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Variable order A-stable strongly regular second derivative general linear methods for stiff ODEs

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Abstract

This paper considers a variable order family of strongly regular second derivative general linear methods (SR-SD-GLMs), which are employed for the numerical integration of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The new methods derived herein are characterized by abscissae vectors, which are shown to achieve A-stability up to order eight accuracy for different step values. Results of numerical experiments presented show that the proposed SR-SD-GLMs preserve the asymptotic solution of the underlying differential equations.

Keywords: SD-GLMs, Strong regularity, A-stability, Superconvergence, Equilibrium solution.

1 Introduction

In Butcher and Hojjati [1], general linear methods that incorporate the second derivative functions as an extension of the traditional GLMs [2–5] were introduced for the numerical solution of stiff IVPs in ODEs,

$$\begin{cases} y' = f\left(y(x)\right), \quad x \in [x_0,\ X] \\ y(x_0) = y_0 \in \mathbb{R}^m, \end{cases} \tag{1}$$

The second derivative GLMs (SD-GLMs) offer higher-order accuracy and improved stability, details can be found in [1, 6–10], and the references therein. This paper

aims to introduce a new class of second-derivative general linear methods (SD-GLMs) with regularity properties. Numerical methods possessing the inherent property that preserves the asymptotic solution of ODEs are referred to as regular methods [11–14]. Hairer et al. in [13] consider the concept of essentially one stage RKM, to which [11] gave a generalization to DIMSIMs following [15]. In particular, Jackiewicz et al. in [11] derived an equivalent criterion for GLM regularity with a detailed discussion in [12, 13, 16–20].

2 SD-GLMs

The compact form of the SD-GLM in [1] for problem (1) takes the form

$$\left(\begin{array}{c} Y \\ \hline y^{[n]} \end{array}\right) = \left(\begin{array}{c|c|c} A & \hat{A} & U \\ \hline B & \hat{B} & V \end{array}\right) \left(\begin{array}{c} hF(Y) \\ h^2G(Y) \\ \hline y^{[n-1]} \end{array}\right). \tag{2}$$

where

$$Y = [Y_1, Y_2, \ldots, Y_s]^T, \quad y^{[n-1]} = \left[y_1^{[n-1]}, y_2^{[n-1]}, \ldots, y_k^{[n-1]}\right]^T,$$
$$y^{[n]} = \left[y_1^{[n]}, y_2^{[n]}, \ldots, y_k^{[n]}\right]^T, \quad F(Y) = [f(Y_1), f(Y_2), \ldots, f(Y_s)]^T$$
$$F'(Y) = G(Y) = [f'(Y_1), f'(Y_2), \ldots, f'(Y_s)]^T = [g(Y_1), g(Y_2), \ldots, g(Y_s)]^T$$

$A = \{a_{ij}\} \in \mathbb{R}^{s\times s}$, $\hat{A} = \{\hat{a}_{ij}\} \in \mathbb{R}^{s\times s}$, $B = \{b_{ij}\} \in \mathbb{R}^{k\times s}$, $\hat{B} = \{\hat{b}_{ij}\} \in \mathbb{R}^{k\times s}$, $U = \{u_{ij}\} \in \mathbb{R}^{s\times k}$, $V = \{v_{ij}\} \in \mathbb{R}^{k\times k}$. The s stages Y_i, $i = 1, \cdots, s$ and the k outgoing approximations at the end of step number n, $y_i^{[n]}$, $i = 1, \cdots, k$, are given by

$$\begin{cases} Y_i = h^2 \sum_{j=1}^{s} \hat{a}_{ij} G_j + h \sum_{j=1}^{s} a_{ij} F_j + \sum_{j=1}^{k} u_{ij} y_j^{[n-1]} & i = 1, 2, \ldots, s \\ y_i^{[n]} = h^2 \sum_{j=1}^{s} \hat{b}_{ij} G_j + h \sum_{j=1}^{s} b_{ij} F_j + \sum_{j=1}^{k} v_{ij} y_j^{[n-1]} & i = 1, 2, \ldots, k \end{cases} \tag{3}$$

The SD-GLM (3) requires an input value $y^{[n-1]}$ in order to compute an approximation to $y(x_n)$ so as to move from step number $n-1$ to the next number n. The stage and derivatives values in (3) are described by $Y_i = y(x_{n-1} + hc_i)$, $F_i = y'(x_{n-1} + hc_i)$, and $F_i' = G_i = y''(x_{n-1} + hc_i)$. Moreover, the stages $Y_i = y(x_{n-1} + c_i h) + O(h^{q+1})$, $i = 1, 2, \ldots, s$, in (3) are of order q, while the components of the output vector, $y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(x_n) + O(h^{p+1})$, $i = 1, 2, \ldots, k$, are of order p. The coefficients of the method are a_{ij}, $\hat{a}_{ij}$, $b_{ij}, \hat{b}_{ij}, u_{ij}$ and v_{ij}, and are defined by the matrices A, $\hat{A}$, B, $\hat{B}$, U and V respectively, given by $(s+r) \times (2s+r)$ matrix in (2).

2.1 Preconsistency, consistency, stage-consistency, and zero-stability of SD-GLMs

To develop SD-GLMs (2) for IVPs (1), certain algebraic conditions must be satisfied to guarantee practical convergence and stability [1]. Suppose there exist vectors q_0

and q_1 defined by

$$q_0 = \begin{bmatrix} q_{1,0} & q_{2,0} & \cdots & q_{k,0} \end{bmatrix}^T, \quad q_1 = \begin{bmatrix} q_{1,1} & q_{2,1} & \cdots & q_{k,1} \end{bmatrix}^T.$$

The components of the input vector, stage vector, and output vector of the SD-GLM (2) satisfy

$$y_i^{[n-1]} = q_{i,0} y(x_{n-1}) + q_{i,1} h y'(x_{n-1}) + O(h^2), \quad i = 1, \ldots, k, \tag{4}$$

$$Y_i^{[n]} = y(x_{n-1} + c_i h) + O(h^2), \quad i = 1, \ldots, s, \tag{5}$$

$$y_i^{[n]} = q_{i,0} y(x_n) + q_{i,1} h y'(x_n) + O(h^2), \quad i = 1, \ldots, k. \tag{6}$$

Substituting these expansions into the SD-GLM (2), we obtain

$$Y^{[n]} = hAF(Y^{[n]}) + h^2 \hat{A} G(Y^{[n]}) + U y^{[n-1]},$$

$$y^{[n]} = hBF(Y^{[n]}) + h^2 \hat{B} G(Y^{[n]}) + V y^{[n-1]},$$

Collecting terms up to $O(h)$, we obtain the following algebraic relations

$$\sum_{j=1}^{k} u_{ij} q_{j,0} = 1, \quad \sum_{j=1}^{k} v_{ij} q_{j,0} = q_{i,0}, \quad i = 1, \ldots, k, \tag{7}$$

$$\sum_{j=1}^{s} a_{ij} + \sum_{j=1}^{k} u_{ij} q_{j,1} = c_i, \quad \sum_{j=1}^{s} b_{ij} + \sum_{j=1}^{k} v_{ij} q_{j,1} = q_{i,0} + q_{i,1}, \quad i = 1, \ldots, k. \tag{8}$$

Thus, the definitions follows from [1],

Definition 1 (Preconsistency) The SD-GLM (2) is preconsistent if there exists a vector q_0 such that

$$U q_0 = e, \qquad V q_0 = q_0,$$

where $e = [1, \ldots, 1]^T \in \mathbb{R}^s$.

