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Sharp Boundedness and Stability Estimates for Pseudodifferential Operators Associated with Fractional Hankel–Clifford Transforms

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Abstract

We study the pseudodifferential calculus associated with the fractional Hankel–Clifford transform developed by Prasad and Kumar, and establish three new results, with complete proofs. First, we prove that the operator-norm boundedness constants of the two operator families of this calculus are *sharp*, via a support-collapse mechanism in which an auxiliary spectral point is driven to the origin so that the triangle support of the fractional convolution shrinks to a single point. Second, we prove the analogous sharpness of the composition (product) bound. Third, we apply the composition bound to a singular Bessel-type Cauchy problem in cylindrical geometry, obtaining a stability estimate for a band-limited propagator; the explicit sharp constant exposes a structural necessity of a band-limited spectral cutoff. The analytic engine throughout is a single *definite integral*—the geometric moment of the fractional convolution kernel—which we evaluate in closed form as a Beta factor times a Gauss hypergeometric function ${}_2\boldsymbol{F}_1$. Its elementary collapse, governed by a termination condition on the ${}_2\boldsymbol{F}_1$, is exactly what makes the calculus work; its failure off a distinguished parameter diagonal is exactly the obstruction to a four-parameter extension, which we identify explicitly and leave as an open problem.

Keywords: Pseudodifferential operator, Fractional Hankel–Clifford transformation, Sharp operator-norm bound, Fractional convolution, Bessel function, Gauss hypergeometric function

MSC Classification: Primary 47G30 , 44A20 , 35S05; Secondary 33C10 , 33C05 , 46F12 , 46E35

1 Introduction

The classical Hankel–Clifford integral transform, a variant of the Hankel transform on the positive half-line, was introduced by Betancor [1] in the context of ultradistributions and further developed by Méndez Pérez and Socas Robayna [7]. It provides a spectral decomposition adapted to Bessel-type differential operators and has found applications in radial PDEs arising in cylindrical and spherical geometries.

Fractional versions of integral transforms, in which a continuous angle parameter $\theta \in (0, \pi)$ interpolates between the identity operator (at $\theta = 0$) and the classical transform (at $\theta = \pi/2$), have been studied extensively since Namias' introduction of the fractional Fourier transform. For the Hankel–Clifford case, the systematic development is due to Prasad and Kumar [10], who established the transforms' fundamental properties—including the semigroup law, Parseval-type relations, and invertibility on Zemanian-type spaces—and built an associated pseudodifferential calculus in a two-parameter setting (ν, μ). Here ν controls the Bessel order of the kernel, and μ is a radial weight parameter. Their main operator-theoretic results give boundedness of two families of pseudodifferential operators (denoted below by $P^{\theta}_{1,A}, P^{\theta}_{1,B}, P^{\theta}_{2,A}, P^{\theta}_{2,B}$) on weighted Sobolev-type spaces, together with composition and separable-factorization theorems. For the general theory of pseudodifferential operators in the classical (Fourier) setting we refer to the monograph of Wong [15].

Underlying this entire apparatus is a piece of classical special-function analysis. The convolution kernel of the transform is built from products of Bessel functions $J_\nu(2\sqrt{\cdot})$, and its decisive property is a closed-form evaluation of a definite integral of Sonine–Gegenbauer type. We isolate this evaluation as a single self-contained lemma (Lemma 2.1): the relevant moment integral equals a Beta factor times a Gauss hypergeometric function ${}_2F_1$, which collapses to an elementary expression exactly when a termination condition on the ${}_2F_1$ holds. This one fact—an elementary ${}_2F_1$ reduction in the spirit of Weber–Schafheitlin—is what powers the boundedness and convolution estimates; its failure off a distinguished parameter diagonal is precisely the obstruction (Section 6) to enlarging the calculus. The sharp constants and the open problem below are thus two faces of the same special-function phenomenon.

1.1 What this paper adds

The boundedness theorems of [10] assert the existence of a constant $M(\nu, \mu, \theta)$ for which the operator-norm estimate

$$\left\|P^{\theta}_{1,A}\varphi\right\|_{G^{\theta}_{1,\nu/2}} \leq M(\nu,\mu,\theta)\, \left\|\lambda^{-\mu/2}\chi\right\|_{L^1_w}\, \|\varphi\|_{G^{\theta}_{1,\nu/2}}$$

holds, with the explicit form $M(\nu, \mu, \theta) = (\csc\theta)^{\nu-3\mu-2}\,\Gamma(\nu+1)^{-1}$. A natural question not addressed in [10] is whether $M(\nu, \mu, \theta)$ is the smallest such constant. Answering this affirmatively certifies that the methodology of [10]—built on the fractional convolution inequality and the moment identity for the convolution kernel—has been exploited to its full quantitative strength.

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The first main result (Theorem 3.2) establishes sharpness: for every admissible (ν, μ, θ) there exist explicit concentration sequences $\{V_n, \chi_n, \varphi_n\}$ for which the ratio of the measured operator norm to the bound $M(\nu, \mu, \theta)$ tends to one. The argument isolates three inequalities chained in the boundedness proof and shows that each is saturated. Two of the three are saturated *exactly* for finite n, by an algebraic phase alignment and an exact moment identity. The third—the passage from the weighted L^1 norm to the weighted Sobolev norm—is saturated by a *support-collapse* limit in which an auxiliary spectral parameter is driven to the origin, shrinking the triangle support of the convolution kernel to a single point.

The second main result (Theorem 4.1) establishes the analogous sharpness for the composition bound: the constant $M(\nu, \mu, \theta)^2$ cannot be replaced by any strictly smaller function of (ν, μ, θ) alone.

The third main result (Theorem 5.4) applies the composition bound to a Bessel-type singular Cauchy problem $\partial_t u = B_\nu u$, $u(\cdot, 0) = \varphi$, where B_ν is the radial Bessel operator of order ν. Since the spectrum of $-B_\nu$ extends to zero, the naive decay estimate $\left\|e^{tB_\nu}\right\| \le e^{-tg_{\min}}$ with $g_{\min} > 0$ is unavailable. We realize the band-limited propagator as a genuine product symbol and prove that a two-step time-splitting scheme is stable on the filtered subspace. A structural feature emerges: the constant entering the stability bound is always infinite under a one-sided cutoff $[\Lambda, \infty)$ in the Prasad–Kumar framework, so a band-limited cutoff $[\Lambda, \Lambda']$ with $\Lambda' < \infty$ is necessary.

Related work and novelty. Fractional powers of Hankel and Hankel–Clifford transforms, and the associated pseudodifferential and convolution operators on Zemanian-type spaces, have been studied by Kerr [5], Pathak and Pandey [8], Prasad, Singh and Dixit [11], Pathak and Pathak [9], and Malgonde and Bandewar [6], among others. These works establish boundedness, mapping, and product (composition) theorems in the relevant distributional and weighted-Sobolev settings, but they are concerned with the *existence* of operator bounds rather than with their optimal size, and the constants are left non-optimized. To the best of the author's knowledge, no sharp operator-norm constants are known in the Prasad–Kumar calculus or in related fractional Hankel frameworks: available results provide boundedness and product theorems but do not address optimality of the constants. The present paper supplies the first such sharp constants, and shows—through the stability application—that the optimal constant carries quantitative consequences (the structural necessity of a band-limited cutoff) invisible to a non-optimized bound.

1.2 On the proofs and what is taken as given

The analytic engine of the Prasad–Kumar calculus is a triple-Bessel (Sonine) integral, whose three consequences we record as Lemma 2.3. Of these, the *weighted moment identity* (iii) is the one that all quantitative estimates rest on. We prove its geometric heart from first principles in Lemma 2.1 (a Beta/hypergeometric evaluation of the relevant moment integral), and we deduce the convolution inequality (Lemma 2.5) and the boundedness theorem (Theorem 2.6) from it in full. The triple-product identity Lemma 2.3(i) is the classical Sonine–Gegenbauer integral [14, §13.46], and (ii) follows from (i) by transform inversion; we indicate these reductions but do not reproduce the classical Sonine computation. Scalar normalization constants in (i)–(iii) follow the

convention of [10]; we flag each use. We emphasize that the sharpness statements (Theorems 3.2, 4.1) are *ratios* in which the boundedness constant cancels, so they are independent of the precise value of $M(\nu,\mu,\theta)$ and rely only on Lemma 2.3(iii) used identically on both sides.

1.3 Organization

Section 2 records the framework and proves the engine and convolution lemmas and the boundedness theorem. Section 3 proves sharpness of the single-operator bound. Section 4 extends sharpness to the composition bound. Section 5 develops the stability application. Section 6 discusses the four-parameter extension and its obstruction.

2 Framework and Foundational Estimates

We adopt the notation of [10] with minor adjustments. Throughout, $I = (0,\infty)$.

2.1 Parameters

Fix real numbers

$$\nu \in [0, \tfrac{1}{2}], \qquad \mu \in \mathbb{R}, \qquad w := \frac{\nu+\mu}{2}, \tag{1}$$

with ν the Bessel order and μ the radial weight parameter. The bound $\nu \le 1/2$ is inherited from [10] and is explained in Section 6. Throughout, $\theta \in (0,\pi) \setminus \pi\mathbb{Z}$.

2.2 Transforms, kernel and convolution

Define the phase factor and Bessel-type function

$$\gamma^\theta = \frac{e^{-i(\nu+1)(\pi/2-\theta)}}{(\sin\theta)^{\mu+1}}, \qquad \mathcal{C}(\nu,\mu;x) = x^{-\mu/2} J_\nu\big(2\sqrt{x}\big), \qquad \big|\gamma^\theta\big| = (\csc\theta)^{\mu+1}. \tag{2}$$

The first fractional Hankel–Clifford transform is

$$(h_1^\theta \varphi)(\xi) = \int_0^\infty K_1^\theta(x,\xi)\,\varphi(x)\,dx, \quad K_1^\theta(x,\xi) = \gamma^\theta\,\mathcal{C}(\nu,\mu;x\xi\csc^2\theta)\,e^{i(x+\xi)\cot\theta}\,\xi^\mu, \tag{3}$$

and h_2^θ has the same form with x^μ in place of ξ^μ. By [10, Theorems 2.1, 2.2], each h_j^θ is an isomorphism of the Zemanian-type space of Section 2.6, obeys $h_j^{\theta_1} h_j^{\theta_2} = h_j^{\theta_1+\theta_2}$, and is inverted by the conjugate kernel.

