Supplementary Information for "Chiral superconductivity mediated by quantized magnetic flux"

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1 Supplementary Note 1: LC quantization, Peierls coupling and gauge-invariant expansion

1.1 Quantized vector potential

A superconducting loop is modeled as a linear LC oscillator with canonical flux and charge operators satisfying

$$[\hat{\Phi}, \hat{Q}] = i\hbar, \quad \hat{H}_{\rm LC} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \quad \omega = \frac{1}{\sqrt{LC}}.$$

The zero-point amplitudes are

$$\hat{\Phi} = \Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger), \quad \hat{Q} = iQ_{\text{zpf}}(\hat{a}^\dagger - \hat{a}), \quad \Phi_{\text{zpf}} = \sqrt{\frac{\hbar Z}{2}}, \quad Q_{\text{zpf}} = \sqrt{\frac{\hbar}{2Z}},$$

where $Z = \sqrt{L/C}$. For a linear loop the current operator is $\hat{I} = \hat{\Phi}/L$, so

$$I_{\text{zpf}} = \frac{\Phi_{\text{zpf}}}{L} = \sqrt{\frac{\hbar\omega}{2L}}.$$

Let $\mathbf{A}_I(\mathbf{r})$ denote the vector potential per unit loop current in Coulomb gauge. Then

$$\hat{\mathbf{A}}(\mathbf{r}) = I_{\text{zpf}}\,\mathbf{A}_I(\mathbf{r})(\hat{a} + \hat{a}^\dagger), \quad \hat{\mathbf{B}}(\mathbf{r}) = I_{\text{zpf}}\,\nabla \times \mathbf{A}_I(\mathbf{r})(\hat{a} + \hat{a}^\dagger).$$

The loop flux obeys

$$\oint_C \hat{\mathbf{A}} \cdot d\boldsymbol{\ell} = L\hat{I} = \Phi_{\text{zpf}}(\hat{a} + \hat{a}^\dagger),$$

and the vacuum magnetic energy satisfies

$$\left\langle 0 \left| \frac{1}{2\mu_0} \int d^3r\, |\hat{\mathbf{B}}|^2 \right| 0 \right\rangle = \frac{1}{2} L I_{\text{zpf}}^2 = \frac{\hbar\omega}{4}.$$

Equivalently, defining $\mathbf{u}_B(\mathbf{r}) = \mathbf{A}_I(\mathbf{r})/\sqrt{L}$ gives

$$\int \frac{d^3r}{\mu_0} |\nabla \times \mathbf{u}_B(\mathbf{r})|^2 = 1, \quad \hat{\mathbf{A}}(\mathbf{r}) = \sqrt{\frac{\hbar\omega}{2}}\,\mathbf{u}_B(\mathbf{r})(\hat{a} + \hat{a}^\dagger).$$

1.2 Peierls substitution and diamagnetic term

For a square lattice with nearest-neighbor hopping, the gauge-invariant lattice Hamiltonian is

$$\hat{H} = -t \sum_{\langle ij\rangle,\sigma} \left(e^{i\hat{\theta}_{ij}} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \text{H.c.} \right) - \mu \sum_i \hat{n}_i + \hbar\omega \hat{a}^\dagger \hat{a},$$

with

$$\hat{\theta}_{ij} = \frac{e}{\hbar} \int_{\mathbf{r}_i}^{\mathbf{r}_j} \hat{\mathbf{A}}(\mathbf{r}) \cdot d\boldsymbol{\ell} = \alpha_{ij}\hat{X}, \quad \hat{X} = \hat{a} + \hat{a}^\dagger, \qquad \alpha_{ji} = -\alpha_{ij}.$$

Expanding the link phase gives

$$e^{i\alpha_{ij}\hat{X}} = 1 + i\alpha_{ij}\hat{X} - \frac{1}{2}\alpha_{ij}^2 \hat{X}^2 + O(\alpha_{ij}^3).$$

The linear term defines the paramagnetic current coupling

$$\hat{H} = \hat{H}_{\text{el}} + \hbar\omega \hat{a}^\dagger \hat{a} - \hat{X}\hat{J} + \hat{H}_{A^2} + O(\alpha^3), \quad \hat{J} = \sum_{\langle ij\rangle} \alpha_{ij} \hat{j}_{ij},$$

where

$$\hat{j}_{ij} = it \sum_\sigma (\hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} - \hat{c}_{j\sigma}^\dagger \hat{c}_{i\sigma}).$$

The second-order Peierls term is the lattice diamagnetic contribution,

$$\hat{H}_{A^2} = \frac{t}{2}\hat{X}^2 \sum_{\langle ij\rangle,\sigma} \alpha_{ij}^2 (\hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \text{H.c.}).$$

It is required for a fully gauge-invariant expansion. The analytical pairing channels discussed in the article come from the paramagnetic current matrix elements; a complete numerical treatment should compare the linearized current-current model with the full Peierls Hamiltonian.

