Additional file 1
Mathematical details for the imprecise-probability VOI definitions

This file accompanies the manuscript "Value of Information with Imprecise Probabilities for Decision Modeling." It records the mathematical facts underlying the imprecise-probability VOI estimands defined in the Methods. For continuity, the equations referenced below are restated here with the same labels used in the main text.

Equations referenced from the main text. For a bounded scalar output X with support in $[\ell, u]$ and p-box bounds $\underline{F}_X, \overline{F}_X$, the band expectation bounds are

$$\begin{aligned} \underline{\mathbb{E}}^{\square}(X) &= \ell + \int_{\ell}^{u} \{1 - \overline{F}_X(x)\}\, dx, \\ \overline{\mathbb{E}}^{\square}(X) &= \ell + \int_{\ell}^{u} \{1 - \underline{F}_X(x)\}\, dx, \end{aligned} \tag{1}$$

and, because the induced output distributions are contained in the band,

$$\underline{\mathbb{E}}^{\square}(X) \leq \underline{\mathbb{E}}_{\mathcal{P}}(X) \leq \overline{\mathbb{E}}_{\mathcal{P}}(X) \leq \overline{\mathbb{E}}^{\square}(X). \tag{2}$$

The lower-expectation EVPI and the general policy-based lower-expectation VOI are

$$\mathrm{EVPI}^{\mathrm{LE}}_{\mathcal{P}} = \inf_{P \in \mathcal{P}} \mathbb{E}_P \left\{ \max_{a \in \mathcal{A}} \mathrm{NB}_a(\theta) \right\} - \max_{a \in \mathcal{A}} \inf_{P \in \mathcal{P}} \mathbb{E}_P \{\mathrm{NB}_a(\theta)\}, \tag{3}$$

$$\mathrm{VOI}^{\rho}_{Z} = V^{\rho}_{Z}(\mathcal{P}_Z) - V^{\rho}_{0}(\mathcal{P}). \tag{4}$$

This file records the main mathematical facts used in the Methods (subsection "From incomplete evidence to IP-VOI estimands"). Let $\mathcal{C}$ denote stated evidence constraints, such as parameter ranges, distributional bounds, dependence bounds, or a prespecified set of admissible model structures. These constraints define

$$\mathcal{P}(\mathcal{C}) = \{P : P \text{ is a probability measure on } \Theta \text{ satisfying } \mathcal{C}\}. \tag{5}$$

Assume $\mathcal{P}(\mathcal{C})$ is nonempty and all model outputs considered below are bounded unless otherwise stated.

Proposition 1 (Expectation bounds for the full p-box band)**.** *Let X be a bounded scalar output with support contained in $[\ell, u]$. Let $\mathcal{B}(\underline{F}_X, \overline{F}_X)$ be the set of all cumulative distribution functions F satisfying $\underline{F}_X(x) \leq F(x) \leq \overline{F}_X(x)$ for all x. Then the lower and upper expectations over this full p-box band are given by Equation* (1)*. If $\{F_{X,P} : P \in \mathcal{P}\} \subseteq \mathcal{B}(\underline{F}_X, \overline{F}_X)$, the inequalities in Equation* (2) *follow.*

Proof. For any cumulative distribution function F supported on $[\ell, u]$,

$$\mathbb{E}_F(X) = \ell + \int_{\ell}^{u} \{1 - F(x)\}\, dx.$$

The expectation is decreasing in F pointwise. Therefore the smallest expectation over the full band is obtained by using the largest admissible cumulative distribution function in the band, $\overline{F}_X$, and the largest expectation is obtained by using the smallest one, $\underline{F}_X$. This gives Equation (1). If the output distributions induced by $\mathcal{P}$ form a proper subset of the band, optimizing over the larger band can only decrease the lower endpoint and increase the upper endpoint, which gives Equation (2). $\square$

Proposition 2 (Lower-expectation EVPI)**.** *For a finite strategy set $\mathcal{A}$, the lower-expectation EVPI in Equation* (3) *is nonnegative.*

Proof. Let $M(\theta) = \max_{a\in\mathcal{A}} \mathrm{NB}_a(\theta)$. For every strategy a and every θ, $M(\theta) \geq \mathrm{NB}_a(\theta)$. Monotonicity of expectation implies $\mathbb{E}_P M(\theta) \geq \mathbb{E}_P\{\mathrm{NB}_a(\theta)\}$ for every $P \in \mathcal{P}$. Taking infima over P gives

$$\underline{\mathbb{E}}_{\mathcal{P}} M(\theta) \geq \underline{\mathbb{E}}_{\mathcal{P}}\{\mathrm{NB}_a(\theta)\}$$

for every a. Taking the maximum over a on the right gives

$$\underline{\mathbb{E}}_{\mathcal{P}} M(\theta) \geq \max_{a\in\mathcal{A}} \underline{\mathbb{E}}_{\mathcal{P}}\{\mathrm{NB}_a(\theta)\}.$$