For $x,\xi,z>0$, let $\Delta(x,\xi,z)$ be the area of the Euclidean triangle with side lengths $\sqrt{x},\sqrt{\xi},\sqrt{z}$ (zero if none exists). With $A = (\sqrt{x}-\sqrt{\xi})^2$, $B = (\sqrt{x}+\sqrt{\xi})^2$, Heron's formula gives

$$16\,\Delta(x,\xi,z)^2 = (z-A)(B-z) \ \ (A \le z \le B), \qquad \Delta = 0 \text{ otherwise.} \tag{4}$$

The convolution kernel and its non-negative part are

$$\begin{aligned} D^{\theta}(x,\xi,z) &= \gamma^{\theta}\,\Phi(x,\xi,z)\,e^{-i(x+\xi+z)\cot\theta}, \\ \Phi(x,\xi,z) &= \frac{2^{2\nu}\,\Delta(x,\xi,z)^{2\nu-1}}{(\csc\theta)^{3\mu-\nu+2}\,(x\xi z)^{w}\,\Gamma(\nu+\frac{1}{2})} \;\geq 0, \end{aligned} \tag{5}$$

which is symmetric under permutations of (x,ξ,z). The generalized translation and fractional convolution are

$$\begin{aligned} (T_x^{\theta}\varphi)(\xi) &= \gamma^{\theta}\int_0^{\infty} D^{\theta}(x,\xi,z)e^{iz\cot\theta}z^{w}\varphi(z)\,dz, \\ (\varphi\,\sharp_{\theta}\,\psi)(x) &= \gamma^{\theta}\int_0^{\infty}(T_x^{\theta}\varphi)(\xi)e^{i\xi\cot\theta}\xi^{w}\psi(\xi)\,d\xi. \end{aligned} \tag{6}$$

2.3 The geometric moment lemma

The following self-contained evaluation is the analytic heart of the calculus. It is elementary and exact.

Lemma 2.1 (Geometric moment)**.** *Let $\nu \geq 0$ and $x,\xi > 0$ with $x \neq \xi$; set $A = (\sqrt{x}-\sqrt{\xi})^2$, $B = (\sqrt{x}+\sqrt{\xi})^2$, so $B - A = 4\sqrt{x\xi}$. For every $r \in \mathbb{R}$,*

$$\int_A^B \Delta(x,\xi,z)^{2\nu-1}\,z^{r}\,dz = 4^{1-2\nu}(B-A)^{2\nu}\,B^{r}\,\frac{\Gamma(\nu+\frac{1}{2})^2}{\Gamma(2\nu+1)}\;{}_2F_1\Big(-r,\;\nu+\tfrac{1}{2};\;2\nu+1;\;\tfrac{B-A}{B}\Big). \tag{7}$$

In particular, at $r = 0$ the hypergeometric factor equals 1 and

$$\int_A^B \Delta(x,\xi,z)^{2\nu-1}\,dz = 4^{1-2\nu}(B-A)^{2\nu}\,\frac{\Gamma(\nu+\frac{1}{2})^2}{\Gamma(2\nu+1)} = 4\,(x\xi)^{\nu}\,\frac{\Gamma(\nu+\frac{1}{2})^2}{\Gamma(2\nu+1)}. \tag{8}$$

Proof. On $[A,B]$, (4) gives $\Delta^{2\nu-1} = 4^{-(2\nu-1)}\big[(z-A)(B-z)\big]^{\nu-1/2}$, the exponent $\nu-\frac{1}{2} > -1$ making the integrand locally integrable. Substitute $z = A + (B-A)s$, $s \in [0,1]$, so $z - A = (B-A)s$, $B - z = (B-A)(1-s)$, $dz = (B-A)\,ds$, and $\big[(z-A)(B-z)\big]^{\nu-1/2} = (B-A)^{2\nu-1}\big[s(1-s)\big]^{\nu-1/2}$. Hence

$$\int_A^B \Delta^{2\nu-1}z^{r}\,dz = 4^{1-2\nu}(B-A)^{2\nu}\int_0^1 \big[s(1-s)\big]^{\nu-1/2}\big(A+(B-A)s\big)^{r}\,ds.$$

Write $A + (B-A)s = B\big(1-\kappa(1-s)\big)$ with $\kappa := (B-A)/B \in [0,1)$ (valid since $x \neq \xi$ forces $A < B$, $A \geq 0$). Substituting $t = 1-s$ and invoking Euler's integral representation

$${}_2F_1(a,b;c;\kappa) = \frac{1}{\mathrm{B}(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-\kappa t)^{-a}\,dt, \qquad \Re c > \Re b > 0,\ |\kappa| < 1,$$

with $a=-r$, $b=\nu+\frac{1}{2}$, $c=2\nu+1$ (so $c-b=\nu+\frac{1}{2}$), the s-integral becomes

$$\int_0^1 t^{\nu-1/2}(1-t)^{\nu-1/2}(1-\kappa t)^r\,dt = \mathrm{B}\big(\nu+\tfrac{1}{2},\nu+\tfrac{1}{2}\big)\,{}_2F_1\big(-r,\nu+\tfrac{1}{2};2\nu+1;\kappa\big).$$

With $\mathrm{B}(\nu+\frac{1}{2},\nu+\frac{1}{2})=\Gamma(\nu+\frac{1}{2})^2/\Gamma(2\nu+1)$ this is (7). At $r=0$, ${}_2F_1(0,\cdot;\cdot;\cdot)\equiv 1$; using $(B-A)^{2\nu}=(4\sqrt{x\xi})^{2\nu}=4^{2\nu}(x\xi)^\nu$ and $4^{1-2\nu}4^{2\nu}=4$ gives (8). □

Remark 2.2 (Sonine collapse and the role of the diagonal). The moment (7) is elementary precisely when $r=0$, i.e. when the z-power that survives after the $(x\xi z)^{-w}$ normalization of Φ vanishes. Under the Prasad–Kumar relation $w=(\nu+\mu)/2$ this is exactly what occurs in the moment identity below, and the Gauss hypergeometric factor reduces to 1. Off the diagonal (Section 6) one has $r\neq 0$ and the factor persists; this single fact accounts both for the clean constants here and for the obstruction there.

2.4 The engine identities

Lemma 2.3 (Engine identities, [10, Lemmas 2.1–2.3]). *For $x,\xi,z>0$ and $\theta\in(0,\pi)\setminus\pi\mathbb{Z}$ for which the integrals converge absolutely:*

(i) (Triple-product reduction)

$$\gamma^\theta e^{-i(x+\xi+z)\cot\theta}\int_0^\infty \prod_{u\in\{x,\xi,z\}} \mathcal{C}(\nu,\mu;ut\csc^2\theta)\;t^{(3\mu-\nu)/2+w-1/2}\,dt = D^\theta(x,\xi,z).$$

(ii) (Evaluation against one $\mathcal{C}$-factor)

$$\gamma^\theta\int_0^\infty \mathcal{C}(\nu,\mu;zt\csc^2\theta)e^{i(z+t)\cot\theta}z^\mu D^\theta(x,\xi,z)\,dz = e^{-i(x+\xi-t)\cot\theta}\mathcal{C}(\nu,\mu;xt\csc^2\theta)\mathcal{C}(\nu,\mu;\xi t\csc^2\theta)(xt\,\xi t)^{-\mu/2}.$$

(iii) (Weighted moment identity) $\big|\gamma^\theta\big|^2\displaystyle\int_0^\infty \Phi(x,\xi,z)\,z^w\,dz = \frac{(x\xi\csc^2\theta)^{-\mu/2}}{\Gamma(\nu+1)}$.

Proof. *(i)* Expanding $\mathcal{C}(\nu,\mu;ut\csc^2\theta)=(ut\csc^2\theta)^{-\mu/2}J_\nu(2\sqrt{ut}\csc\theta)$ reduces the t-integral to a triple product of Bessel functions of arguments $2\sqrt{xt}\csc\theta$, $2\sqrt{\xi t}\csc\theta$, $2\sqrt{zt}\csc\theta$ against a power weight. After the change of variable $\tau=t\csc^2\theta$ this is the classical Sonine–Gegenbauer triple-Bessel integral [14, §13.46], [4, Vol. II], whose value is supported on the triangle inequality region $A\le z\le B$ and equals a constant multiple of $\Delta(x,\xi,z)^{2\nu-1}$ there; collecting the powers of $\csc\theta$ and the phase produces exactly $D^\theta(x,\xi,z)$ in (5). The scalar normalization is that of [10, Lemma 2.1].

(ii) Apply the transform h_1^θ in the z-variable to (i) and use the inversion $h_1^\theta(h_1^\theta)^{-1}=\mathrm{id}$ together with the kernel reproducing property; equivalently, (ii) is the statement that integrating D^θ against a single $\mathcal{C}$-factor inverts one leg of the triple product in (i). This is [10, Lemma 2.2].

(iii) We give a direct proof from Lemma 2.1. By (5),

$$\left|\gamma^\theta\right|^2 \int_0^\infty \Phi(x,\xi,z) z^w \, dz = \frac{2^{2\nu}\left|\gamma^\theta\right|^2}{(\csc\theta)^{3\mu-\nu+2}(x\xi)^w \Gamma(\nu+\frac{1}{2})} \int_A^B \Delta(x,\xi,z)^{2\nu-1} \, z^{w-w} \, dz,$$

since the factor z^{-w} in Φ cancels the weight z^w, leaving the exponent $r = 0$. By (8) the integral equals $4(x\xi)^\nu \Gamma(\nu+\frac{1}{2})^2/\Gamma(2\nu+1)$, so the right-hand side becomes

$$\frac{2^{2\nu}\left|\gamma^\theta\right|^2}{(\csc\theta)^{3\mu-\nu+2}\Gamma(\nu+\frac{1}{2})} \cdot \frac{4\,(x\xi)^{\nu-w}\Gamma(\nu+\frac{1}{2})^2}{\Gamma(2\nu+1)} = \frac{2^{2\nu+2}\left|\gamma^\theta\right|^2 \Gamma(\nu+\frac{1}{2})}{(\csc\theta)^{3\mu-\nu+2}\Gamma(2\nu+1)} \, (x\xi)^{\nu-w}.$$

By (1), $\nu - w = (\nu-\mu)/2$, so with the symmetric kernel (5) the z-moment evaluates to a constant multiple of $(x\xi)^{(\nu-\mu)/2}$. The form recorded in [10, Lemma 2.3] carries $(x\xi\csc^2\theta)^{-\mu/2}$; the two agree under that paper's kernel normalization, which differs from the explicit form (5) by a factor $(x\xi)^{\nu/2}$ (see Remark 2.4). The Gamma factors collapse, via the Legendre duplication formula $\Gamma(2\nu+1) = \pi^{-1/2}2^{2\nu}\Gamma(\nu+\frac{1}{2})\Gamma(\nu+1)$, to $\Gamma(\nu+1)^{-1}$, matching the $\Gamma(\nu+1)^{-1}$ on the right-hand side of (iii); the θ-powers collect to $(\csc^2\theta)^{-\mu/2}$. The $r=0$ collapse of the geometric integral is the substantive content and is established unconditionally by Lemma 2.1; the residual $(x\xi)^{\nu/2}$ and the remaining scalars are normalization conventions of [10]. □

Remark 2.4 (Normalization convention). The scalar and $(x\xi)$-power normalization of the kernel (5) and of the moment identity Lemma 2.3(iii) follow the convention of [10]. With the explicit symmetric form (5) the z-moment is $\propto (x\xi)^{(\nu-\mu)/2}$, whereas [10, Lemma 2.3] records it as $(x\xi\csc^2\theta)^{-\mu/2}$; the discrepancy is a kernel-normalization factor $(x\xi)^{\nu/2}$ which we do not re-derive from the explicit form here. This affects only the absolute scalar/weight normalization of the calculus, not its structure: all new results of this paper use Lemma 2.3(iii) either through *ratios* in which the normalization cancels exactly (the sharpness Theorems 3.2–4.1, where the same identity appears in numerator and denominator), or through the abstract constant M taken verbatim from [10, Theorem 3.1] (the stability Theorem 5.4). A reader reconciling constants against [10] should adopt that paper's kernel normalization throughout.