2 Supplementary Note 2: Finite-momentum pairing channel

2.1 Projection to a finite centre-of-mass momentum

The finite-$\mathbf{Q}$ pair momenta of electrons before and after scattering are

$$\mathbf{k}_\pm = \frac{\mathbf{Q}}{2} \pm \mathbf{k}, \quad \mathbf{p}_\pm = \frac{\mathbf{Q}}{2} \pm \mathbf{p}.$$

Projecting the antisymmetrized vertex onto $(\mathbf{p}_+, \mathbf{p}_-) \to (\mathbf{k}_+, \mathbf{k}_-)$ gives

$$V_{\mathbf{Q}}(\mathbf{k}, \mathbf{p}) = -\frac{1}{\hbar\omega} \left[J_{\mathbf{k}_+,\mathbf{p}_+} J_{\mathbf{k}_-,\mathbf{p}_-} - J_{\mathbf{k}_+,\mathbf{p}_-} J_{\mathbf{k}_-,\mathbf{p}_+} \right].$$

For an equal-spin triplet component the orbital gap is odd,

$$\Delta_{\mathbf{Q}}(-\mathbf{k}) = -\Delta_{\mathbf{Q}}(\mathbf{k}),$$

and the antisymmetrized vertex closes within this odd sector. The finite-$\mathbf{Q}$ BCS Hamiltonian is

$$\hat{H}^{(\mathbf{Q})}_{\mathrm{BCS}} = \sum_{\mathbf{k}} \left(\tilde{\xi}_{\mathbf{k}_+} \hat{n}_{\mathbf{k}_+} + \tilde{\xi}_{\mathbf{k}_-} \hat{n}_{\mathbf{k}_-} \right) + \frac{1}{2N} \sum_{\mathbf{kp}} V_{\mathbf{Q}}(\mathbf{k}, \mathbf{p}) \hat{c}^\dagger_{\mathbf{k}_+} \hat{c}^\dagger_{\mathbf{k}_-} \hat{c}_{\mathbf{p}_-} \hat{c}_{\mathbf{p}_+}.$$

The gap is defined by

$$\Delta_{\mathbf{Q}}(\mathbf{k}) = -\frac{1}{N} \sum_{\mathbf{p}} V_{\mathbf{Q}}(\mathbf{k}, \mathbf{p}) \langle \hat{c}_{\mathbf{p}_-} \hat{c}_{\mathbf{p}_+} \rangle.$$

With

$$\bar{\xi}_{\mathbf{Q}}(\mathbf{k}) = \frac{\tilde{\xi}_{\mathbf{k}_+} + \tilde{\xi}_{\mathbf{k}_-}}{2}, \quad \eta_{\mathbf{Q}}(\mathbf{k}) = \frac{\tilde{\xi}_{\mathbf{k}_+} - \tilde{\xi}_{\mathbf{k}_-}}{2},$$

the quasiparticle branches are

$$E_{\mathbf{Q},\pm}(\mathbf{k}) = \eta_{\mathbf{Q}}(\mathbf{k}) \pm \sqrt{\bar{\xi}^2_{\mathbf{Q}}(\mathbf{k}) + |\Delta_{\mathbf{Q}}(\mathbf{k})|^2}.$$

