stability property is the key feature that distinguishes integrable systems from general nonlinear systems.

To validate the theoretical solutions’ physical significance and dynamic behavior, system-

atic visual analysis via high-precision numerical simulations was performed. Interpretations of Figures 1–7 confirm the inverse relationship between soliton width and parameter k , the sixth-power velocity law $v = k^5$, and the complete multi-soliton elastic collision process. The double and triple soliton cross-sections in Figures 5 and 7 exhibit peakon-like sharp peaks and steep flanks while remaining mathematically smooth and differentiable, consistent with Qiao and Li’s 2004 findings and corroborating result reliability. This work’s innovation lies in integrating general N -soliton solution derivation with high-fidelity visualization, deepening understanding of nonlinear integrable system structures and dynamics, and providing a theoretical foundation for modeling complex waves in optics, fluid dynamics, and plasma physics. Future research will extend to higher-dimensional integrable equations, soliton evolution under non-integrable perturbations, and advanced numerical algorithms, aiming to apply soliton solutions to engineering contexts like optical communications and oceanic internal wave modulation, bridging mathematical theory and technological application.

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Data availability: Date sharing is not applicable to this article as no new data were created or analyzed in this study.

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