Definition 2 (Consistency) The SD-GLM (2) is consistent if it is preconsistent with vector q_0 and there exists a vector q_1 such that

$$Be + V q_1 = q_0 + q_1.$$

Definition 3 (Stage-consistency) The SD-GLM (2) is stage-consistent if

$$Ae + U q_1 = c.$$

Definition 4 (Zero-stability) The SD-GLM (2) is zero-stable if there exists a constant C such that

$$\|V^n\| \leq C, \qquad \forall n \in \mathbb{Z}^+.$$

2.2 Convergence of SD-GLMs

Let a starting procedure $S_h : \mathbb{R}^m \to \mathbb{R}^{mr}$ be used to generate the first input vector $y^{[0]} = S_h(y_0)$, satisfying $\lim_{h\to 0} y^{[0]} = (q_0 \otimes I)y(x_0)$, where $y(x_0) \in \mathbb{R}^m$ is the initial data.

Definition 5 (Convergence) The SD-GLM (2) is convergent if for any initial value problem satisfying a Lipschitz condition, there exists $q_0 \neq 0$ and a starting procedure S_h such that the sequence $y^{[n]}$ computed using n steps with $h = (\bar{x} - x_0)/n$ converges to $q_0 y(\bar{x})$ for any fixed $\bar{x} \in [x_0, X]$.

Theorem 1 *The SD-GLM (2) is convergent if and only if it is consistent and zero-stable.*

2.3 Order and stage order conditions

We give the following order conditions for the SD-GLM (2)

Theorem 2 *The SD-GLM (2) has stage order q and order of method p if and only if*

$$\begin{aligned} e^{cz} &= zAe^{cz} + z^2\hat{A}e^{cz} + Uw + O(z^{q+1}), \\ e^{cz}w &= zBe^{cz} + z^2\hat{B}e^{cz} + Vw + O(z^{p+1}), \end{aligned} \tag{9}$$

where $e^{cz} = [e^{c_1 z}, \cdots, e^{c_s z}]^T$ and $w_j = \sum_{m=0}^{p} w_{jm} z^m;\ j = 1, 2, \cdots, k.$

Proof Assume that the components of the input vector satisfy

$$y_j^{[n-1]} = \sum_{m=0}^{p} w_{jm} h^m y^{(m)}(x_{n-1}) + O(h^{p+1}), \qquad j = 1, 2, \ldots, k.$$

The internal stages approximate the exact solution at the points $x_{n-1} + c_i h$ with stage order q,

$$Y_i^{[n]} = y(x_{n-1} + c_i h) + O(h^{q+1}), \qquad i = 1, 2, \ldots, s.$$

Expanding the exact solution in Taylor series about x_{n-1},

$$y(x_{n-1} + c_i h) = \sum_{m=0}^{q} \frac{c_i^m}{m!} h^m y^{(m)}(x_{n-1}) + O(h^{q+1}).$$

Since the SD-GLM (2) contains both first and second derivatives, we expand

$$hf(Y_i^{[n]}) = \sum_{m=1}^{q} \frac{c_i^{m-1}}{(m-1)!} h^m y^{(m)}(x_{n-1}) + O(h^{q+1}),$$

$$h^2 g(Y_i^{[n]}) = \sum_{m=2}^{q} \frac{c_i^{m-2}}{(m-2)!} h^m y^{(m)}(x_{n-1}) + O(h^{q+1}).$$

Substituting into the SD-GLM (2) stage relation

$$Y^{[n]} = hAF(Y^{[n]}) + h^2\hat{A}G(Y^{[n]}) + Uy^{[n-1]},$$

and collecting terms in powers of h, we obtain

$$\sum_{m=0}^{q}\Big(c_i^m-\sum_{j=1}^{s}ma_{ij}c_j^{m-1}-\sum_{j=1}^{s}m(m-1)\hat{a}_{ij}c_j^{m-2}-\sum_{j=1}^{k}m!u_{ij}w_{jm}\Big)\frac{h^m}{m!}y^{(m)}(x_{n-1})=O(h^{q+1}).$$

Similarly, the output relation

$$y^{[n]}=hBF(Y^{[n]})+h^2\hat{B}G(Y^{[n]})+Vy^{[n-1]},$$

gives

$$\sum_{m=0}^{p}\Big(\sum_{l=0}^{m}\frac{m!}{l!}w_{i,m-l}-\sum_{j=1}^{s}mb_{ij}c_j^{m-1}-\sum_{j=1}^{s}m(m-1)\hat{b}_{ij}c_j^{m-2}-\sum_{j=1}^{k}m!v_{ij}w_{jm}\Big)\frac{h^m}{m!}y^{(m)}(x_{n-1})=O(h^{p+1}).$$

Equating coefficients of $\frac{h^m y^{(m)}(x_{n-1})}{m!}$ to zero gives the algebraic order conditions. Multiplying these relations by $z^m/m!$ and summing over m yields the generating function of the form:

$$e^{cz}-zAe^{cz}-z^2\hat{A}e^{cz}-Uw=O(z^{q+1}),$$
$$e^{cz}w-zBe^{cz}-z^2\hat{B}e^{cz}-Vw=O(z^{p+1}).$$

□

where $e^{cz}=[e^{c_1z},\cdots,e^{c_sz}]^T$ and $w_j=\sum_{m=0}^{p}w_{jm}z^m;\ j=1,2,\cdots,k.$

2.4 Stability analysis of SD-GLMs

Applying the SD-GLM (2) to the test problem $y''(x)=\lambda^2y(x),\quad R(\lambda)>0$ yields the stage and output relations [1],

$$\begin{aligned}Y^{[n]}&=h^2\lambda^2\hat{A}Y^{[n]}+h\lambda AY^{[n]}+Uy^{[n-1]}, &&\text{(a)},\quad z=\lambda h\\ y^{[n]}&=h^2\lambda^2\hat{B}Y^{[n]}+h\lambda BY^{[n]}+Vy^{[n-1]}, &&\text{(b)}\end{aligned}\tag{10}$$

where $Y^{[n]}$ is the stage vector and $y^{[n]}$ is the output vector. Assuming that $I-zA-z^2\hat{A}$ is invertible, we solve (10)(a) for $Y^{[n]}$:

$$Y^{[n]}=\Big(I-zA-z^2\hat{A}\Big)^{-1}Uy^{[n-1]}.\tag{11}$$

Substituting (11) into (10)(b) gives

$$y^{[n]}=M(z)y^{[n-1]},\tag{12}$$

where

$$M(z)=V+\Big(zB+z^2\hat{B}\Big)\Big(I-zA-z^2\hat{A}\Big)^{-1}U,\quad Re(z)<0\tag{13}$$

The stability polynoimal can be obtained by computing

$$\pi(w,z)=\det(wI-M(z))\tag{14}$$

Thus, we give the following definitions. See [1, 6, 7]

Definition 6 The SD-GLM (2) is said to be zero-stable if the matrix V has spectral radius less or equals one with roots on the unit circle being simple.

Definition 7 The SD-GLM (2) is said to be absolutely stable at z if the stability polynomial (14) has roots $|\{w_j\}_{j=0}^{k}| \leq 1$ with roots on the unit circle being simple.

Definition 8 The SD-GLM (2) is said to be A-stable if the absolute stability region of (14) lies in the entire open left half of the z-plane.

Definition 9 The SD-GLM (2) is said to be L-stable if it is A-stable and if the stability polynomial $\pi(w, z)$ in (14) has absolute vanishing roots as $z \to -\infty$.

Definition 10 The SD-GLM (2) is said to be $A(\alpha)$-stable for some $\alpha \in \left[0, \frac{\pi}{2}\right)$ if the wedge

$$S_\alpha = \{z : |Arg(-z)| < \alpha,\ z \neq 0\} \tag{15}$$

is contained in its region of absolute stability.