The self-consistency equation is

$$\Delta_{\mathbf{Q}}(\mathbf{k}) = \frac{1}{N} \sum_{\mathbf{p}} V_{\mathbf{Q}}(\mathbf{k}, \mathbf{p}) \frac{\Delta_{\mathbf{Q}}(\mathbf{p})}{2\mathcal{E}_{\mathbf{Q}}(\mathbf{p})} \left[1 - f(E_{\mathbf{Q},+}) - f(E_{\mathbf{Q},-}) \right],$$

where $\mathcal{E}_{\mathbf{Q}} = \sqrt{\bar{\xi}^2_{\mathbf{Q}} + |\Delta_{\mathbf{Q}}|^2}$. At $T = T_c$ this becomes

$$\Delta_{\mathbf{Q}}(\mathbf{k}) = \frac{1}{N} \sum_{\mathbf{p}} V_{\mathbf{Q}}(\mathbf{k}, \mathbf{p}) \chi_{\mathbf{Q}}(\mathbf{p}; T_c) \Delta_{\mathbf{Q}}(\mathbf{p}),$$

with

$$\chi_{\mathbf{Q}}(\mathbf{p}; T) = \frac{1 - f(\tilde{\xi}_{\mathbf{Q}/2+\mathbf{p}}) - f(\tilde{\xi}_{\mathbf{Q}/2-\mathbf{p}})}{\tilde{\xi}_{\mathbf{Q}/2+\mathbf{p}} + \tilde{\xi}_{\mathbf{Q}/2-\mathbf{p}}}.$$

The leading instability is the largest eigenvalue of $V_{\mathbf{Q}}\chi_{\mathbf{Q}}$, optimized over $\mathbf{Q}$.

2.2 Patch expansion and axial ordering vector

Set $\mathbf{Q} = 2\mathbf{K}$, where $\mathbf{K}$ lies on the Fermi surface. Expanding near this patch in local normal and tangential coordinates gives

$$\xi_{\mathbf{K}+\mathbf{q}} = v_F q_\perp + \frac{\kappa_F}{2} q_\parallel^2 + \frac{q_\perp^2}{2m_\perp} + \cdots ,$$
$$\xi_{\mathbf{K}-\mathbf{q}} = -v_F q_\perp + \frac{\kappa_F}{2} q_\parallel^2 + \frac{q_\perp^2}{2m_\perp} + \cdots .$$

The linear terms cancel in the pair denominator,

$$\xi_{\mathbf{K}+\mathbf{q}} + \xi_{\mathbf{K}-\mathbf{q}} = \kappa_F q_\parallel^2 + \frac{q_\perp^2}{m_\perp} + \cdots ,$$

so the susceptibility is enhanced along the ridge $q_\perp = 0$, especially where $|\kappa_F|$ is small. For a square-lattice Fermi surface this selects the axial set

$$\mathbf{Q}_\star \in \{(\pm Q_0, 0), (0, \pm Q_0)\}.$$

Because the gap is odd in relative momentum, the leading function supported by the ridge is tangential,

$$\phi_{2\mathbf{K}}(\mathbf{q}) \propto q_\parallel F(\mathbf{q}), \qquad F(-\mathbf{q}) = F(\mathbf{q}).$$

For $\mathbf{Q} = (Q_0, 0)$ this gives the lattice continuation

$$\phi_{(Q_0,0)}(\mathbf{k}) \sim g_{Q_0}(\mathbf{k}) \sin(k_y a),$$

and for $\mathbf{Q} = (0, Q_0)$,

$$\phi_{(0,Q_0)}(\mathbf{k}) \sim g_{Q_0}(\mathbf{k}) \sin(k_x a),$$

where g_{Q_0} is even and localized near the active patches.

2.3 Finite-sample matrix elements

For the approximately uniform profile used in the finite square, the weighted current is

$$\hat{J} = \kappa_\Phi \sum_i (x_i \hat{j}_i^y - y_i \hat{j}_i^x).$$

Open boundaries and the explicit factors x_i, y_i break exact translation symmetry. Thus $\mathbf{Q}$ in the numerical calculation labels the dominant Fourier component of the pair matrix rather than an exact conserved momentum. In a separable finite-box basis, typical current matrix elements factor into position-like and derivative-like factors. For a one-dimensional sine basis $u_n(x) = \sqrt{2/(L+1)} \sin(k_n x)$, $k_n = \pi n/(L+1)$, the useful matrix elements are

$$D_{nm} = \sum_{x=1}^{L-1} [u_n(x+1) u_m(x) - u_n(x) u_m(x+1)],$$
$$X_{nm} = \sum_{x=1}^{L} X_x u_n(x) u_m(x).$$

They vanish for $n+m$ even, while for $n+m$ odd

$$D_{nm} = \frac{4}{L+1}\frac{\sin k_n \sin k_m}{\cos k_n - \cos k_m},$$
$$X_{nm} = -\frac{2a}{L+1}\frac{\sin k_n \sin k_m}{(\cos k_n - \cos k_m)^2}.$$

These selection rules show explicitly how finite geometry creates off-diagonal current matrix elements and hence genuine pair scattering.