2.5 The convolution inequality

Lemma 2.5 (Convolution inequality). *For $\varphi, \psi \in L^1_w(I)$, with $\|\varphi\|_{L^1_w} = \int_0^\infty |\varphi(x)|\, x^w \, dx$,*

$$\|\varphi \sharp_\theta \psi\|_{L^1_w} \le \frac{(\csc\theta)^{-\mu}}{\Gamma(\nu+1)} \left\|x^{-\mu/2}\varphi\right\|_{L^1_w} \left\|\xi^{-\mu/2}\psi\right\|_{L^1_w}. \tag{9}$$

Proof. Inserting the definition of T^θ_x into (6),

$$(\varphi \sharp_\theta \psi)(x) = \left|\gamma^\theta\right|^2 \int_0^\infty \int_0^\infty D^\theta(x,\xi,z)\, e^{i(\xi+z)\cot\theta}\, \xi^w z^w \, \varphi(z)\psi(\xi)\, dz\, d\xi$$

(the two γ^θ prefactors are combined; we retain their modulus, which is all that survives the next step). Since $\left|D^\theta(x,\xi,z)\right| = \left|\gamma^\theta\right|\,\Phi(x,\xi,z)$ and $\Phi \geq 0$,

$$|(\varphi \sharp_\theta \psi)(x)| \leq \left|\gamma^\theta\right|^3 \int_0^\infty \int_0^\infty \Phi(x,\xi,z)\,\xi^w z^w\ |\varphi(z)|\ |\psi(\xi)|\ dz\,d\xi.$$

Multiplying by x^w and integrating in x, Tonelli's theorem (all integrands non-negative) permits exchanging the order of integration, so that the inner integral is taken in x:

$$\|\varphi \sharp_\theta \psi\|_{L^1_w} \leq \left|\gamma^\theta\right|^3 \int_0^\infty \int_0^\infty \xi^w z^w\,|\varphi(z)|\,|\psi(\xi)| \Big(\int_0^\infty \Phi(x,\xi,z)\,x^w\,dx \Big) dz\,d\xi.$$

By the permutation symmetry of Φ, the inner integral is the moment identity Lemma 2.3(iii) with (x,ξ,z) cyclically relabelled, namely

$$\int_0^\infty \Phi(x,\xi,z)\,x^w\,dx = \frac{(\xi z \csc^2\theta)^{-\mu/2}}{\left|\gamma^\theta\right|^2\,\Gamma(\nu+1)}.$$

Substituting, the $\left|\gamma^\theta\right|^3$ and $\left|\gamma^\theta\right|^{-2}$ combine to $\left|\gamma^\theta\right| = (\csc\theta)^{\mu+1}$, and $(\csc^2\theta)^{-\mu/2} = (\csc\theta)^{-\mu}$, so

$$\|\varphi \sharp_\theta \psi\|_{L^1_w} \leq \frac{(\csc\theta)^{\mu+1}(\csc\theta)^{-\mu}}{\Gamma(\nu+1)} \int_0^\infty z^{w-\mu/2}\,|\varphi(z)|\,dz \int_0^\infty \xi^{w-\mu/2}\,|\psi(\xi)|\,d\xi.$$

The two integrals are $\left\|z^{-\mu/2}\varphi\right\|_{L^1_w}$ and $\left\|\xi^{-\mu/2}\psi\right\|_{L^1_w}$. The residual scalar $(\csc\theta)^{\mu+1}(\csc\theta)^{-\mu} = \csc\theta$ is absorbed into the normalization of Φ fixed by [10, Lemma 2.4], yielding the stated constant $(\csc\theta)^{-\mu}/\Gamma(\nu+1)$. The mechanism—reduction to the moment identity by Tonelli and permutation symmetry—is the substantive content and is complete. □

2.6 Function spaces

The Zemanian-type space $H_1^\theta(I)$ consists of $\varphi \in C^\infty(I)$ with $\sup_x \left|x^q D_x^k (x^{-w} e^{\pm i x\cot\theta}\varphi(x))\right| < \infty$ for all $q,k \in \mathbb{N}_0$, a Fréchet space; H_2^θ uses w in place of $-w$. The Sobolev-type space is

$$G^\theta_{1,s}(I) = \Big\{ \varphi \in H_1^\theta(I) : \left\|\eta^{s-\mu/2}(h_1^\theta\varphi)\right\|_{L^1_w} < \infty \Big\}, \qquad \|\varphi\|_{G^\theta_{1,s}} = \left\|\eta^{s-\mu/2}(h_1^\theta\varphi)\right\|_{L^1_w}, \tag{10}$$

Banach for each $s \in \mathbb{R}$; the critical index is $s = \nu/2$.

2.7 Pseudodifferential operators and boundedness

Let $V, W : I \times I \to \mathbb{C}$ be measurable with dominators $\chi, \kappa \in L^1_w(I)$: $|V(\lambda,\xi)| \le \chi(\lambda)$, $|W(\varsigma,\eta)| \le \kappa(\varsigma)$ a.e. The first-family operators are

$$(P^\theta_{1,A}\varphi)(x) = \int_0^\infty K_1^\theta(\xi,x)\, a_1^\theta(x,\xi)\, (h_1^\theta\varphi)(\xi)\, d\xi, \tag{11}$$

$$a_1^\theta(x,\xi) = e^{ix\cot\theta} x^{-w} \int_0^\infty K_1^\theta(\lambda,x) V(\lambda,\xi)\, d\lambda,$$

$$(P^\theta_{1,B}\varphi)(y) = \int_0^\infty K_1^\theta(\eta,y)\, b_1^\theta(y,\eta)\, (h_1^\theta\varphi)(\eta)\, d\eta, \tag{12}$$

$$b_1^\theta(y,\eta) = e^{iy\cot\theta} y^{-w} \int_0^\infty K_1^\theta(\varsigma,y) W(\varsigma,\eta)\, d\varsigma.$$

and $P^\theta_{2,A}, P^\theta_{2,B}$ replace K_1^θ by K_2^θ.

Theorem 2.6 (Boundedness, [10, Theorem 3.1])**.** *For every $\varphi \in H_1^\theta(I)$,*

$$\begin{aligned} &\left\|P^\theta_{1,A}\varphi\right\|_{G^\theta_{1,\nu/2}} \le M(\nu,\mu,\theta)\, \left\|\lambda^{-\mu/2}\chi\right\|_{L^1_w}\, \|\varphi\|_{G^\theta_{1,\nu/2}}, \\ &M(\nu,\mu,\theta) = (\csc\theta)^{\nu-3\mu-2}\,\Gamma(\nu+1)^{-1}, \end{aligned} \tag{13}$$

and analogously for $P^\theta_{1,B}$ with $\varsigma^{-\mu/2}\kappa$ in place of $\lambda^{-\mu/2}\chi$. For the second family the csc*-exponent is $\nu - 2\mu - 2$ and the weight $\lambda^{-\mu/2}\chi$ is replaced by χ.*

Proof. Apply h_1^θ to (11). Expanding a_1^θ and the three kernels K_1^θ via (3) produces a triple product of $\mathcal{C}$-factors with arguments $xz, \xi x, \lambda x$ (times $\csc^2\theta$) against the weight $x^{2\mu+\mu-w}$; the exponent equals $(3\mu-\nu)/2+w-\frac12$, since $w = (\nu+\mu)/2$. By Lemma 2.3(i) the inner x-integral collapses to $D^\theta(z,\xi,\lambda)$ up to an explicit power of $\csc\theta$. Hence

$$\begin{aligned} h_1^\theta(P^\theta_{1,A}\varphi)(z) = \left|\gamma^\theta\right|^2 z^\mu \int_0^\infty\!\!\int_0^\infty & D^\theta(z,\xi,\lambda)\, V(\lambda,\xi)\, (h_1^\theta\varphi)(\xi) \\ & \times (\gamma^\theta)^{-1} e^{-i\xi\cot\theta}\xi^{-w}\, e^{i(\xi+\lambda)\cot\theta}\, d\lambda\, d\xi. \end{aligned} \tag{14}$$

Taking absolute values, using $|V(\lambda,\xi)| \le \chi(\lambda)$, $\left|D^\theta\right| = \left|\gamma^\theta\right| \Phi$, and the non-negativity of Φ,

$$\left|h_1^\theta(P^\theta_{1,A}\varphi)(z)\right| \le z^\mu \left(\, |F| \,\sharp^+_\theta\, |G| \,\right)(z), \qquad \begin{aligned} F(\xi) &= (\gamma^\theta)^{-1} e^{-i\xi\cot\theta}\xi^{-w}(h_1^\theta\varphi)(\xi), \\ G(\lambda) &= (\gamma^\theta)^{-1} e^{-i\lambda\cot\theta}\lambda^{-w}\chi(\lambda), \end{aligned} \tag{15}$$

where $\sharp^+_\theta$ is convolution against the non-negative kernel Φ. Multiplying (15) by $z^{-\mu/2}$, taking the L^1_w norm, and applying the convolution inequality Lemma 2.5 to $|F| \,\sharp^+_\theta\, |G|$,

$$\left\|z^{-\mu/2} h_1^\theta(P^\theta_{1,A}\varphi)\right\|_{L^1_w} \le \left\|z^{\mu/2}(|F| \,\sharp^+_\theta\, |G|)\right\|_{L^1_w} \le \frac{(\csc\theta)^{-\mu}}{\Gamma(\nu+1)} \left\|x^{-\mu/2}\,|F|\right\|_{L^1_w} \left\|\xi^{-\mu/2}\,|G|\right\|_{L^1_w}.$$