Definition 11 The SD-GLM (2) is said to be $L(\alpha)$-stable if it is $A(\alpha)$-stable and if the stability polynomial $\pi(w, z)$ in (14) has absolute vanishing roots as $z \to -\infty$.

The largest angle α is regarded as the angle of absolute stability of the method and a stiffly-stable method is also $A(\alpha)$-stable (see [7]), and the references therein.

3 New strongly regular SD-GLM

Consider the methods with stage values,

$$y_{n+v_1} = \sum_{j=0}^{k-1} \alpha_j^{(v_1)} y_{n+j} + h\left(\bar{\beta}_{v_1}^{(v_1)} f_{n+v_1} + \bar{\beta}_{v_2}^{(v_1)} f_{n+v_2}\right) + h^2 \bar{\lambda}_{v_1}^{(v_1)} f'_{n+v_1}, \tag{16}$$

$$y_{n+v_2} = \sum_{j=0}^{k-1} \alpha_j^{(v_2)} y_{n+j} + h\left(\bar{\beta}_{v_1}^{(v_2)} f_{n+v_1} + \bar{\beta}_{v_2}^{(v_2)} f_{n+v_2}\right) + h^2\left(\bar{\lambda}_{v_1}^{(v_2)} f'_{n+v_1} + \bar{\lambda}_{v_1}^{(v_2)} f'_{n+v_2}\right), \tag{17}$$

and an output approximation,

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_j^{(k)} y_{n+j} + h\left(\beta_{v_1}^{(k)} f_{n+v_1} + \beta_{v_2}^{(k)} f_{n+v_2}\right) + h^2\left(\lambda_{v_1}^{(k)} f'_{n+v_1} + \lambda_{v_1}^{(k)} f'_{n+v_2}\right). \tag{18}$$

The representation of (16), (17) and (18) in SD-GLM (2) formalism we obtain

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+k} \\ y_{n+k-1} \\ \vdots \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|ccc} \bar{\beta}_{v_1}^{(v_1)} & \bar{\beta}_{v_2}^{(v_1)} & \lambda_{v_1}^{(v_1)} & 0 & \alpha_{k-1}^{(v_1)} & \cdots & \alpha_0^{(v_1)} \\ \bar{\beta}_{v_1}^{(v_2)} & \bar{\beta}_{v_2}^{(v_2)} & \lambda_{v_1}^{(v_2)} & \lambda_{v_2}^{(v_2)} & \alpha_{k-1}^{(v_2)} & \cdots & \alpha_0^{(v_2)} \\ \hline \beta_{v_1}^{(k)} & \beta_{v_2}^{(k)} & \lambda_{v_1}^{(k)} & \lambda_{v_2}^{(k)} & \alpha_{k-1}^{(k)} & \cdots & \alpha_0^{(k)} \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2 f'_{n+v_1} \\ h^2 f'_{n+v_2} \\ \hline y_{n+k-1} \\ y_{n+k-2} \\ \vdots \\ y_n \end{array}\right] \tag{19}$$

where y_{n+v_i} is the numerical approximation of the solution $y(x_n + v_i h)$, $i = 1, 2$, y_{n+k} is the output numerical solution to the analytical solution $y(x_{n+k})$, and $h = x_n - x_{n-1}$ is the step-size. Here, the abscissae vector is chosen according to the generalization,

$$v(k) = \left(k - \frac{3}{2}\delta k_1,\ k - \frac{1}{2}\right)^T \tag{20}$$

where $\delta k_1 = 1$ if $k = 1$, else zero.

The stability polynomial of the SD-GLM (2) is determined by computing (14) from which we obtain the boundary locus of the stability region of the method (19).

3.1 Regularity and strong regularity condition of the SD-GLM

The extention of strong regularity to SD-GLMs is analogous to regularity theory developed for GLMs in [11, 21]. Suppose Γ denote the set of all bounded asymptotic values of (1):

$$\Gamma = \{y \in \mathbb{R}^m : f(y) = 0, f'(y) = 0\}.$$

Also define

$$\hat{\Gamma} = \left\{(y, \ldots, y) \in \mathbb{R}^{m(r+s)} : y \in \Gamma\right\}.$$

Now consider the system where $Y_i^{[n]} \to \hat{Y}_i, \quad y_i^{[n]} \to \hat{y}_i; \quad n \to \infty$; in (19),

$$\begin{aligned} \hat{Y}_i &= h\sum_{j=1}^{s} a_{ij} f(\hat{Y}_j) + h^2 \sum_{j=1}^{s} \bar{a}_{ij} f'(\hat{Y}_j) + \sum_{j=1}^{r} u_{ij}\hat{y}_j, \quad i = 1, \ldots, s, \quad s = 2, \\ \hat{y}_i &= h\sum_{j=1}^{s} b_{ij} f(\hat{Y}_j) + h^2 \sum_{j=1}^{s} \bar{b}_{ij} f'(\hat{Y}_j) + \sum_{j=1}^{r} v_{ij}\hat{y}_j, \quad i = 1, \ldots, r, \end{aligned} \tag{21}$$

Here, $\hat{Y}_i$ and $\hat{y}_i$ are fixed points of the discrete dynamics in (21) and (19) . Define

$$\Gamma_h = \left\{\sum_{j=1}^{r} v_{1j}\hat{y}_j : (\hat{y}_1, \ldots, \hat{y}_r, \hat{Y}_1, \ldots, \hat{Y}_s) \text{ satisfies} (21)\right\}, \quad s = 2,$$

and

$$\hat{\Gamma}_h = \left\{(\hat{y}_1, \cdots, \hat{y}_r, \hat{Y}_1, \hat{Y}_2, \cdots, \hat{Y}_s) : (\hat{y}_1, \cdots, \hat{y}_r, \hat{Y}_1, \hat{Y}_2, \cdots, \hat{Y}_s) \text{ satisfies}(21)\right\}, \quad s = 2.$$

Here, $\Gamma \subseteq \Gamma_h$ and $\hat{\Gamma} \subseteq \hat{\Gamma}_h$ for any $h > 0$, but the differences $\Gamma_h - \Gamma$ and $\hat{\Gamma}_h - \hat{\Gamma}$ need not be empty as noted by Jackiewicz [21]. Hence, the SD-GLM (21) is *regular* if, for every $h > 0$, $\Gamma_h = \Gamma$, and *strongly regular* if, for every $h > 0$, $\hat{\Gamma}_h = \hat{\Gamma}$.

Thus, this follows that if $Y_i^{[n]} \to \hat{Y}_i$ as $n \to \infty$, then $y_i^{[n]} \to \hat{y}$ and $y_i^{[n-1]} \to \hat{y}$ as $n \to \infty$. Also, we know that $\sum_{j=1}^{r} u_{ij} = 1$, $i = 1(1)s$ and $\sum_{j=1}^{r} v_{1j} = 1$, $i = 1(1)r$. Therefore, (21) can now take the form

$$\begin{aligned} \hat{Y}_i &= h^2 \sum_{j=1}^{s} \bar{a}_{ij} f'(\hat{Y}_j) + h \sum_{j=1}^{s} a_{ij} f(\hat{Y}_j) + \hat{y}_i \\ \hat{y}_i &= h^2 \sum_{j=1}^{s} \bar{b}_{ij} f'(\hat{Y}_j) + h \sum_{j=1}^{s} b_{ij} f(\hat{Y}_j) + \hat{y}_i, \end{aligned} \tag{22}$$