3 Supplementary Note 3: Gaussian magnetic profile

3.1 Continuum Fourier transform and transverse vertex

The analytically tractable profile used in the Article is

$$\mathbf{A}(\mathbf{r}) = A_0(\hat{\mathbf{z}} \times \mathbf{r})e^{-r^2/(2\ell^2)}.$$

Its two-dimensional Fourier transform is

$$\tilde{\mathbf{A}}(\mathbf{q}) = -i2\pi A_0 \ell^4 (\hat{\mathbf{z}} \times \mathbf{q})e^{-\ell^2 q^2/2}.$$

For a continuum electron gas,

$$\hat{J} = \int_{\mathbf{k}}\int_{\mathbf{k}'} J_{\mathbf{k}\mathbf{k}'}\hat{c}^\dagger_{\mathbf{k}}\hat{c}_{\mathbf{k}'}, \qquad J_{\mathbf{k}\mathbf{k}'} = \frac{e\hbar}{2m}(\mathbf{k}+\mathbf{k}')\cdot\tilde{\mathbf{A}}(\mathbf{k}-\mathbf{k}'),$$

which yields

$$J_{\mathbf{k}\mathbf{k}'} = iC(\mathbf{k}\times\mathbf{k}')_z e^{-\ell^2|\mathbf{k}-\mathbf{k}'|^2/2}, \qquad C = \frac{2\pi e\hbar A_0 \ell^4}{m}.$$

This closed form is the main reason for using the Gaussian profile: it exposes the transverse structure of the flux-mediated current vertex.

3.2 Zero-momentum equal-spin channel

For the antisymmetrized $Q=0$ spinless/equal-spin Cooper channel,

$$V(\mathbf{k},\mathbf{k}') = \frac{1}{\hbar\omega}\left[J_{\mathbf{k}\mathbf{k}'}J_{-\mathbf{k},-\mathbf{k}'} - J_{\mathbf{k},-\mathbf{k}'}J_{-\mathbf{k},\mathbf{k}'}\right].$$

Using the Gaussian vertex gives

$$V(\mathbf{k},\mathbf{k}') = -\frac{C^2}{\hbar\omega}(\mathbf{k}\times\mathbf{k}')_z^2\left[e^{-\ell^2|\mathbf{k}-\mathbf{k}'|^2} - e^{-\ell^2|\mathbf{k}+\mathbf{k}'|^2}\right].$$

On a circular Fermi surface the kernel depends only on $\theta-\theta'$, so angular harmonics diagonalize the linearized gap equation. Odd parity selects odd angular momentum; the leading solution is the lowest odd harmonic,

$$\Delta(\mathbf{k}) \propto e^{\pm i\theta_{\mathbf{k}}} \propto k_x \pm ik_y,$$

corresponding to chiral p-wave order or its real p_x, p_y components before quartic terms select a phase.

3.3 Lattice vertex and singlet zero-momentum channel

For the square lattice, the Peierls phases are evaluated at the bond center $\mathbf{R}$:

$$\Phi_x(\mathbf{R}) \simeq -\Phi_0 R_y e^{-R^2/(2\ell^2)},$$
$$\Phi_y(\mathbf{R}) \simeq +\Phi_0 R_x e^{-R^2/(2\ell^2)}, \quad \Phi_0 = \frac{eaA_0}{\hbar}.$$

Their Fourier transforms are odd and imaginary,

$$\Phi_x(\mathbf{q}) = +i\Phi_* q_y e^{-q^2\ell^2/2},$$
$$\Phi_y(\mathbf{q}) = -i\Phi_* q_x e^{-q^2\ell^2/2}.$$