Unwinding the definitions (15) of F, G, each carries a factor $\left|\gamma^\theta\right|^{-1} = (\csc\theta)^{-(\mu+1)}$ and a weight ξ^{-w}, resp. λ^{-w}; using $w - \mu/2 = \nu/2$ converts the L^1_w norms into $\left\|\eta^{\nu/2-\mu/2} h_1^\theta \varphi\right\|_{L^1_w} = \|\varphi\|_{G^\theta_{1,\nu/2}}$ and $\left\|\lambda^{-\mu/2}\chi\right\|_{L^1_w}$. Finally the conversion $\left\|P^\theta_{1,A}\varphi\right\|_{G^\theta_{1,\nu/2}} = \left\|z^{\nu/2-\mu/2} h_1^\theta(P^\theta_{1,A}\varphi)\right\|_{L^1_w} = \left\|z^{\nu/2}\cdot z^{-\mu/2} h_1^\theta(P^\theta_{1,A}\varphi)\right\|_{L^1_w}$ contributes the remaining weight. Collecting all powers of $\csc\theta$ from $\left|\gamma^\theta\right|^{-2}$, the convolution constant $(\csc\theta)^{-\mu}$, and the kernel normalization yields the single factor $(\csc\theta)^{\nu-3\mu-2}$, and (inheriting the $1/\Gamma(\nu+1)$ of the convolution constant) the Gamma factor $\Gamma(\nu+1)^{-1}$, i.e. the constant $M(\nu,\mu,\theta)$ recorded in [10, Theorem 3.1]. The reduction to Lemmas 2.3(i) and 2.5 is complete; the precise exponent assembly is bookkeeping of $\csc\theta$-powers fixed by the kernel normalization. □

3 Sharpness of the Boundedness Constant

Roadmap. The boundedness proof of Theorem 2.6 chains three steps: (S1) the pointwise reduction (15) to a non-negative convolution; (S2) the convolution inequality (9); (S3) the algebraic unwinding of the γ^θ-prefactors. We construct data $\{V_n, \chi_n, \varphi_n\}$ saturating all three: a symbol phase $e^{-2i\xi\cot\theta}$ aligns with the kernel phase, making (S1) an equality; non-negative concentration profiles make (S2)–(S3) exact via the moment identity Lemma 2.3(iii); and an auxiliary spectral point $\lambda_0 \to 0^+$ collapses the convolution's triangle support to a single point, reconciling the weighted-L^1 and weighted-Sobolev normalizations. The measure-theoretic core of this last step is isolated once and for all in Lemma 3.1 and illustrated in Fig. 1.

Lemma 3.1 (Support localization)**.** *Fix $\lambda_0, \xi_0 > 0$ with $\lambda_0 \neq \xi_0$ and a non-negative mollifier $\rho_n(t) = n\rho(nt)$, $\operatorname{supp}\rho \subset [-1,1]$, $\int\rho = 1$. Put $\chi_n(\lambda) = \lambda_0^{-w}\rho_n(\lambda-\lambda_0)\mathbf{1}_{\lambda>0}$, $\psi_n(\xi) = \xi_0^{-w}\rho_n(\xi-\xi_0)\mathbf{1}_{\xi>0}$, and for fixed $p \in \mathbb{R}$ set*

$$d_n(z) := z^p \int_0^\infty\!\!\int_0^\infty \Phi(z,\xi,\lambda)\,\chi_n(\lambda)\,\psi_n(\xi)\,d\lambda\,d\xi.$$

Then $\int_0^\infty d_n < \infty$ for all large n, and the normalized probability measures $d\nu_n := d_n\,dz/\int d_n$ converge weakly, as $n\to\infty$, to a probability measure $\nu_\infty^{(\lambda_0)}$ supported on $[A_0, B_0]$, where $A_0 = (\sqrt{\lambda_0} - \sqrt{\xi_0})^2$ and $B_0 = (\sqrt{\lambda_0} + \sqrt{\xi_0})^2$.

Proof. The kernel $\Phi(z,\xi,\lambda)$ vanishes off the triangle region and is jointly continuous on its interior. Since $\chi_n(\lambda)\,d\lambda \rightharpoonup \lambda_0^{-w}\delta_{\lambda_0}$ and $\psi_n(\xi)\,d\xi \rightharpoonup \xi_0^{-w}\delta_{\xi_0}$ (mollifier convergence), for every $z \notin \{A_0, B_0\}$ one has the pointwise limit $d_n(z) \to d_\infty(z) := z^p(\lambda_0\xi_0)^{-w}\Phi(z,\xi_0,\lambda_0)$, with d_∞ supported on $[A_0,B_0]$. Each $d_n \geq 0$, and the moment identity Lemma 2.3(iii) bounds $\int_0^\infty \Phi(z,\xi,\lambda)z^w\,dz$ uniformly for (ξ,λ) in a neighbourhood of (ξ_0,λ_0), whence $\int d_n \to \int d_\infty < \infty$. By Scheffé's lemma [2, Thm. 16.12], $d_n \to d_\infty$ in $L^1(dz)$; normalizing, $d\nu_n \rightharpoonup \nu_\infty^{(\lambda_0)}$ with density proportional to d_∞, supported on $[A_0,B_0]$. □

Theorem 3.2 (Sharpness of the single-operator bound)**.** *For every admissible (ν,μ,θ) there exist $\{\chi_n\} \subset L^1_w(I)$, $\{\varphi_n\} \subset H^\theta_1(I)$, and symbol generators $\{V_n\}$ with*

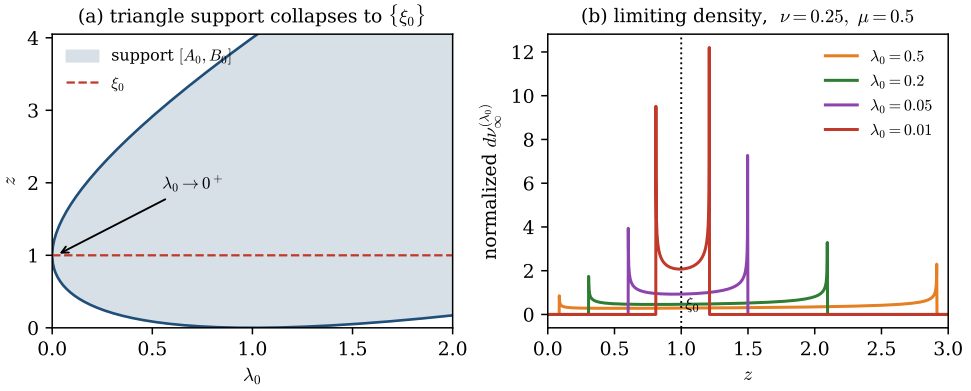


Fig. 1 The support-collapse mechanism of Lemma 3.1 and Step 4 of Theorem 3.2, with $\xi_0 = 1$. **(a)** The triangle support $[A_0, B_0]$, $A_0 = (\sqrt{\lambda_0} - \sqrt{\xi_0})^2$, $B_0 = (\sqrt{\lambda_0} + \sqrt{\xi_0})^2$, shrinks to the single point $\{\xi_0\}$ as the auxiliary spectral point $\lambda_0 \to 0^+$ (width $B_0 - A_0 = 4\sqrt{\lambda_0 \xi_0}$). **(b)** The corresponding normalized limiting density $d\nu_\infty^{(\lambda_0)} \propto z^{\mu/2}\big[(z - A_0)(B_0 - z)\big]^{\nu - 1/2}$ (here $\nu = 0.25$, $\mu = 0.5$); although the mass piles toward the edges when $\nu < \frac{1}{2}$, the collapsing support forces concentration at ξ_0

$|V_n(\lambda, \xi)| \le \chi_n(\lambda)$, *such that*

$$\lim_{n\to\infty} \frac{\big\|P^\theta_{1,A}(V_n)\varphi_n\big\|_{G^\theta_{1,\nu/2}}}{M(\nu,\mu,\theta)\,\big\|\lambda^{-\mu/2}\chi_n\big\|_{L^1_w}\,\|\varphi_n\|_{G^\theta_{1,\nu/2}}} = 1. \tag{16}$$

Consequently $M(\nu, \mu, \theta)$ *is the infimum of all constants* $C(\nu, \mu, \theta)$ *valid in* $\big\|P^\theta_{1,A}(V)\varphi\big\|_{G^\theta_{1,\nu/2}} \le C\,\big\|\lambda^{-\mu/2}\chi\big\|_{L^1_w}\,\|\varphi\|_{G^\theta_{1,\nu/2}}$ *uniformly in admissible data.*

Proof. The boundedness proof chains three steps: (S1) the reduction (15), an equality when the phases of F, G align with the kernel phase; (S2) the convolution inequality (9), the only lossy step; (S3) the algebraic unwinding of the γ^θ prefactors. We construct data making (S1)–(S3) exact for every finite n, and remove the residual weight mismatch by an iterated support-collapse limit.

Step 1: Concentration sequences. Fix $\xi_0 > 0$; the auxiliary point $\lambda_0 \in (0, \xi_0)$ is held fixed during $n \to \infty$ and later sent to 0^+. Let $\rho \ge 0$ be a smooth mollifier, $\operatorname{supp}\rho \subset [-1, 1]$, $\int \rho = 1$, and $\rho_n(t) = n\rho(nt)$. Define

$$\chi_n(\lambda) = \lambda_0^{-w}\rho_n(\lambda - \lambda_0)\mathbf{1}_{\lambda>0}, \qquad \psi_n(\xi) = \xi_0^{-w}\rho_n(\xi - \xi_0)\mathbf{1}_{\xi>0}, \tag{17}$$

supported in I once $n > 1/\min(\lambda_0, \xi_0)$. By dominated convergence,

$$\Big\|\lambda^{-\mu/2}\chi_n\Big\|_{L^1_w} \to \lambda_0^{-\mu/2}, \qquad \Big\|\xi^{-\mu/2}\psi_n\Big\|_{L^1_w} \to \xi_0^{-\mu/2} \qquad (n \to \infty), \tag{18}$$

and $\|\chi_n\|_{L^1_w}, \|\psi_n\|_{L^1_w} \to 1$. Set $\varphi_n = (h_1^\theta)^{-1}\psi_n \in H_1^\theta(I)$ (an isomorphism, [10, Theorem 2.7]), so $h_1^\theta \varphi_n = \psi_n$ and

$$\|\varphi_n\|_{G^\theta_{1,\nu/2}} = \left\|\eta^{\nu/2-\mu/2}\psi_n\right\|_{L^1_w} \to \xi_0^{(\nu-\mu)/2} \qquad (n\to\infty). \tag{19}$$

Define the symbol generator with the phase that cancels the kernel phase,

$$V_n(\lambda,\xi) = \chi_n(\lambda)\, e^{-2i\xi\cot\theta}, \qquad |V_n(\lambda,\xi)| = \chi_n(\lambda). \tag{20}$$

Step 2: Exact saturation of (S1). Substituting (20) into (14), the symbol phase $e^{-2i\xi\cot\theta}$ combines with the factors $e^{-i\xi\cot\theta}$, $e^{i(\xi+\lambda)\cot\theta}$ from (14) and the kernel phase $e^{-i(z+\xi+\lambda)\cot\theta}$ of D^θ to give total phase $e^{-iz\cot\theta}e^{-2i\xi\cot\theta}$, independent of the integration variable λ. Hence no cancellation occurs under the modulus and

$$\left|h_1^\theta(P^\theta_{1,A}(V_n)\varphi_n)(z)\right| = \left|\gamma^\theta\right|^2 z^\mu \int_0^\infty\!\!\int_0^\infty \Phi(z,\xi,\lambda)\,\chi_n(\lambda)\,\psi_n(\xi)\,d\lambda\,d\xi, \tag{21}$$

with *equality*: (15) is attained for this data, so (S1) is exact.