The difference between the two sides is Equation (3), so lower-expectation EVPI is nonnegative. $\square$

Proposition 3 (Nonnegativity of policy-based lower-expectation VOI)**.** *Let Z be information observed before the final strategy is chosen, and let the policy class include all constant policies. Assume the admissible joint set $\mathcal{P}_Z$ extends $\mathcal{P}$, so that its θ-marginal set is $\mathcal{P}$. Then the lower-expectation VOI in Equation* (4) *is nonnegative.*

Proof. A constant policy chooses the same strategy regardless of Z. Because the θ-marginals of $\mathcal{P}_Z$ are exactly $\mathcal{P}$, the lower expected value of any constant policy computed over $\mathcal{P}_Z$ equals its current lower expected value computed over $\mathcal{P}$. Therefore every strategy available before observing Z is also available as a post-information policy with the same guaranteed value. Taking the supremum over all post-information policies cannot give a guaranteed value smaller than the maximum over constant policies. Hence $V_Z^{\mathrm{LE}}(\mathcal{P}_Z) \geq V_0^{\mathrm{LE}}(\mathcal{P})$, and the difference in Equation (4) is nonnegative. $\square$

Proposition 4 (Sharp fixed-measure VOI envelope)**.** *Let $T(P)$ be a conventional fixed-measure VOI functional, such as EVPI, EVPPI for a prespecified parameter group, or EVSI for a prespecified design. The set*

$$\mathcal{V}_T(\mathcal{C}) = \{T(P) : P \in \mathcal{P}(\mathcal{C})\}$$

is the sharp set of fixed-measure VOI values implied by the evidence constraints $\mathcal{C}$. If $\mathcal{P}(\mathcal{C})$ is compact and T is continuous on $\mathcal{P}(\mathcal{C})$, then the lower and upper endpoints are attained. If, in addition, $\mathcal{P}(\mathcal{C})$ is connected, then $\mathcal{V}_T(\mathcal{C})$ is an interval in $\mathbb{R}$.

Proof. For every $P \in \mathcal{P}(\mathcal{C})$, P satisfies the evidence constraints, and therefore $T(P)$ is a conventional VOI value that is compatible with the evidence. Conversely, if a number v is not equal to $T(P)$ for any $P \in \mathcal{P}(\mathcal{C})$, then no probability measure satisfying the evidence constraints can produce v through the conventional VOI calculation. Thus $\mathcal{V}_T(\mathcal{C})$ is exactly the set of fixed-measure VOI values left possible by $\mathcal{C}$.

If $\mathcal{P}(\mathcal{C})$ is compact and T is continuous, the extreme value theorem implies that T attains its minimum and maximum on $\mathcal{P}(\mathcal{C})$. If $\mathcal{P}(\mathcal{C})$ is connected, the continuous image of a connected set in $\mathbb{R}$ is connected. The connected subsets of $\mathbb{R}$ are intervals. Therefore $\mathcal{V}_T(\mathcal{C})$ is an interval. When $\mathcal{P}(\mathcal{C})$ is not connected, the reported envelope is the smallest closed interval containing $\mathcal{V}_T(\mathcal{C})$. $\square$

A simple continuity result is useful for the fixed-measure EVPI envelope. Suppose $\mathcal{A}$ is finite and there is $B < \infty$ such that $|\,\mathrm{NB}_a(\theta)| \leq B$ for all $a \in \mathcal{A}$ and all $\theta \in \Theta$. Let $\|P - Q\|_{\mathrm{TV}} = \sup_{A \subseteq \Theta} |P(A) - Q(A)|$ denote total variation distance. Then

$$|\,\mathrm{EVPI}(P) - \mathrm{EVPI}(Q)| \leq 4B\|P - Q\|_{\mathrm{TV}}. \tag{6}$$

To see this, write $M(\theta) = \max_{a \in \mathcal{A}} \mathrm{NB}_a(\theta)$. Because $|M(\theta)| \leq B$,

$$|\mathbb{E}_P M(\theta) - \mathbb{E}_Q M(\theta)| \leq 2B\|P - Q\|_{\mathrm{TV}}. \tag{7}$$

Also,

$$\begin{aligned} \left|\max_{a \in \mathcal{A}} \mathbb{E}_P\{\mathrm{NB}_a(\theta)\} - \max_{a \in \mathcal{A}} \mathbb{E}_Q\{\mathrm{NB}_a(\theta)\}\right| &\leq \max_{a \in \mathcal{A}} |\mathbb{E}_P\{\mathrm{NB}_a(\theta)\} - \mathbb{E}_Q\{\mathrm{NB}_a(\theta)\}| \\ &\leq 2B\|P - Q\|_{\mathrm{TV}}. \end{aligned} \tag{8}$$

Adding the two bounds gives the result. Thus, under bounded net benefits, small changes in the probability measure cannot produce arbitrarily large changes in conventional EVPI.