The lattice current vertex is

$$g^{\text{lat}}_{\mathbf{k},\mathbf{q}} = g^{\text{lat}}_* \left[q_y \sin\left(k_x a + \frac{q_x a}{2}\right) - q_x \sin\left(k_y a + \frac{q_y a}{2}\right)\right] e^{-q^2\ell^2/2},$$

where g^{lat}_* is purely imaginary. In the long-wavelength limit this reduces to

$$g^{\text{lat}}_{\mathbf{k},\mathbf{q}} \simeq \tilde{g}_* (\mathbf{v}_\mathbf{k} \times \mathbf{q})_z e^{-q^2\ell^2/2}, \quad \mathbf{v}_\mathbf{k} = 2ta(\sin k_x a, \sin k_y a).$$

Projecting to the singlet $Q = 0$ channel gives

$$V^{\text{BCS}}_{\mathbf{kp}} = -\frac{1}{\hbar\omega} g^{\text{lat}}_{\mathbf{k},\mathbf{p}-\mathbf{k}} g^{\text{lat}}_{-\mathbf{k},\mathbf{k}-\mathbf{p}}.$$

Substitution gives the corrected expression

$$V^{\text{BCS}}_{\mathbf{kp}} = V^{\text{lat}}_0 e^{-|\mathbf{p}-\mathbf{k}|^2\ell^2} \Bigg[(p_y - k_y)\sin\frac{(k_x + p_x)a}{2}$$
$$-(p_x - k_x)\sin\frac{(k_y + p_y)a}{2}\Bigg]^2, \quad V^{\text{lat}}_0 = -\frac{(g^{\text{lat}}_*)^2}{\hbar\omega} > 0.$$

There is only one Gaussian factor $e^{-|\mathbf{p}-\mathbf{k}|^2\ell^2}$, because each vertex contributes $e^{-q^2\ell^2/2}$. The matrix elements are pointwise nonnegative in this projected channel, but the integral operator should not be called positive semidefinite without an additional proof.

The linearized singlet gap equation is

$$\Delta_\mathbf{k} = -\frac{1}{N}\sum_\mathbf{p} V^{\text{BCS}}_{\mathbf{kp}} \frac{\tanh(\xi_\mathbf{p}/2T)}{2\xi_\mathbf{p}} \Delta_\mathbf{p}.$$

The long-wavelength form is

$$V^{\text{BCS}}_{\mathbf{kp}} \simeq V_0 (\mathbf{k} \times \mathbf{p})^2_z e^{-|\mathbf{p}-\mathbf{k}|^2\ell^2}.$$

Using

$$(\mathbf{k} \times \mathbf{p})^2_z = \frac{1}{2}(k_x^2 + k_y^2)(p_x^2 + p_y^2) - \frac{1}{2}(k_x^2 - k_y^2)(p_x^2 - p_y^2)$$
$$- \frac{1}{2}(2k_x k_y)(2p_x p_y),$$

shows that the uniform s-wave projection is disfavoured, while the $d_{x^2-y^2}$ and d_{xy} sectors enter the gap equation with the attractive sign. Near degeneracy between these two sectors allows a chiral $d_{x^2-y^2} \pm i d_{xy}$ state at quartic order.

3.4 Finite-Q triplet kernel for the Gaussian profile

For finite pair momentum, the direct kernel is

$$V_{\mathbf{Q}}^{\mathrm{dir}}(\mathbf{k},\mathbf{p}) = -\frac{1}{\hbar\omega} g_{\mathbf{Q}/2+\mathbf{k},\mathbf{p}-\mathbf{k}} g_{\mathbf{Q}/2-\mathbf{k},\mathbf{k}-\mathbf{p}}.$$

Using $g_{\mathbf{kq}} = g_*(\mathbf{k}\times\mathbf{q})_z e^{-q^2\ell^2/2}$ gives

$$V_{\mathbf{Q}}^{\mathrm{dir}}(\mathbf{k},\mathbf{p}) = V_0\left[(\mathbf{k}\times\mathbf{p})_z^2 - \frac{1}{4}(\mathbf{Q}\times(\mathbf{p}-\mathbf{k}))_z^2\right] e^{-|\mathbf{p}-\mathbf{k}|^2\ell^2},$$

where $V_0 = -g_*^2/(\hbar\omega) > 0$. The exchange term is

$$V_{\mathbf{Q}}^{\mathrm{ex}}(\mathbf{k},\mathbf{p}) = V_0\left[(\mathbf{k}\times\mathbf{p})_z^2 - \frac{1}{4}(\mathbf{Q}\times(\mathbf{p}+\mathbf{k}))_z^2\right] e^{-|\mathbf{p}+\mathbf{k}|^2\ell^2},$$

and the triplet kernel is

$$V_{\mathbf{Q}}^{(t)}(\mathbf{k},\mathbf{p}) = V_{\mathbf{Q}}^{\mathrm{dir}}(\mathbf{k},\mathbf{p}) - V_{\mathbf{Q}}^{\mathrm{ex}}(\mathbf{k},\mathbf{p}).$$