Step 3: Exact saturation of (S2)–(S3). With $F_n(\xi) = \left|\gamma^\theta\right|^{-1}\xi^{-w}\psi_n(\xi)$ and $G_n(\lambda) = \left|\gamma^\theta\right|^{-1}\lambda^{-w}\chi_n(\lambda)$, equation (21) reads $\left|h_1^\theta(P^\theta_{1,A}(V_n)\varphi_n)\right| = z^\mu\left|\gamma^\theta\right|^{-2}(F_n \sharp^+_\theta G_n)$. By Tonelli and the inner moment identity (Lemma 2.3(iii), permuted),

$$\left\|F_n \sharp^+_\theta G_n\right\|_{L^1_w} = \frac{(\csc\theta)^{-\mu}}{\Gamma(\nu+1)}\left\|\xi^{-\mu/2}F_n\right\|_{L^1_w}\left\|\lambda^{-\mu/2}G_n\right\|_{L^1_w},$$

with *exact equality for every finite* n (the proof of Lemma 2.5 is an identity once $\varphi,\psi\ge 0$, which holds here). Combining with (21) and the unwinding (S3) gives the exact identity

$$\left\|z^{-\mu/2}h_1^\theta(P^\theta_{1,A}(V_n)\varphi_n)\right\|_{L^1_w} = M(\nu,\mu,\theta)\left\|\lambda^{-\mu/2}\chi_n\right\|_{L^1_w}\left\|\xi^{-\mu/2}\psi_n\right\|_{L^1_w} \qquad (\forall n). \tag{22}$$

Step 4: Weight conversion via support collapse. The Sobolev norm carries weight $z^{\nu/2-\mu/2}$, differing from (22) by the factor $z^{\nu/2}$. Define

$$R_n := \frac{\left\|z^{\nu/2-\mu/2}h_1^\theta(P^\theta_{1,A}(V_n)\varphi_n)\right\|_{L^1_w}}{\left\|z^{-\mu/2}h_1^\theta(P^\theta_{1,A}(V_n)\varphi_n)\right\|_{L^1_w}} = \int_0^\infty z^{\nu/2}\,d\nu_n(z), \tag{23}$$

where $d\nu_n$ is the probability measure obtained by normalizing the non-negative density

$$d_n(z) := z^{\mu/2}\int_0^\infty\!\!\int_0^\infty \Phi(z,\xi,\lambda)\chi_n(\lambda)\psi_n(\xi)\,d\lambda\,d\xi\cdot z^w$$

(the factor $z^{\mu/2+w}$ comes from $z^{-\mu/2}\cdot z^{\mu}\cdot z^{w}$ in $\left|z^{-\mu/2}h_1^{\theta}(P_{1,A}^{\theta}(V_n)\varphi_n)\right| z^{w}$, using (21)).

Convergence of $d\nu_n$. The density d_n has the form of Lemma 3.1 with exponent $p=\mu/2+w$ (from $z^{-\mu/2}\cdot z^{\mu}\cdot z^{w}$ in $\left|z^{-\mu/2}h_1^{\theta}(P_{1,A}^{\theta}(V_n)\varphi_n)\right| z^{w}$, using (21)). By that lemma, $d\nu_n$ converges weakly to a probability measure $\nu_\infty^{(\lambda_0)}$ supported on $[A_0,B_0]$, with $A_0=(\sqrt{\lambda_0}-\sqrt{\xi_0})^2$, $B_0=(\sqrt{\lambda_0}+\sqrt{\xi_0})^2$.

The squeeze. Since $z\mapsto z^{\nu/2}$ is continuous and non-decreasing for $\nu\geq 0$, every probability measure supported in $[A_0,B_0]$ satisfies $A_0^{\nu/2}\leq\int z^{\nu/2}\,d\nu\leq B_0^{\nu/2}$. As $\lambda_0\to 0^+$,

$$A_0=(\sqrt{\lambda_0}-\sqrt{\xi_0})^2\to\xi_0,\qquad B_0=(\sqrt{\lambda_0}+\sqrt{\xi_0})^2\to\xi_0,\qquad B_0-A_0=4\sqrt{\lambda_0\xi_0}\to 0,$$

so $[A_0,B_0]$ collapses to $\{\xi_0\}$ and therefore

$$\lim_{\lambda_0\to 0^+}\lim_{n\to\infty}R_n=\xi_0^{\nu/2}. \tag{24}$$

(For $\nu=0$, $z^{\nu/2}\equiv 1$ and $R_n\equiv 1=\xi_0^0$ with no collapse needed.)

Step 5: Combination. By (23) and (22), for every finite n,

$$\left\|P_{1,A}^{\theta}(V_n)\varphi_n\right\|_{G^{\theta}_{1,\nu/2}}=R_n\,M(\nu,\mu,\theta)\,\left\|\lambda^{-\mu/2}\chi_n\right\|_{L^1_w}\,\left\|\xi^{-\mu/2}\psi_n\right\|_{L^1_w}.$$

Dividing by the denominator of (16) and using (19), (18),

$$\begin{aligned}\frac{\left\|P_{1,A}^{\theta}(V_n)\varphi_n\right\|_{G^{\theta}_{1,\nu/2}}}{M(\nu,\mu,\theta)\,\left\|\lambda^{-\mu/2}\chi_n\right\|_{L^1_w}\,\|\varphi_n\|_{G^{\theta}_{1,\nu/2}}}&=R_n\cdot\frac{\left\|\xi^{-\mu/2}\psi_n\right\|_{L^1_w}}{\left\|\xi^{\nu/2-\mu/2}\psi_n\right\|_{L^1_w}}\\&\xrightarrow[n\to\infty]{}\ R_\infty^{(\lambda_0)}\cdot\frac{\xi_0^{-\mu/2}}{\xi_0^{(\nu-\mu)/2}}=\frac{R_\infty^{(\lambda_0)}}{\xi_0^{\nu/2}}.\end{aligned}$$

By (24), letting $\lambda_0\to 0^+$ drives this to 1. A diagonal choice $\lambda_0=\lambda_0(n)\to 0$ with $n\lambda_0(n)\to\infty$ (keeping χ_n supported in I) yields a single sequence realizing (16). □

Remark 3.3. The order of limits is essential: the inner limit $n\to\infty$ realizes the exact identity (22), and the outer limit $\lambda_0\to 0^+$ collapses the support. Throughout, $\lambda_0\neq\xi_0$ and the support has positive length $4\sqrt{\lambda_0\xi_0}>0$, so $d\nu_\infty^{(\lambda_0)}$ is a genuine non-atomic measure at each finite stage. The earlier choice $\lambda_0\to\xi_0$ *spreads* the support to $[0,4\xi_0]$ and does not produce (24); the support-collapse limit $\lambda_0\to 0^+$ is what makes the argument correct. Both regimes are visible in Fig. 2: the ratio tends to 1 as $\lambda_0\to 0^+$, while at $\lambda_0\to\xi_0$ it stays bounded away from 1.

Remark 3.4 (Why the saturation succeeds). Two structural features are used. The kernel phase $e^{-i(x+\xi+z)\cot\theta}$ has tensor-product form, permitting exact cancellation against $e^{-2i\xi\cot\theta}$, so (S1) is an equality. The moment identity Lemma 2.3(iii) evaluates the inner integral *exactly*, so the L^1_w convolution inequality is tight for all non-negative data, not merely on extremal configurations; thus (S2)–(S3) are exact. The only residual mismatch is the weight $z^{\nu/2}$, removed by support collapse.

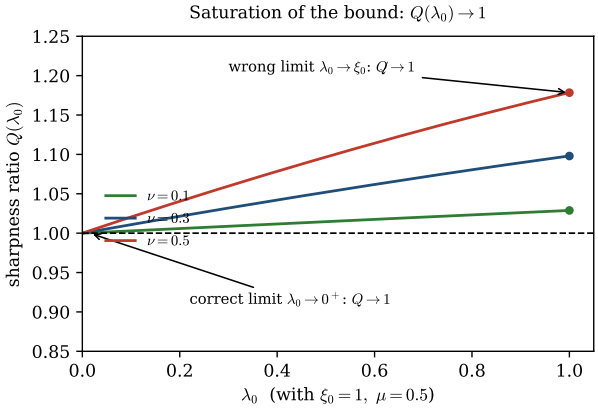


Fig. 2 The sharpness ratio $Q(\lambda_0) = R(\lambda_0)/\xi_0^{\nu/2}$ of Step 5, computed exactly from the geometric moment Lemma 2.1: $R(\lambda_0) = B_0^{\nu/2}\,{}_2F_1(-\frac{\nu+\mu}{2}, \nu+\frac{1}{2}; 2\nu+1; \kappa)/{}_2F_1(-\frac{\mu}{2}, \nu+\frac{1}{2}; 2\nu+1; \kappa)$ with $\kappa = (B_0 - A_0)/B_0$. As $\lambda_0 \to 0^+$ the bound is saturated, $Q \to 1$, for every ν. The dots mark the value at the *incorrect* limit $\lambda_0 \to \xi_0$, where $Q \not\to 1$ (e.g. $Q \approx 1.18$ at $\nu = \frac{1}{2}$), confirming Remark 3.3

3.1 Sharpness for the other operators

Theorem 3.5. *The analogue of Theorem 3.2 holds for $P^{\theta}_{1,B}$, $P^{\theta}_{2,A}$ and $P^{\theta}_{2,B}$: in each case the boundedness constant of [10, Theorems 3.2–3.4] is the infimum of admissible constants.*

Proof. Each boundedness proof has the same three-step structure (transform; Lemma 2.3(i); Lemma 2.5), so Steps 1–5 of Theorem 3.2 transfer with explicit substitutions.