Here, (21) becomes

$$\begin{aligned} \hat{Y}_i = \hat{y}_i + h^2 \sum_{j=1}^{s} \bar{a}_{ij} f'(\hat{Y}_j) + h \sum_{j=1}^{s} a_{ij} f(\hat{Y}_j), \\ h^2 \sum_{j=1}^{s} \bar{b}_{ij} f'(\hat{Y}_j) + h \sum_{j=1}^{s} b_{ij} f(\hat{Y}_j) = 0, \end{aligned} \tag{23}$$

This implies by regularity that

$$\begin{pmatrix} A + UB & \bar{A} + U\bar{B} \\ V_1 B & V_1 \bar{B} \end{pmatrix} \begin{pmatrix} hf(Y) \\ h^2 f'(Y) \end{pmatrix} = \begin{pmatrix} Y - \xi e \\ 0 \end{pmatrix} \tag{24}$$

The V_1 means the first row of V. $\xi \in \mathbb{R}$, $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s$ with solutions $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_r, \hat{Y}_1, \hat{Y}_2, \ldots, \hat{Y}_s)$ such that $f(\hat{y}_i) = f'(\hat{y}_i) = 0$. Here, we consider a GLM structure with a fixed $s = 2$, so that we have solutions $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_r, \hat{Y}_1, \hat{Y}_2)$. The SD-GLM (21) is regular on the condition of (24), giving the criterion that admits the attraction of the asymptotic values of $\hat{y}_i$ of (23). Thus, we have the following lemmas and theorems.

Lemma 1 The SD-GLM (21) is regular if and only if for every $n \geq 1$ and for every step size $h > 0$, every solution of (23), satisfying $f(\hat{Y}_1) = f(\hat{Y}_2) = \cdots = f(\hat{Y}_s)$; $f'(\hat{Y}_1) = f'(\hat{Y}_2) = \cdots = f'(\hat{Y}_s)$ whenever $\hat{Y}_1 = \hat{Y}_2 = \cdots = \hat{Y}_s$, admits the fact that $f(\hat{Y}_1) = f(\hat{Y}_2) = \cdots = f(\hat{Y}_s) = 0$; $f'(\hat{Y}_1) = f'(\hat{Y}_2) = \cdots = f'(\hat{Y}_s) = 0$ and $\hat{y}_i = \hat{Y}_1 = \hat{Y}_2 = \cdots = \hat{Y}_s$.

Proof By regularity from (23), the continuous function f is constructed such that

$$f(\hat{Y}_i) = f_i, \quad i = 1, 2, \ldots, s$$

$$f'(\hat{Y}_i) = f'_i, \quad i = 1, 2, \ldots, s$$

if and only if $Y_i, f_i, f'_i,\ i = 1, 2, \ldots, s$ satisfy $f(\hat{Y}_1) = f(\hat{Y}_2) = \cdots = f(\hat{Y}_s)$ and that $f'(\hat{Y}_1) = f'(\hat{Y}_2) = \cdots = f'(\hat{Y}_s)$ whenever $\hat{Y}_1 = \hat{Y}_2 = \cdots = \hat{Y}_s$. □

Lemma 2 The SD-GLM (21) is strongly regular if and only if for every $n \geq 1$ and for every step size $h > 0$, every solution of (23) admits $\hat{Y}_1 = \hat{Y}_2 = \cdots = \hat{Y}_s$ and for all such solutions, we have $\hat{y}_i = \hat{y}_j, i, j = 1, 2, \ldots, r$ and $f(\hat{Y}_1) = f(\hat{Y}_2) = \cdots = f(\hat{Y}_s) = 0$, together with the fact that $f'(\hat{Y}_1) = f'(\hat{Y}_2) = \cdots = f'(\hat{Y}_s) = 0$.

Following the idea in [11, 13, 21], the SD-GLM (21) is EOS if there exists an index $\nu \in \{1, 2, \cdots, s\}$ such that

$$r_\nu(U)B = V_1 B = e_\nu^T, \qquad r_\nu(A) = c_\nu e_\nu^T$$

$$r_\nu(U)\bar{B} = V_1 \bar{B} = e_\nu^T, \qquad r_\nu(\bar{A}) = c_\nu e_\nu^T$$

for some $c_\nu \in \mathbb{R}$, where $r_\nu(U)$ and $r_\nu(A)$, $r_\nu(\bar{A})$ represent the νth rows of matrices U and A, $\bar{A}$ respectively, and V_1 is the first row of V.

Theorem 3 *Assume that the SD-GLM (21) is strongly regular and non-EOS, then*

$$r_1(Q^{(m)}) - r_2(Q^{(m)}) = \begin{cases} \lambda^{(1)} V_1 B & ; \quad m = 1, \\ \lambda^{(2)} V_1 \bar{B} & ; \quad m = 2, \end{cases} \quad \lambda^{(m)} \in \mathbb{R}$$

where $Q^{(1)} = A + UB$, $Q^{(2)} = \bar{A} + U\bar{B}$. Here, $r_1(Q^{(m)})$ and $r_2(Q^{(m)})$ are the first and second row of $Q^{(m)}$ respectively and V_1 is the first row of the matrix V.

Proof Consider the system (21) with $s = 2$, which solution is given by

$$(I - V)\hat{y} = hB\hat{F} + h^2 \bar{B}\hat{F}' \tag{25}$$

where $\hat{F}$ and $\hat{F}'$ satisfy $\hat{F} = F(\hat{Y}) = 0$ and $\hat{F}' = F'(\hat{Y}) = 0$, respectively. Also, consider the folded method which is the system corresponding to (21) given by

$$\left[\begin{array}{c|c|c} A^* & \bar{A}^* & U^* \\ \hline B^* & \bar{B}^* & V^* \end{array}\right] = \left[\begin{array}{c|c|c} [I\ 0]\, A \begin{bmatrix} I \\ e_1^T \end{bmatrix} & [I\ 0]\, \bar{A} \begin{bmatrix} I \\ e_1^T \end{bmatrix} & [I\ 0]\, U \\ \hline B \begin{bmatrix} I \\ e_1^T \end{bmatrix} & \bar{B} \begin{bmatrix} I \\ e_1^T \end{bmatrix} & V \end{array}\right]$$

Thus,

$$\begin{aligned} \hat{Y}^* &= hA^*\hat{F}^* + h^2\bar{A}^*\hat{F}^* + U^*\hat{y}^* \\ (I - V^*)\hat{y}^* &- hB^*\hat{F}^* - h^2\bar{B}^*\hat{F}^* = 0 \end{aligned} \tag{26}$$

Assuming $V\hat{y} = \xi e;\ \xi \in \mathbb{R}$, then the second equation of (26) takes the form

$$\hat{y}^* = hB^*\hat{F}^* + h^2\bar{B}^*\hat{F}^* + \xi^* e.$$

Then substituting for $\hat{y}^*$ in the first equation of (26), and also multiplying by V_1,

$$hV_1\bar{B}^*\hat{F}^* + h^2 V_1 \bar{B}^* \hat{F'}^* = 0,$$

$$h(A^* + U^*B^*)\hat{F}^* + h^2(\bar{A}^* + U^*\bar{B}^*)\hat{F'}^* = hQ^{(1)*}\hat{F}^* + h^2Q^{(2)*}\hat{F'}^* = \hat{Y}^* - \xi^* e,$$

$r_1(Q^{(1)}) - r_2(Q^{(1)}) = \lambda^{(1)}V_1B;\ \lambda^{(1)} \in \mathbb{R}$ and $r_1(Q^{(2)}) - r_2(Q^{(2)}) = \lambda^{(2)}V_1\bar{B};\ \lambda^{(2)} \in \mathbb{R}$. This implies that there can be only one solution, $\hat{Y}_1 = \hat{Y}_2$. Therefore, the SD-GLM (21) is strongly regular. □

Next, we give the following theorem to establish the order and stage order condition of the strongly regular SD-GLM (19). This is analogous to a similar one in Jackiewicz [21].