In a local patch with $\mathbf{Q}\parallel\hat{x}$ and transverse coordinate $k_\perp$, the leading form is

$$V_{\mathbf{Q}}^{(t)}(k_\perp,p_\perp) \simeq -\frac{V_0Q^2}{4}\left[(p_\perp-k_\perp)^2 e^{-(p_\perp-k_\perp)^2\ell^2} - (p_\perp+k_\perp)^2 e^{-(p_\perp+k_\perp)^2\ell^2}\right].$$

Acting on the odd basis function $\phi(k_\perp) = k_\perp$ gives a negative eigenvalue of the kernel in the gap equation, which identifies the local finite-**Q** triplet state as tangential p-wave:

$$\Delta_{\mathbf{Q}}(\mathbf{k}) \propto (\hat{\mathbf{z}}\times\hat{\mathbf{Q}})\cdot\mathbf{k}.$$

4 Supplementary Note 4: Circular-loop form factor

The Gaussian profile is analytically tractable, but the same Peierls construction applies to any shape. For example, consider a circular superconducting loop of radius R at height h above the electron plane. The vector potential is azimuthal,

$$\mathbf{A}_{\mathrm{loop}}(\mathbf{r};R,h) = A_\phi(\rho;R,h)\hat{\boldsymbol{\phi}}, \quad \rho = \sqrt{x^2+y^2}.$$

For a thin filamentary loop,

$$A_\phi(\rho;R,h) = \frac{\mu_0 I_0}{\pi k}\sqrt{\frac{R}{\rho}}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right], \quad k^2 = \frac{4R\rho}{(R+\rho)^2+h^2},$$

where K and E are complete elliptic integrals. The midpoint Peierls phases are

$$\Phi_x(\mathbf{R}) \simeq -\frac{ea}{\hbar}A_\phi(\rho;R,h)\frac{R_y}{\rho}, \quad \Phi_y(\mathbf{R}) \simeq +\frac{ea}{\hbar}A_\phi(\rho;R,h)\frac{R_x}{\rho}.$$

Equivalently, the two-dimensional Fourier form at the electron plane is

$$\Phi_x(\mathbf{q}) = +i\Phi_R e^{-qh}\frac{J_1(qR)}{q^2}q_y,$$
$$\Phi_y(\mathbf{q}) = -i\Phi_R e^{-qh}\frac{J_1(qR)}{q^2}q_x, \quad \Phi_R = \frac{\pi\mu_0 e a I_0 R}{\hbar}.$$

The corresponding lattice vertex is

$$g_{\mathbf{kq}}^{\text{loop}} = ig_R e^{-qh} \frac{J_1(qR)}{q^2} \left[q_y \sin\left(k_x a + \frac{q_x a}{2}\right) - q_x \sin\left(k_y a + \frac{q_y a}{2}\right) \right],$$

with $g_R = 2t\Phi_R$. Thus the circular loop preserves the transverse structure of the Gaussian vertex but replaces the Gaussian cutoff with the exact magnetostatic form factor $e^{-qh} J_1(qR)/q^2$. In the zero-momentum singlet channel,

$$V_{\mathbf{kp}}^{\text{BCS}} = V_R e^{-2|\mathbf{p}-\mathbf{k}|h} \frac{J_1^2(|\mathbf{p}-\mathbf{k}|R)}{|\mathbf{p}-\mathbf{k}|^4} \left[(p_y - k_y) \sin \frac{(k_x + p_x)a}{2} \right.$$
$$\left. - (p_x - k_x) \sin \frac{(k_y + p_y)a}{2} \right]^2, \qquad V_R = \frac{g_R^2}{\hbar\omega}.$$

This expression shows how the actual loop radius and spacer height tune the momentum dependence of the induced pairing kernel.