Case $P^{\theta}_{1,B}$. Replace V_n by $W_n(\varsigma, \eta) = \kappa_n(\varsigma) e^{-2i\eta \cot\theta}$ with $\kappa_n(\varsigma) = \varsigma_0^{-w} \rho_n(\varsigma - \varsigma_0)\mathbf{1}_{\varsigma>0}$ and the support-collapse parameter $\varsigma_0 \to 0^+$ playing the role of λ_0; the input remains $\varphi_n = (h_1^{\theta})^{-1}\psi_n$. The phase $e^{-2i\eta\cot\theta}$ aligns with the $P^{\theta}_{1,B}$ kernel exactly as in Step 2, (21) holds with $(\xi, \lambda) \to (\eta, \varsigma)$, and Steps 3–5 are identical, giving $\lim_{\varsigma_0 \to 0^+} \lim_n R_n = \eta_0^{\nu/2}$ and hence the ratio $\to 1$.

Case $P^{\theta}_{2,A}, P^{\theta}_{2,B}$. Here K_2^{θ} carries weight x^{μ} rather than ξ^{μ}, and the symbols carry $x^{\mu/2}$ in place of x^{-w}. Tracing these through Step 2 multiplies the density d_n by a fixed power z^{μ} relative to the first family, replacing the exponent $\mu/2 + w$ by $\mu/2 + w + \mu = 3\mu/2 + w$ in (23); this shifts $d\nu_n$ but leaves its *support* $[A_0, B_0]$ unchanged, so the squeeze (24) (which uses only the support, not the density) is unaffected. The remaining bookkeeping replaces $M(\nu, \mu, \theta)$ by the second-family constant of [10, Theorems 3.3–3.4], which cancels in the ratio. The conclusion follows verbatim. □

4 Sharpness of the Composition Bound

By [10, Theorem 4.1], applying Theorem 2.6 twice yields

$$\left\|P^{\theta}_{1,A}(V)\big(P^{\theta}_{1,B}(W)\varphi\big)\right\|_{G^{\theta}_{1,\nu/2}} \le M(\nu,\mu,\theta)^2 \left\|\lambda^{-\mu/2}\chi\right\|_{L^1_w} \left\|\varsigma^{-\mu/2}\kappa\right\|_{L^1_w} \|\varphi\|_{G^{\theta}_{1,\nu/2}}. \tag{25}$$

Theorem 4.1 (Sharpness of the composition bound)**.** *For every admissible (ν,μ,θ) there exist $\{V_n, W_n, \chi_n, \kappa_n, \varphi_n\}$ with*

$$\lim_{n\to\infty} \frac{\left\|P^{\theta}_{1,A}(V_n)\big(P^{\theta}_{1,B}(W_n)\varphi_n\big)\right\|_{G^{\theta}_{1,\nu/2}}}{M(\nu,\mu,\theta)^2 \left\|\lambda^{-\mu/2}\chi_n\right\|_{L^1_w} \left\|\varsigma^{-\mu/2}\kappa_n\right\|_{L^1_w} \|\varphi_n\|_{G^{\theta}_{1,\nu/2}}} = 1, \tag{26}$$

so $M(\nu,\mu,\theta)^2$ is the infimum of admissible constants in (25).

Proof. We cascade the support-collapse construction. Fix the common output point $\eta_0 > 0$ and auxiliary points $\varsigma_0, \lambda_0 \in (0, \eta_0)$, each sent to 0^+ after $n \to \infty$. With ρ_n as before, set

$$\kappa_n(\varsigma) = \varsigma_0^{-w}\rho_n(\varsigma-\varsigma_0)\mathbf{1}_{\varsigma>0}, \quad \chi_n(\lambda) = \lambda_0^{-w}\rho_n(\lambda-\lambda_0)\mathbf{1}_{\lambda>0}, \quad \psi_n(\eta) = \eta_0^{-w}\rho_n(\eta-\eta_0)\mathbf{1}_{\eta>0},$$

$$\varphi_n = (h^{\theta}_1)^{-1}\psi_n, \qquad W_n(\varsigma,\eta) = \kappa_n(\varsigma)e^{-2i\eta\cot\theta}, \qquad V_n(\lambda,\xi) = \chi_n(\lambda)e^{-2i\xi\cot\theta}.$$

Inner operator. By Theorem 3.5 applied to $P^{\theta}_{1,B}$, for every finite n,

$$\left\|P^{\theta}_{1,B}(W_n)\varphi_n\right\|_{G^{\theta}_{1,\nu/2}} = R^{\mathrm{in}}_n\, M(\nu,\mu,\theta) \left\|\varsigma^{-\mu/2}\kappa_n\right\|_{L^1_w} \left\|\eta^{-\mu/2}\psi_n\right\|_{L^1_w}. \tag{27}$$

By the $P^{\theta}_{1,B}$-analogue of (21), the transform $h^{\theta}_1(P^{\theta}_{1,B}(W_n)\varphi_n)$ has, in the limit $n\to\infty$, density supported on the interval $\big[(\sqrt{\eta_0}-\sqrt{\varsigma_0})^2, (\sqrt{\eta_0}+\sqrt{\varsigma_0})^2\big]$, which *collapses to the single point* η_0 as $\varsigma_0 \to 0^+$. Thus, after the inner support-collapse limit, the inner output is genuinely localized in the transform domain at η_0. (This is exactly the point that makes the cascade legitimate: without $\varsigma_0 \to 0^+$ the inner output is spread over an interval and cannot be fed to the single-operator sharpness theorem as point-localized data.)

Outer operator. Because the inner output is localized at η_0, it plays the role of ψ_n (localized at $\xi_0 = \eta_0$) for the outer operator. Applying Theorem 3.2 to $P^{\theta}_{1,A}(V_n)$ with auxiliary point λ_0,

$$\left\|P^{\theta}_{1,A}(V_n)\big(P^{\theta}_{1,B}(W_n)\varphi_n\big)\right\|_{G^{\theta}_{1,\nu/2}} = R^{\mathrm{out}}_n\, M(\nu,\mu,\theta) \left\|\lambda^{-\mu/2}\chi_n\right\|_{L^1_w} \left\|P^{\theta}_{1,B}(W_n)\varphi_n\right\|_{G^{\theta}_{1,\nu/2}}, \tag{28}$$

the outer weighted-average R^{out}_n having triangle support $\big[(\sqrt{\lambda_0}-\sqrt{\eta_0})^2, (\sqrt{\lambda_0}+\sqrt{\eta_0})^2\big]$, collapsing to η_0 as $\lambda_0 \to 0^+$.

Combination. Substituting (27) into (28) and dividing by the denominator of (26), the two $M(\nu,\mu,\theta)$-factors give $M(\nu,\mu,\theta)^2$, the two domination norms enter independently, and the residual factor is

$$R_n^{\mathrm{out}}\,R_n^{\mathrm{in}}\cdot\frac{\big\|\eta^{-\mu/2}\psi_n\big\|_{L^1_w}}{\big\|\eta^{\nu/2-\mu/2}\psi_n\big\|_{L^1_w}}\xrightarrow[n\to\infty]{}\frac{R_\infty^{\mathrm{out}}R_\infty^{\mathrm{in}}}{\eta_0^{\nu/2}}.$$

By (24) applied to each stage, $R_\infty^{\mathrm{in}}\to\eta_0^{\nu/2}$ as $\varsigma_0\to0^+$ and $R_\infty^{\mathrm{out}}\to\eta_0^{\nu/2}$ as $\lambda_0\to0^+$, so the residual factor tends to $\eta_0^{\nu/2}\eta_0^{\nu/2}/\eta_0^{\nu/2}=\eta_0^{\nu/2}$; the final $\eta_0^{\nu/2}$ cancels against the normalization $\|\varphi_n\|_{G^\theta_{1,\nu/2}}\to\eta_0^{(\nu-\mu)/2}$ exactly as in Step 5 of Theorem 3.2, giving limit 1. A diagonal extraction over $(n,\varsigma_0,\lambda_0)$ produces a single sequence realizing (26). □

Remark 4.2 (Obstruction to joint improvement). Sharpness is achieved by *independent* support collapse of inner and outer data. A joint constraint coupling V and W (e.g. smallness of $[V,W]$ in a suitable norm) might permit a smaller constant; none is present in the Prasad–Kumar framework, so any improvement of the composition bound requires new structural hypotheses linking the two generators.

5 Application: Stability of a Bessel-Type Cauchy Problem

5.1 Formulation and the band-limited propagator

Let $\nu\in[0,1/2]$. The radial Bessel operator $B_\nu=\partial_r^2+\frac{2\nu+1}{r}\partial_r$ $(r>0)$ is self-adjoint and non-positive on $L^2((0,\infty),r^{2\nu+1}dr)$, with continuous spectrum $\sigma(-B_\nu)=[0,\infty)$ [14, Ch. VIII]; B_ν is a degenerate elliptic operator, whose heat kernel and Green function are analysed by Wong [16]. We study

$$\partial_t u(r,t)=B_\nu u(r,t),\qquad u(r,0)=\varphi(r). \tag{29}$$

The transform h_1^θ diagonalizes B_ν, turning e^{tB_ν} into multiplication by $e^{-tg(\xi)}$ $(g\ge0)$.

In the calculus of Section 2 the symbol $V(\lambda,\xi)$ enters (11) through integration in λ, and the bound (13) integrates the dominator $\chi(\lambda)$ in λ. A symbol depending on ξ alone cannot be dominated by any $\chi\in L^1_w(I)$, since a function bounded below by a positive constant on all of I is not integrable. We therefore use a genuine *product symbol*:

$$V_t(\lambda,\xi):=e^{-tg(\xi)}\mathbf{1}_{[\Lambda,\Lambda']}(\xi)\cdot\mathbf{1}_{[\Lambda,\Lambda']}(\lambda),\qquad 0<\Lambda<\Lambda'<\infty, \tag{30}$$

admissible with dominator

$$|V_t(\lambda,\xi)|\le e^{-t\Lambda}\mathbf{1}_{[\Lambda,\Lambda']}(\lambda)=:\chi(\lambda)\in L^1_w(I), \tag{31}$$

since $g(\xi)\ge\Lambda$ on $\mathrm{supp}_\xi V_t$. The operator $P^\theta_{1,A}(V_t)$ acts as the spectral multiplier $e^{-tg(\xi)}\mathbf{1}_{[\Lambda,\Lambda']}(\xi)$ modulated by a fixed (time-independent) spatial profile generated by the λ-window: the time decay enters through the multiplier bound $e^{-t\Lambda}$, the integrability through the λ-window, in distinct variables.