Theorem 4 *Assume that $p, q \geq 1$ being the order and stage order of (19) respectively, and that $(A + UB)v + (\bar{A} + U\bar{B})u = \xi e, \quad \xi \in \mathbb{R}$ where $v, u \in \mathbb{R}^s$, and $e = (1, 1, \cdots, 1)^T$, for any $v, u \in S$ where $S = \{v, u : V_1Bv + V_1\bar{B}u = 0\}$. Then the SD-GLM (19) is strongly regular*

This is seen from (24) by letting $v = hf(Y),\ u = h^2f'(Y)$, so that

$$\begin{pmatrix} A + UB & \bar{A} + U\bar{B} \\ V_1B & V_1\bar{B} \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} Y - \xi e \\ 0 \end{pmatrix} \tag{27}$$

Then at equilibrium point of (1), $f(Y) = 0,\ f'(Y) = 0 \implies Y = y = \xi e$. The following Corollary is a direct consequence of Theorem 2

Corollary 1 For the step number $k = 1, 2, \cdots$, the conditions for stage order q and output order p of (19) are

$$c_i^m = m\sum_{j=1}^{s} a_{ij}c_j^{m-1} + m(m-1)\sum_{j=1}^{s} \bar{a}_{ij}c_j^{m-2} + \sum_{j=1}^{k}(k-j)^m u_{ij};$$

$i = 1, 2, \cdots s; \quad m = 0, 1, \cdots q$ and

$$k^m = m\sum_{j=1}^{s} b_{1j}c_j^{m-1} + m(m-1)\sum_{j=1}^{s} \bar{b}_{1j}c_j^{m-2} + \sum_{j=1}^{k}(k-j)^m v_{1j}; \quad m = 0, 1, \cdots, p$$

respectively.

3.2 Construction of strongly regular SD-GLMs

In this section, we apply the strong regularity conditions established in Theorem 3, together with the order and stage order conditions derived in Corollary 1 to construct variable order strongly regular SD-GLMs (19).

3.2.1 2-stage 1-value strongly regular SD-GLM ($q = p = 3$)

Setting $k = 1$ in (19), and transforming to (2) we obtain,

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|c} a_{11} & a_{12} & \bar{a}_{11} & 0 & u_{11} \\ a_{21} & a_{22} & \bar{a}_{21} & \bar{a}_{22} & u_{21} \\ \hline b_{11} & b_{12} & \bar{b}_{11} & \bar{b}_{12} & v_{11} \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2f'_{n+v_1} \\ \hline h^2f'_{n+v_2} \\ y_n \end{array}\right]; \ c = v = [v_1,\ v_2]^T. \tag{28}$$

Using the order condition in Corollary 1, with $q = p = 3$, we obtain

$$\begin{cases} u_{11} = 1, \\ a_{11} + a_{12} = v_1, \\ 2a_{11}v_1 + 2a_{12}v_2 + 2\bar{a}_{11} = v_1^2, \\ 3a_{11}v_1^2 + 3a_{12}v_2^2 + 6\bar{a}_{11}v_1 = v_1^3, \\ u_{21} = 1, \\ a_{21} + a_{22} = v_2, \\ 2a_{21}v_1 + 2a_{22}v_2 + 2\bar{a}_{21} + 2\bar{a}_{22} = v_2^2, \\ 3a_{21}v_1^2 + 3a_{22}v_2^2 + 6\bar{a}_{21}v_1 + 6\bar{a}_{22}v_2 = v_2^3, \\ v_{11} = 1, \\ b_{11} + b_{12} = 1, \\ 2b_{11}v_1 + 2b_{12}v_2 + 2\bar{b}_{11} + 2\bar{b}_{12} = 1, \\ 3b_{11}v_1^2 + 3b_{12}v_2^2 + 6\bar{b}_{11}v_1 + 6\bar{b}_{12}v_2 = 1, \end{cases} \tag{29}$$

For strong regularity, Theorem 3 must be satisfied. Thus, we obtain

$$\begin{aligned} (a_{11} - a_{21})\, b_{12}v_{11} &= (a_{12} - a_{22})\, b_{11}v_{11}, \\ (\bar{a}_{11} - \bar{a}_{21})\, \bar{b}_{12}v_{11} &= -\bar{a}_{22}\bar{b}_{11}v_{11}. \end{aligned} \tag{30}$$

Solving (29) and (30) simultaneously, we obtain

$$
\begin{aligned}
a_{11} &= \frac{v_1\left(2v_1^2-6v_2v_1+3v_2^2\right)}{3\left(v_1-v_2\right)^2},\ u_{11}=u_{21}=v_{11}=1,\\
a_{12} &= \frac{v_1^3}{3\left(v_1-v_2\right)^2},\\
a_{21} &= \frac{2v_1^5-4v_2v_1^4-3v_1^4-4v_2^2v_1^3+9v_2v_1^3+v_1^3-3v_2^3v_1^2-3v_2v_1^2+6v_2^3v_1-3v_2^2v_1+3v_2^5-6v_2^4+3v_2^3}{3\left(v_1-v_2\right)^2\left(v_1^2+4v_2v_1-3v_1+v_2^2-3v_2+2\right)},\\
a_{22} &= \frac{-2v_1^5+7v_2v_1^4+3v_1^4+10v_2^2v_1^3-18v_2v_1^3-v_1^3-15v_2^3v_1^2+9v_2^2v_1^2+9v_2v_1^2+6v_2^4v_1+3v_2^3v_1-9v_2^2v_1-3v_2^4+3v_2^3}{3\left(v_1-v_2\right)^2\left(v_1^2+4v_2v_1-3v_1+v_2^2-3v_2+2\right)},\\
\bar{a}_{11} &= \frac{3v_1^2v_2-v_1^3}{6\left(v_1-v_2\right)},\\
\bar{a}_{21} &= \frac{-2v_1^5+4v_2v_1^4+3v_1^4+4v_2^2v_1^3-9v_2v_1^3-v_1^3+4v_2^3v_1^2+3v_2v_1^2+4v_2^4v_1-9v_2^3v_1+3v_2^2v_1-2v_2^5+3v_2^4-v_2^3}{6\left(v_1^3+3v_2v_1^2-3v_1^2-3v_2^2v_1+2v_1-v_2^3+3v_2^2-2v_2\right)},\\
\bar{a}_{22} &= -\frac{\left(v_1-v_2\right)^2\left(2v_1^2+2v_2v_1-3v_1-v_2^2+1\right)}{6\left(v_1^2+4v_2v_1-3v_1+v_2^2-3v_2+2\right)},\\
b_{11} &= \frac{v_2^2+2v_1v_2-2v_2-v_1+1}{v_1^2+4v_2v_1-3v_1+v_2^2-3v_2+2},\\
b_{12} &= \frac{v_1^2+2v_2v_1-2v_1-v_2+1}{v_1^2+4v_2v_1-3v_1+v_2^2-3v_2+2},\\
\bar{b}_{11} &= \frac{-6v_2v_1^3+3v_1^3+12v_2^2v_1^2-3v_2v_1^2-2v_1^2+12v_2^3v_1-30v_2^2v_1+19v_2v_1-3v_1-6v_2^3+13v_2^2-9v_2+2}{6\left(v_1^3+3v_2v_1^2-3v_1^2-3v_2^2v_1+2v_1-v_2^3+3v_2^2-2v_2\right)},\\
\bar{b}_{12} &= \frac{-12v_2v_1^3+6v_1^3-12v_2^2v_1^2+30v_2v_1^2-13v_1^2+6v_2^3v_1+3v_2^2v_1-19v_2v_1+9v_1-3v_2^3+2v_2^2+3v_2-2}{6\left(v_1^3+3v_2v_1^2-3v_1^2-3v_2^2v_1+2v_1-v_2^3+3v_2^2-2v_2\right)}.
\end{aligned}
\tag{31}
$$