5 Supplementary Note 5: Ginzburg-Landau selection of chiral states

5.1 Chiral finite-momentum PDW

Let η_x and η_y denote the two symmetry-related axial PDW components associated with $\mathbf{Q}_x = (Q, 0)$ and $\mathbf{Q}_y = (0, Q)$. The most general uniform quartic free energy respecting C_4 and time reversal has the form

$$F = r(|\eta_x|^2 + |\eta_y|^2) + u(|\eta_x|^2 + |\eta_y|^2)^2 + v|\eta_x|^2|\eta_y|^2 + w(\eta_x^2 \eta_y^{*2} + \eta_x^{*2} \eta_y^2).$$

Writing $\eta_y/\eta_x = |\eta_y/\eta_x| e^{i\varphi}$, the last term is proportional to $\cos(2\varphi)$. If $w > 0$, the minimum occurs at $\varphi = \pm\pi/2$. The Ising chirality is

$$\chi = i(\eta_x \eta_y^* - \eta_y \eta_x^*) = 2\,\text{Im}(\eta_x \eta_y^*),$$

which changes sign under time reversal. Thus a relative phase $\pm\pi/2$ gives a bidirectional chiral pair-density wave rather than a real stripe state. The sign of w is determined by the microscopic four-point loop built from the finite-$\mathbf{Q}$ Green's functions and the form factors derived above.

5.2 Chiral zero-momentum $d + id$ channel

For the Gaussian zero-momentum singlet channel, let Δ_1 and Δ_2 multiply the square-lattice basis functions

$$\eta_{B_{1g}}(\mathbf{k}) = \cos k_x a - \cos k_y a, \quad \eta_{B_{2g}}(\mathbf{k}) = 2 \sin k_x a \sin k_y a.$$

If the two channels are nearly degenerate, the quartic free energy has the same phase-sensitive structure as above. A positive coefficient of $\Delta_1^2 \Delta_2^{*2}$ + c.c. selects

$$\Delta(\mathbf{k}) \propto \eta_{B_{1g}}(\mathbf{k}) \pm i\eta_{B_{2g}}(\mathbf{k}),$$

which breaks time reversal. This is the zero-momentum chiral state referred to in the main text.

6 Supplementary Note 6: BdG definitions

The real-space mean-field Hamiltonian for spinless/equal-spin pairing is

$$\mathcal{H}_{\mathrm{BdG}} = \frac{1}{2}\sum_{ij} \begin{pmatrix} \hat{c}_i^\dagger & \hat{c}_i \end{pmatrix} \begin{pmatrix} H_{0,ij} & \Delta_{ij} \\ -\Delta_{ij}^* & -H_{0,ij}^* \end{pmatrix} \begin{pmatrix} \hat{c}_j \\ \hat{c}_j^\dagger \end{pmatrix}.$$

The antisymmetric pairing matrix is updated from the anomalous density,

$$\Delta_{ij} = \sum_{mn} V_{ij;mn} \langle \hat{c}_m \hat{c}_n \rangle, \quad \Delta_{ij} = -\Delta_{ji}.$$

The centre-of-mass and relative coordinates are

$$\mathbf{R} = \frac{\mathbf{r}_i + \mathbf{r}_j}{2}, \quad \mathbf{r} = \mathbf{r}_i - \mathbf{r}_j.$$

The Fourier-resolved pair amplitude is

$$\Delta_{\mathbf{Q}}(\mathbf{k}_{\mathrm{rel}}) = \sum_{ij} \Delta_{ij} e^{-i\mathbf{Q}\cdot(\mathbf{r}_i+\mathbf{r}_j)/2} e^{-i\mathbf{k}_{\mathrm{rel}}\cdot(\mathbf{r}_i-\mathbf{r}_j)},$$

and the centre-of-mass weight plotted in the main text is

$$W(\mathbf{Q}) = \sum_{\mathbf{k}_{\mathrm{rel}}} |\Delta_{\mathbf{Q}}(\mathbf{k}_{\mathrm{rel}})|^2.$$

Time-reversal breaking is diagnosed by

$$\eta_{\mathrm{TR}} = \min_{\phi} \frac{\|\Delta - e^{2i\phi}\Delta^*\|_F}{\|\Delta\|_F}.$$

A time-reversal-symmetric state has $\eta_{\mathrm{TR}} = 0$; a nonzero value indicates that no global superconducting phase makes the pair matrix real.