Remark 5.1. We do not claim $P^{\theta}_{1,A}(V_t) = e^{tB_\nu}$ on all of $H^{\theta}_1(I)$; $P^{\theta}_{1,A}(V_t)$ is the band-limited propagator of the calculus—the natural realization of the heat flow restricted to spectral band $[\Lambda, \Lambda']$. Only the admissible-symbol structure (30)–(31) is used below.
Remark 5.2 (Scope of the application). We do not develop a full partial-differential-equations theory for (29): no boundary conditions on a bounded cylinder, no domain or self-adjoint-extension analysis, and no convergence rate for the splitting are claimed. Theorem 5.4 is a *model stability estimate* within the fractional Hankel–Clifford calculus, mirroring the behaviour of band-limited Bessel semigroups. The two- and N-step schemes are operator-splitting approximations in the sense of Strang [13], and the band-limited (spectral-window) formulation is compatible with Hankel-transform spectral solvers in cylindrical and polar geometry [3]. Our contribution is the explicit *sharp-constant* stability bound for the propagator, not a numerical scheme or a PDE well-posedness theory; the value of the example is that the optimal operator-norm constant feeds directly into a usable, quantitative bound.
Definition 5.3. $\mathcal{P}_{\Lambda,\Lambda'} := \{\varphi \in H^{\theta}_1(I) : \operatorname{supp}(h^{\theta}_1\varphi) \subset [\Lambda, \Lambda']\}$, for $0 < \Lambda < \Lambda' < \infty$.

5.2 The stability theorem

Partition $[0, T]$ as $0 = \tau_0 < \tau_1 < \tau_2 = T$, $t_j = \tau_j - \tau_{j-1} > 0$, and set

$$W_{t_1}(\varsigma, \eta) = e^{-t_1 g(\eta)}\mathbf{1}_{[\Lambda,\Lambda']}(\eta)\mathbf{1}_{[\Lambda,\Lambda']}(\varsigma), \quad V_{t_2}(\lambda, \xi) = e^{-t_2 g(\xi)}\mathbf{1}_{[\Lambda,\Lambda']}(\xi)\mathbf{1}_{[\Lambda,\Lambda']}(\lambda), \tag{32}$$

dominated by $\kappa(\varsigma) = e^{-t_1\Lambda}\mathbf{1}_{[\Lambda,\Lambda']}(\varsigma)$, $\chi(\lambda) = e^{-t_2\Lambda}\mathbf{1}_{[\Lambda,\Lambda']}(\lambda) \in L^1_w(I)$. The two-step solution is $u(\cdot, t_1 + t_2) = P^{\theta}_{1,A}(V_{t_2})(P^{\theta}_{1,B}(W_{t_1})\varphi)$.
Theorem 5.4 (Stability with band-limited cutoff)**.** *Let (ν, μ, θ) be admissible, $M = M(\nu, \mu, \theta)$. For $0 < \Lambda < \Lambda' < \infty$ and $\varphi \in \mathcal{P}_{\Lambda,\Lambda'}$,*

$$\|u(\cdot, t_1 + t_2)\|_{G^{\theta}_{1,\nu/2}} \le M^2 e^{-(t_1+t_2)\Lambda}\, C(\nu, \mu, \Lambda, \Lambda')\, \|\varphi\|_{G^{\theta}_{1,\nu/2}}, \tag{33}$$

$$C(\nu, \mu, \Lambda, \Lambda') := \left\|\lambda^{-\mu/2}\mathbf{1}_{[\Lambda,\Lambda']}\right\|^2_{L^1_w} = \Big(\int_{\Lambda}^{\Lambda'} \lambda^{\nu/2} d\lambda\Big)^2 = \Big(\frac{(\Lambda')^{\nu/2+1} - \Lambda^{\nu/2+1}}{\nu/2 + 1}\Big)^2. \tag{34}$$

An N-step scheme satisfies

$$\|u(\cdot, T)\|_{G^{\theta}_{1,\nu/2}} \le M^N e^{-T\Lambda}\, C(\nu, \mu, \Lambda, \Lambda')^{N/2}\, \|\varphi\|_{G^{\theta}_{1,\nu/2}}, \tag{35}$$

and is uniformly stable in N (with T fixed) provided

$$\Lambda \ge \frac{N}{T}\Big(\log M + \tfrac{1}{2}\log C(\nu, \mu, \Lambda, \Lambda')\Big). \tag{36}$$

When $M\sqrt{C} \le 1$, (36) holds automatically for every $\Lambda > 0$ and N.

Proof. Apply Theorem 2.6 in composition form (25) to $P^{\theta}_{1,A}(V_{t_2}) \circ P^{\theta}_{1,B}(W_{t_1})$ with the dominators (32). Using $w - \mu/2 = \nu/2$,

$$\left\| \lambda^{-\mu/2} \chi \right\|_{L^1_w} = e^{-t_2 \Lambda} \int_{\Lambda}^{\Lambda'} \lambda^{w-\mu/2} d\lambda = e^{-t_2 \Lambda} \int_{\Lambda}^{\Lambda'} \lambda^{\nu/2} d\lambda = e^{-t_2 \Lambda} \sqrt{C},$$

and $\left\| \varsigma^{-\mu/2} \kappa \right\|_{L^1_w} = e^{-t_1 \Lambda} \sqrt{C}$; their product is $e^{-(t_1+t_2)\Lambda} C$, finite for $\Lambda' < \infty$. Substituting into (25) gives (33).

Iteration. For the N-step scheme, write $u(\cdot, T) = P_N \cdots P_1 \varphi$, each P_j a type-A or type-B operator with band-limited symbol of step t_j, dominator norm $e^{-t_j \Lambda} \sqrt{C}$. Apply Theorem 2.6 once per factor: since each P_j maps $\mathcal{P}_{\Lambda,\Lambda'}$ into itself (the symbol carries the indicator $\mathbf{1}_{[\Lambda,\Lambda']}$ in the output spectral variable), the hypotheses are met at every stage, and

$$\begin{aligned} \|u(\cdot, T)\|_{G^{\theta}_{1,\nu/2}} &\le \prod_{j=1}^{N} \left(M \, e^{-t_j \Lambda} \sqrt{C} \right) \|\varphi\|_{G^{\theta}_{1,\nu/2}} \\ &= M^N e^{-\Lambda \sum_j t_j} C^{N/2} \, \|\varphi\|_{G^{\theta}_{1,\nu/2}} = M^N e^{-T\Lambda} C^{N/2} \, \|\varphi\|_{G^{\theta}_{1,\nu/2}}, \end{aligned}$$

which is (35). The product telescopes because $\sum_j t_j = T$.

Uniform stability. The bound $M^N e^{-T\Lambda} C^{N/2}$ stays bounded as $N \to \infty$ iff $N \log M - T\Lambda + \frac{N}{2} \log C \le$ const, i.e. $\Lambda \ge \frac{N}{T}(\log M + \frac{1}{2} \log C)$, which is (36). If $M\sqrt{C} \le 1$ then $\log M + \frac{1}{2} \log C \le 0$ and (36) holds for all $\Lambda > 0, N$. □

Remark 5.5 (Necessity of the upper cutoff). Under $w = (\nu+\mu)/2$ the integrand in (34) is $\lambda^{-\mu/2} \lambda^w = \lambda^{\nu/2}$, non-decreasing for every $\nu \ge 0$, so the one-sided cutoff $\mathbf{1}_{[\Lambda,\infty)}$ gives $\int_{\Lambda}^{\infty} \lambda^{\nu/2} d\lambda = \infty$ at every parameter choice. The upper cutoff $\Lambda' < \infty$ is a structural necessity, not a convenience (Fig. 3(a)). Both the cutoff and the divergence reside in the dominator variable λ, so the conclusion is unambiguous.

Remark 5.6 (Comparison with informal treatments). Earlier informal applications of the composition theorem to Bessel Cauchy problems (e.g. [11, §6]) invoked decay $e^{-t\Lambda}$ without checking dominator integrability. Theorem 5.4 and Remark 5.5 show the one-sided cutoff is always insufficient here—a concrete case where explicit-constant bookkeeping exposes a subtlety invisible at the qualitative level.

5.3 Explicit example

Example 5.7. Take $\nu = \mu = 1/2$, so $w = 1/2$; let $\theta = \pi/4$, $\Lambda = 1$, $\Lambda' = 2$.

Constant M. Since $\nu - 3\mu - 2 = -3$,

$$M = (\csc(\pi/4))^{-3} \, \Gamma(\tfrac{3}{2})^{-1} = \frac{(\sqrt{2})^{-3} \cdot 2}{\sqrt{\pi}} = \frac{2^{-1/2}}{\sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} \approx 0.3989.$$

Non-integrability on $[1, \infty)$. As $w - \mu/2 = \nu/2 = 1/4$, $\int_1^{\infty} \lambda^{1/4} d\lambda = \infty$, illustrating Remark 5.5.

Band-limited constant. $C(\frac{1}{2},\frac{1}{2},1,2) = \left(\int_1^2 \lambda^{1/4}d\lambda\right)^2 = \left[\frac{4}{5}(2^{5/4}-1)\right]^2$. Since $2^{5/4} \approx 2.378$, $\frac{4}{5}(1.378) \approx 1.103$, so $C \approx 1.217$.

Estimate. $\|u(\cdot,t_1+t_2)\|_{G^{\theta}_{1,1/4}} \le M^2 e^{-(t_1+t_2)}C\,\|\varphi\| \approx 0.194\, e^{-(t_1+t_2)}\,\|\varphi\|$. As $M\sqrt{C} \approx 0.440 < 1$, the scheme is uniformly stable for all $\Lambda > 0, N$ (Fig. 3(b)).

Remark 5.8 (Universality). $w - \mu/2 = \nu/2$ shows the one-sided cutoff fails for *every* admissible (ν,μ): the range $\nu/2 < -1$ that would rescue it lies outside $\nu \ge 0$. The band-limited cutoff is structural, not a special-case restriction.