Choosing $v=[-\frac{1}{2},\frac{1}{2}]^T$, according to (20), we obtain the A-stable strongly regular SD-GLM,

$$
\left[\begin{array}{c} y_{n-\frac{1}{2}} \\ y_{n+\frac{1}{2}} \\ \hline y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|c} -\frac{11}{24} & -\frac{1}{24} & -\frac{1}{12} & 0 & 1 \\ -\frac{7}{24} & \frac{19}{24} & -\frac{1}{6} & -\frac{1}{4} & 1 \\ \hline \frac{1}{6} & \frac{5}{6} & \frac{1}{24} & \frac{1}{8} & 1 \end{array}\right] \left[\begin{array}{c} hf_{n-\frac{1}{2}} \\ hf_{n+\frac{1}{2}} \\ h^2f'_{n-\frac{1}{2}} \\ h^2f'_{n+\frac{1}{2}} \\ \hline y_n \end{array}\right] \qquad \lambda^{(1)}=-1,\ \lambda^{(2)}=2, \tag{32}
$$

with stability function $R(z)=\frac{z\left(z^3+6z+32\right)+48}{z(z(z(z+2)-2)-16)+48}$. The boundary locus plot is given in Figure 1. The A-stability region is exterior of the closed curve. The corresponding error constants of the stages and the output method are $C_4(-\frac{1}{2})=\frac{5}{1152}h^4y^{(4)}(x_n)$, $C_4(\frac{1}{2})=\frac{37}{1152}h^4y^{(4)}(x_n)$, and $C_4(1)=\frac{1}{144}h^4y^{(4)}(x_n)$, respectively.

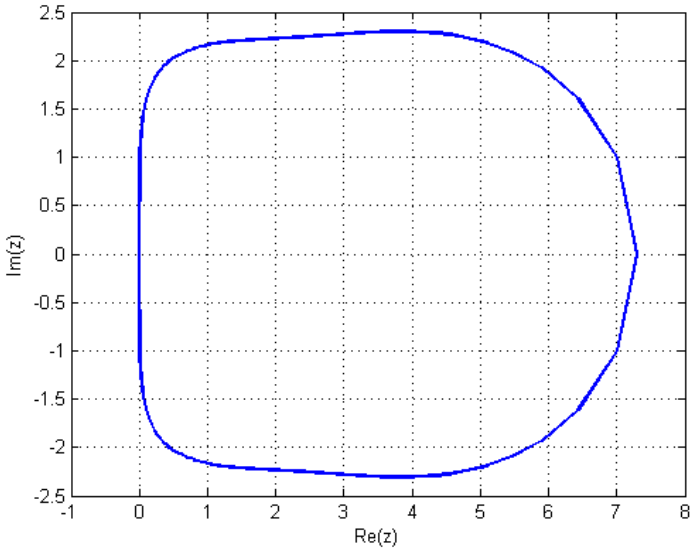


Fig. 1: Boundary locus plot of the strongly regular SD-GLM (32). The stability region is outside the closed curve.

3.2.2 2-stage 2-value strongly regular SD-GLM ($q = p = 4$)

Similarly, setting $k = 2$ in (19), and transforming to (2),

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+2} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|cc} a_{11} & a_{12} & \bar{a}_{11} & 0 & u_{11} & u_{12} \\ a_{21} & a_{22} & \bar{a}_{21} & \bar{a}_{22} & u_{21} & u_{22} \\ \hline b_{11} & b_{12} & b_{11} & b_{12} & v_{11} & v_{12} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2 f'_{n+v_1} \\ h^2 f'_{n+v_2} \\ \hline y_{n+1} \\ y_n \end{array}\right] \tag{33}$$

Using the order condition in Corollary 1, with $q = p = 4$, together with the strong regularity condition in Theorem 3 we obtain

$$\begin{cases} u_{11} + u_{12} = 1, \\ a_{11} + a_{12} + u_{11} = v_1, \\ 2a_{11}v_1 + 2a_{12}v_2 + 2\bar{a}_{11} + u_{11} = v_1^2, \\ 3a_{11}v_1^2 + 3a_{12}v_2^2 + 6\bar{a}_{11}v_1 + u_{11} = v_1^3, \\ 4a_{11}v_1^3 + 4a_{12}v_2^3 + 12c_{11}v_1^2 + u_{11} = v_1^4, \\ u_{21} + u_{22} = 1, \\ a_{21} + a_{22} + u_{21} = v_2, \\ 2a_{21}v_1 + 2a_{22}v_2 + 2\bar{a}_{21} + 2\bar{a}_{22} + u_{21} = v_2^2, \\ 3a_{21}v_1^2 + 3a_{22}v_2^2 + 6\bar{a}_{21}v_1 + 6\bar{a}_{22}v_2 + u_{21} = v_2^3, \\ 4a_{21}v_1^3 + 12\bar{a}_{21}v_1^2 + 4a_{22}v_2^3 + 12\bar{a}_{22}v_2^2 + u_{21} = v_2^4, \\ v_{11} + v_{12} = 1, \\ b_{11} + b_{12} + v_{11} = 2, \\ 2b_{11}v_1 + 2b_{12}v_2 + 2\bar{b}_{11} + 2\bar{b}_{12} + v_{11} = 2, \\ 3b_{11}v_1^2 + 3b_{12}v_2^2 + 6\bar{b}_{11}v_1 + 6\bar{b}_{12}v_2 + v_{11} = 8, \\ 4b_{11}v_1^3 + 4b_{12}v_2^3 + 12\bar{b}_{11}v_1^2 + 12\bar{b}_{12}v_2^2 + v_{11} = 4, \end{cases} \tag{34}$$

For strong regularity from Theorem 3,

$$\begin{aligned} b_{12}\left(a_{11} - a_{21} - b_{11}u_{21} + b_{11}u_{11}\right) &= b_{11}\left(a_{12} - a_{22} - b_{12}u_{21} + b_{12}u_{11}\right), \\ \bar{b}_{12}\left(\bar{a}_{11} - \bar{a}_{21} - \bar{b}_{11}u_{21} + \bar{b}_{11}u_{11}\right) &= \bar{b}_{11}\left(-\bar{a}_{22} - \bar{b}_{12}u_{21} + \bar{b}_{12}u_{11}\right). \end{aligned} \tag{35}$$

Solving (34) and (35) simultaneously, we obtain the required unknowns. Similarly, from (20), we obtain the abscissae characterized by $v = [2, \frac{3}{2}]$, which yields the A-stable strongly regular SD-GLM given by

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+2} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|cc} -\frac{2}{31} & \frac{32}{31} & \frac{2}{31} & 0 & \frac{32}{31} & -\frac{1}{31} \\ -\frac{595}{2976} & \frac{749}{1116} & \frac{3239}{53568} & -\frac{49}{864} & \frac{9185}{8928} & -\frac{257}{8928} \\ \hline \frac{4}{15} & \frac{32}{45} & -\frac{1}{135} & -\frac{14}{135} & \frac{46}{45} & -\frac{1}{45} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2f'_{n+v_1} \\ h^2f'_{n+v_2} \\ \hline y_{n+1} \\ y_n \end{array}\right] \tag{36}$$

$$\lambda^{(1)} = \frac{2945}{5888}, \ \lambda^{(2)} = -\frac{1565}{2944}, \quad V_1 = \left(\tfrac{46}{45} \ -\tfrac{1}{45}\right)$$

The stability polynomial is

$$\pi(w, z) = w^2\left(490z^4 + 2070z^3 - 20795z^2 + 81240z - 133920\right)$$

$$+ w\big(-504z^4 + 1176z^3 + 5796z^2 + 51792z + 136896\big)$$
$$+ \big(14z^4 - 84z^3 + 119z^2 - 2088z - 2976\big).$$

and the stability region is given in Figure 2 for $k = 2$. The corresponding error constants of the stages are $C_5(2) = -\frac{1}{372}h^5y^{(5)}(x_n)$, and $C_5(\frac{3}{2}) = -\frac{7369}{3214080}h^5y^{(5)}(x_n)$, respectively while for the output method we have $C_5(2) = -\frac{23}{16200}h^5y^{(5)}(x_n)$.

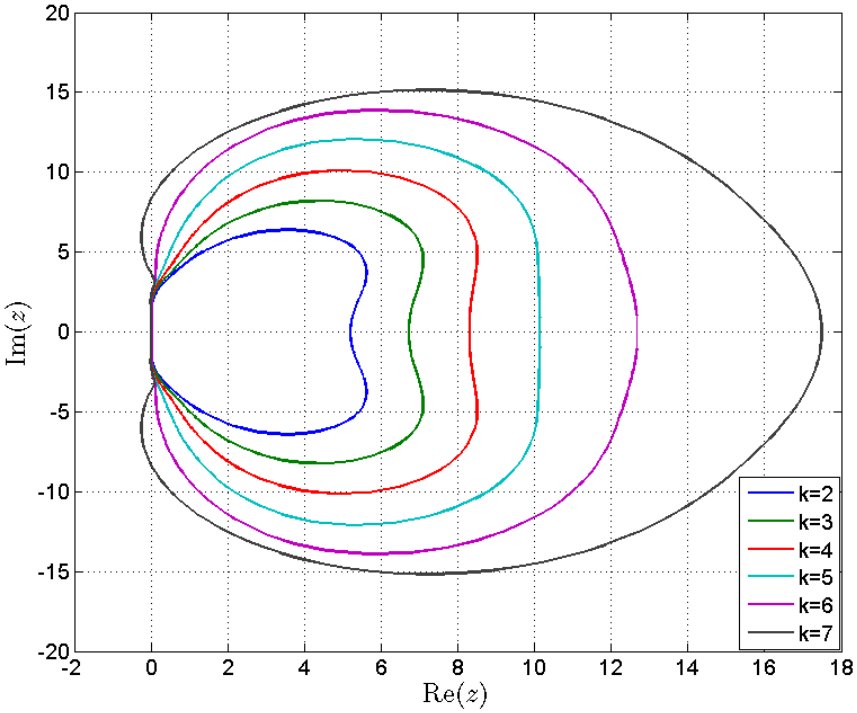


Fig. 2: Stability region of the strongly regular SD-GLM (19) for $k = 2(2)7$. Stability lies outside the boundary locus.

3.2.3 2-stage 3-value strongly regular SD-GLM ($q = p = 5$)

Again, setting $k = 3$, we obtain

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|ccc} a_{11} & a_{12} & \bar{a}_{11} & 0 & u_{11} & u_{12} & u_{13} \\ a_{21} & a_{22} & \bar{a}_{21} & \bar{a}_{22} & u_{21} & u_{22} & u_{23} \\ \hline b_{11} & b_{12} & b_{11} & b_{12} & v_{11} & v_{12} & v_{13} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2 f'_{n+v_1} \\ h^2 f'_{n+v_2} \\ \hline y_{n+2} \\ y_{n+1} \\ y_n \end{array}\right] \tag{37}$$

Using the order condition in Corollary 1, with $q = p = 5$, together with the strong regularity condition in Theorem 3 we obtain the strongly regular A-stable SD-GLM, characterized by the abscissa vector $v = [3,\ \frac{5}{2}]^T$,

$$\left[\begin{array}{cc|cc|ccc} \frac{114}{3773} & \frac{3456}{3773} & \frac{18}{539} & 0 & \frac{3996}{3773} & -\frac{243}{3773} & \frac{20}{3773} \\ -\frac{15703}{181104} & \frac{6133}{11319} & \frac{355}{12936} & -\frac{23}{336} & \frac{379997}{362208} & -\frac{4831}{90552} & \frac{1535}{362208} \\ \hline \frac{2050}{8967} & \frac{6560}{8967} & -\frac{8}{1281} & -\frac{92}{1281} & \frac{9356}{8967} & -\frac{421}{8967} & \frac{32}{8967} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{38}$$

$$\lambda^{(1)} = \frac{1533601}{3068768}, \quad \lambda^{(2)} = -\frac{541253}{598784}, \quad V_1 = \left(\tfrac{9356}{8967}\ \ -\tfrac{421}{8967}\ \ \tfrac{32}{8967}\right).$$

The stability polynomial is given by

$$\begin{aligned} \pi(w,z) = {} & w^3\big(176778z^4 + 704550z^3 - 10118863z^2 + 44237200z - 77331408\big) \\ & + w^2\big(-186300z^4 + 707940z^3 + 4075002z^2 + 31919760z + 80686144\big) \\ & + w\big(10350z^4 - 77280z^3 + 12443z^2 - 2079920z - 3630704\big) \\ & + \big(-828z^4 + 6900z^3 - 5382z^2 + 175600z + 275968\big). \end{aligned}$$

while the corresponding boundary locus plot is given in Figure 2 for $k = 3$. The error constants are $C_6(3) = -\frac{3}{3430}h^6y^{(6)}(x_n)$, $C_6(\frac{5}{2}) = -\frac{63029}{94832640}h^6y^{(6)}(x_n)$, and $C_6(3) = -\frac{1321}{2582496}h^6y^{(6)}(x_n)$, respectively.

3.2.4 2-stage 4-value strongly regular SD-GLM ($q = p = 6$)

In like manner, setting $k = 4$,

$$\left[\begin{array}{c} y_{n+v_1} \\ y_{n+v_2} \\ \hline y_{n+4} \\ y_{n+3} \\ y_{n+2} \\ y_{n+1} \end{array}\right] = \left[\begin{array}{cc|cc|cccc} a_{11} & a_{12} & \bar{a}_{11} & 0 & u_{11} & u_{12} & u_{13} & u_{14} \\ a_{21} & a_{22} & \bar{a}_{21} & \bar{a}_{22} & u_{21} & u_{22} & u_{23} & u_{24} \\ \hline b_{11} & b_{12} & b_{11} & b_{12} & v_{11} & v_{12} & v_{13} & v_{14} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} hf_{n+v_1} \\ hf_{n+v_2} \\ h^2 f'_{n+v_1} \\ h^2 f'_{n+v_2} \\ \hline y_{n+3} \\ y_{n+2} \\ y_{n+1} \\ y_n \end{array}\right] \tag{39}$$

Using the order condition in Corollary 1, with $q = p = 6$, together with the strong regularity condition in Theorem 3 we obtain the A-stable SR-SD-GLM, characterized by the abscissa vector $v = [4,\ \frac{7}{2}]^T$,

$$A = \begin{bmatrix} \frac{5124}{64933} & \frac{55296}{64933} \\ -\frac{8910013534467}{35476390775936} & \frac{1305811139439}{277159302937} \end{bmatrix} \quad \bar{A} = \begin{bmatrix} \frac{1224}{64933} & 0 \\ \frac{4169762814 21}{35476390775936} & -\frac{2675841}{34147112} \end{bmatrix}$$