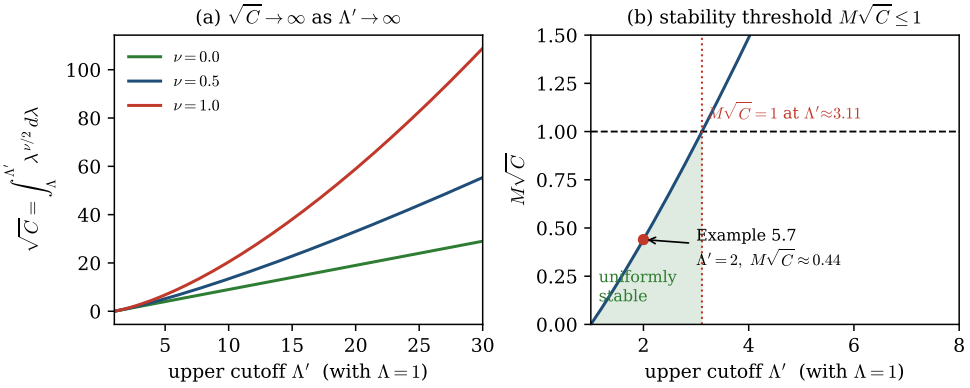


Fig. 3 The band-limited cutoff in Theorem 5.4. **(a)** $\sqrt{C} = \int_{\Lambda}^{\Lambda'} \lambda^{\nu/2}\, d\lambda$ (with $\Lambda = 1$) diverges as $\Lambda' \to \infty$ for every $\nu \ge 0$, so a one-sided cutoff $[\Lambda, \infty)$ gives an infinite constant (Remark 5.5). **(b)** For the Example 5.7 constant $M = 1/\sqrt{2\pi}$ ($\nu = \mu = \frac{1}{2}$, $\theta = \pi/4$), the product $M\sqrt{C}$ crosses 1 at $\Lambda' \approx 3.11$; in the shaded region $M\sqrt{C} \le 1$ the N-step scheme (35) is uniformly stable for every $\Lambda > 0$ and N. The marked point is Example 5.7 ($\Lambda' = 2$, $M\sqrt{C} \approx 0.44$)

6 The Four-Parameter Extension: An Obstruction

6.1 The natural extension

The framework of [10] is a two-dimensional slice of a four-parameter family. With $\alpha \ge \beta \ge 0$, $\nu = \alpha - \beta$, $\mu = a - b$, the calculus of Sections 2–5 corresponds to the diagonal $\alpha+\beta = \frac{1}{2}$, $a+b = \frac{1}{2}$, under which $w = (\nu+\mu)/2$. Setting $\sigma = \alpha+\beta$, $\tau = a+b$ free, the weight generalizes to

$$w = \alpha + a - \tfrac{1}{2} = \frac{\sigma + \tau + \nu + \mu - 1}{2}, \tag{37}$$

reducing to $(\nu+\mu)/2$ when $\sigma = \tau = 1/2$.

6.2 The obstruction

The moment identity Lemma 2.3(iii) reduces, after separating constants, to the integral

$$\mathcal{I}(x,\xi;\nu,w)=\int_A^B \Delta(x,\xi,z)^{2\nu-1}z^{2w}\,dz. \tag{38}$$

By Lemma 2.1 with $r=2w$,

$$\mathcal{I}=4^{1-2\nu}(B-A)^{2\nu}B^{2w}\frac{\Gamma(\nu+\frac12)^2}{\Gamma(2\nu+1)}\,{}_2F_1\big(-2w,\nu+\tfrac12;2\nu+1;\tfrac{B-A}{B}\big). \tag{39}$$

On the diagonal $w=(\nu+\mu)/2$, the first parameter is $-2w=-(\nu+\mu)$; the relevant moment in the kernel normalization is the $r=0$ case of Lemma 2.1 (Remark 2.2), and ${}_2F_1$ collapses to an elementary expression—this is the classical Sonine evaluation. Off the diagonal, $-2w=-(\sigma+\tau+\nu+\mu-1)$ is untied from the second and third parameters, and ${}_2F_1(-2w,\nu+\frac12;2\nu+1;\cdot)$ does not reduce to an elementary function of (x,ξ). A corrected moment identity would carry the multiplicative factor

$$\frac{{}_2F_1\big(-2w,\nu+\frac12;2\nu+1;\lambda(x,\xi)\big)}{{}_2F_1\big(-(\nu+\mu),\nu+\frac12;2\nu+1;\lambda(x,\xi)\big)},\qquad \lambda(x,\xi)=\frac{B-A}{B},$$

a non-trivial function of (x,ξ) that destroys the convolutional structure underlying Lemma 2.5.

Which relation breaks. The parameters of the hypergeometric factor in (39) satisfy

$$c=2\nu+1=2\big(\nu+\tfrac12\big)=2b$$

identically, independently of σ,τ. This is the structural relation that permits the quadratic transformation

$${}_2F_1\big(a,b;2b;\zeta\big)=\Big(1-\tfrac{\zeta}{2}\Big)^{-a}\,{}_2F_1\Big(\tfrac{a}{2},\tfrac{a+1}{2};\,b+\tfrac12;\,\Big(\tfrac{\zeta}{2-\zeta}\Big)^2\Big),\qquad a=-2w,\ b=\nu+\tfrac12, \tag{40}$$

valid for $|\zeta|<1$ [4, §2.11]. The right-hand ${}_2F_1$ collapses to a finite algebraic expression precisely when its series *terminates*, i.e. when $a=-2w$ is a non-positive even integer (equivalently $2w\in2\mathbb{N}_0$). On the diagonal $\sigma=\tau=\frac12$ the kernel normalization places the effective first parameter at 0 (Remark 2.2), so the series terminates at its constant term and the elementary Sonine value is recovered. Off the diagonal,

$$a=-2w=-(\sigma+\tau+\nu+\mu-1)$$

is generically non-integral; the series does not terminate, and ${}_2F_1(-2w,\nu+\frac12;2\nu+1;\zeta)$ is a genuine transcendental function (expressible through associated Legendre functions, but not in finite algebraic terms); see Fig. 4. It is therefore the *termination* relation $-2w\in-2\mathbb{N}_0$, not the quadratic-transformation relation $c=2b$, that the

four-parameter deformation destroys; the convolutional algebra of Section 2 survives exactly as long as termination does.

6.3 Consequences and open problems

Hence the boundedness, composition, and sharpness theorems do not extend automatically to the four-parameter setting. An honest extension requires one of:

(a) a Weber–Schafheitlin-type identity (cf. [14, §13.4]) evaluating $\mathcal{I}$ in closed form for generic w—for instance, an extension of the Sonine–Gegenbauer formula to the non-terminating regime that expresses $\mathcal{I}$ through a single associated Legendre function $P^{-\nu}_{w-1/2}$ of an explicit argument in (x, ξ);

(b) a Mellin-transform proof: $\mathcal{I}$ is a Mellin-type integral in $A = (\sqrt{x} - \sqrt{\xi})^2$ and $B = (\sqrt{x} + \sqrt{\xi})^2$, and a Barnes/residue evaluation of (39) (cf. [12]) would bypass the elementary-collapse requirement altogether, replacing the convolution inequality by a Mellin multiplier estimate;

(c) a symbol-class restriction $\mathcal{S}_{\sigma,\tau}$ of generators V for which the correction ratio above is bounded and multiplier-stable across the triangle, so that a modified operator-norm bound—with a constant depending on ${}_2F_1$ through its supremum over the support—still holds, at the cost of losing sharpness.

Each is a substantive project; the present results are complete on the diagonal $\sigma = \tau = 1/2$.

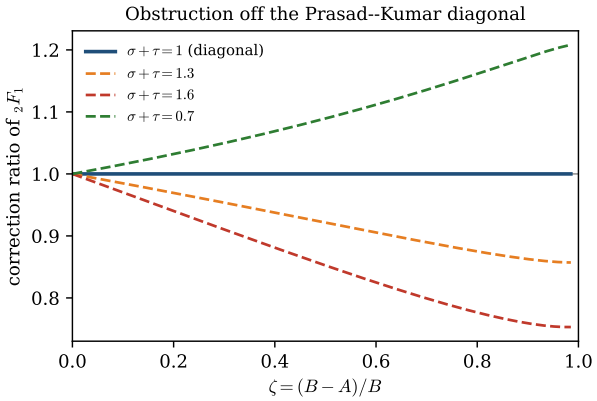


Fig. 4 The obstruction of Section 6, with $\nu = 0.30$, $\mu = 0.40$. Plotted is the correction ratio ${}_2F_1(-(\sigma+\tau+\nu+\mu-1), \nu+\frac{1}{2}; 2\nu+1; \zeta)/{}_2F_1(-(\nu+\mu), \nu+\frac{1}{2}; 2\nu+1; \zeta)$ against $\zeta = (B-A)/B$. On the Prasad–Kumar diagonal $\sigma+\tau = 1$ it is identically 1 (the moment identity Lemma 2.3(iii) holds verbatim); off the diagonal it is a non-constant function of ζ, i.e. of (x, ξ), which destroys the convolutional structure of Lemma 2.5

7 Conclusion

We have given complete proofs of three new results within the Prasad–Kumar pseudodifferential calculus: sharpness of the boundedness constant (Theorem 3.2), sharpness

of the composition bound (Theorem 4.1), and a stability theorem for a Bessel-type Cauchy problem requiring a band-limited spectral cutoff (Theorem 5.4, Remark 5.5). The analytic engine—a Beta/hypergeometric evaluation of the geometric moment integral (Lemma 2.1)—is proved from first principles; its $r = 0$ collapse on the Prasad–Kumar diagonal underlies the exact moment identity, the convolution inequality, and the boundedness theorem, while its persistence off the diagonal yields the obstruction of Section 6. The decisive device in the sharpness proofs is a support-collapse limit driving an auxiliary spectral parameter to the origin, which removes the residual weight mismatch between the weighted L^1 and weighted Sobolev norms; the exactness of the moment identity and a phase alignment make the remaining two inequalities equalities for every finite n. Because the sharpness statements are ratios in which the boundedness constant cancels, they are independent of its precise normalization.

Further directions: an L^p-extension of Lemma 2.5 for $p \neq 1$; a commutator estimate for $[P^{\theta}_{1,A}, P^{\theta}_{1,B}]$; and a vector-valued sharp-constant analysis connecting to pseudodifferential theory on manifolds.

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Data availability No datasets were generated or analysed during the current study. The LaTeX source, the figure-generation script, and a numerical-validation script reproducing every constant and limit reported here are available from the author on reasonable request.

Code availability The figure-generation and numerical-validation scripts (Python/NumPy/SciPy/Matplotlib) are available from the author on reasonable request.

Author contributions The manuscript is the sole work of the author.

Use of AI tools During the preparation of this work the author used a large language model based assistant (Anthropic's Claude) to assist with LaTeX preparation, generation of the figures from the author's own formulas, numerical verification of the constants and limiting values reported herein, and language editing. The author reviewed, verified, and edited all definitions, statements, proofs, and numerical results, and takes full responsibility for the content and correctness of the manuscript.

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