,

$$B = \begin{bmatrix} \frac{865900}{4268389} & \frac{3166720}{4268389} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -\frac{19312}{4268389} & -\frac{213248}{4268389} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

,

$$U = \begin{bmatrix} \frac{70272}{64933} & -\frac{6264}{64933} & \frac{1024}{64933} & -\frac{9}{5903} \\ \frac{18859017509003}{17738195387968} & -\frac{5212931484425}{70952781551872} & \frac{201412580229}{17738195387968} & -\frac{6909756421}{6450252868352} \end{bmatrix}$$

and

$$V = \begin{bmatrix} \frac{13632496}{12805167} & -\frac{320412}{4268389} & \frac{49264}{4268389} & -\frac{13885}{12805167} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$V_1 = r_1(V) = \left(\frac{13632496}{12805167} \; -\frac{320412}{4268389} \; \frac{49264}{4268389} \; -\frac{13885}{12805167}\right)$, $\lambda^{(1)} = \frac{4313209427583}{8634059546624}$, $\lambda^{(2)} = -\frac{690790519167}{474628980736}$. The stability polynomial is given by

$$\begin{aligned} \pi(w,z) = {} & 179928(w-1)\big(w(54609w^2 - 4191w + 709) - 75\big)z^4 \\ & + 1260\big(w(w(38600841w^2 + 40903632w - 5974276) + 1052912) - 104958\big)z^3 \\ & + 120\big(7w(w(4355781105w^2 + 3347537496w - 259637170) + 41770088) - 28137741\big)z \\ & - 3\big(w(w(3w(87271051415w - 48425683152) + 4722922036) - 487589200) + 35110215\big)z^2 \\ & - 519464(w-1)\big(w(w(12805167w - 827329) + 133907) - 13885\big). \end{aligned}$$

and the corresponding boundary locus plot is given in Figure 2 for $k = 4$. The method's error constants are $C_7(4) = -\frac{834}{2272655}h^7y^{(7)}(x_n)$, $C_7(\frac{7}{2}) = -\frac{128979100061}{532145861639040}h^7y^{(7)}(x_n)$, and $C_7(4) = -\frac{648133}{2689085070}h^7y^{(7)}(x_n)$, respectively.

3.2.5 2-stage 5-value strongly regular SD-GLM ($q = p = 7$)

Similarly, for $k = 5$, choosing the abscissae vector $v = [5, \; \frac{9}{2}]^T$, we obtain the A-stable strongly regular SD-GLM whose coefficient matrices are defined by

$$A = \begin{bmatrix} \frac{1234860}{11379047} & \frac{9216000}{11379047} \\ \frac{2312678409723625}{161904619326519872} & \frac{9721437492267580}{22767837092791857} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} \frac{120600}{11379047} & 0 \\ \frac{220273694449625}{80952309663259936} & -\frac{464598175}{5335616616} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{40859140}{222317359} & \frac{1494277120}{2000856231} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -\frac{664640}{222317359} & -\frac{22040320}{666952077} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{12546000}{11379047} & -\frac{1461000}{11379047} & \frac{356000}{11379047} & -\frac{68625}{11379047} & \frac{6672}{11379047} \\ \frac{2083733357018897875}{1942855431918238464} & -\frac{7773521655308671 25}{8742849443632073088} & \frac{6455532911028869}{323809238653039744} & -\frac{3595204639401875}{971427715959119232} & \frac{61579114694743 75}{1748569888726414 6176} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{724501060}{666952077} & -\frac{642211660}{6002568693} & \frac{5550440}{222317359} & -\frac{3145735}{666952077} & \frac{2720548}{6002568693} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\lambda^{(1)} = \frac{483154886087881}{966611926226560}$, $\lambda^{(2)} = -\frac{47674711917347}{19882627489792}$, and
$V_1 = r_1(V) = \begin{pmatrix} \frac{724501060}{666952077} & -\frac{642211660}{6002568693} & \frac{5550440}{222317359} & -\frac{3145735}{666952077} & \frac{2720548}{6002568693} \end{pmatrix}$. The stability polynomial is given by

$$\begin{aligned} \pi(w,z) = {} & 1394739000\,(w-1)\big(361555w^4 - 388890w^3 + 40690w^2 - 3778w + 196\big)\,z^4 \\ & + 8100\,\big(480847987285w^5 + 390635399700w^4 - 71247056300w^3 + 93393789720w^2 \\ & \qquad - 371155450w + 366902452\big)\,z^3 \\ & - 45\,\big(1350815808448805w^5 - 972160890579900w^4 + 71676419595100w^3 \\ & \qquad - 76223002313480w^2 + 2748878315775w - 256820806236\big)\,z^2 \\ & + 3360\,\big(435436243488580w^5 + 342948648516975w^4 - 26594440676650w^3 \\ & \qquad + 5910815811830w^2 - 218992381245w + 20826449247\big)\,z \\ & - 91032376\,(w-1)\big(5999568693w^4 - 6087614574w^3 + 1118437317w^2 - 25591067w + 2720548\big). \end{aligned}$$

and the corresponding boundary locus plot is given in Figure 2 for $k = 5$. The error constants are $C_8(5) = -\frac{14325}{79653329}h^8y^{(8)}(x_n)$, $C_8(\frac{9}{2}) = -\frac{521434495900057975}{52223954009962249912 32}h^8y^{(8)}(x_n)$, and $C_8(5) = -\frac{7451027}{56023974468}h^8y^{(8)}(x_n)$, respectively.

3.2.6 2-stage 6-value strongly regular SD-GLM ($q = p = 8$)

Also, choosing the abscissae $v = [6,\ \frac{11}{2}]^T$, we obtain the A-stable strongly regular SD-GLM whose coefficient matrices are defined by

$$A = \begin{bmatrix} \frac{9093780}{70850681} & \frac{55296000}{70850681} \\ \frac{19902950060113336125}{474416729094014227 84} & \frac{29462808888810648 80}{741276139209397231} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} \frac{379800}{70850681} & 0 \\ -\frac{46936355624259359 85}{15181335331008455290 88} & -\frac{15890930865}{167400200816} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{22964342940}{136012663163} & \frac{101798645760}{136012663163} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -\frac{231146280}{136012663163} & -\frac{2596008960}{136012663163} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} \frac{79401600}{70850681} & -\frac{11344500}{70850681} & \frac{3664000}{70850681} \\ \frac{1635647092455228646671}{151813353310084552908 8} & -\frac{6040706703996871792 95}{6072534132403382116352} & \frac{2164240713605195175 9}{7590667665504227645 44} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} -\frac{1056375}{70850681} & \frac{205056}{70850681} & -\frac{19100}{70850681} \\ -\frac{23587492549207877505}{3036267066201691058176} & \frac{2214923690306892635}{151813353310084552908 8} & -\frac{807533769072720639}{6072534132403382116352} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{150779000448}{136012663163} & -\frac{19447521240}{136012663163} & \frac{6119371840}{136012663163} & -\frac{1743004845}{136012663163} & \frac{335967168}{136012663163} & -\frac{31150208}{136012663163} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\lambda^{(1)} = \frac{4065889694915231053}{812121590193271603 2}$, $\lambda^{(2)} = -\frac{2318207132315465833}{521246623244746752}$, and
$V_1 = r_1(V) = (\frac{150779000448}{136012663163} \ -\frac{19447521240}{136012663163} \ \frac{6119371840}{136012663163} \ -\frac{1743004845}{136012663163} \ \frac{335967168}{136012663163} \ -\frac{31150208}{136012663163})$.
The corresponding boundary locus plot is given in Figure 2 for $k = 6$. The error constants are $C_9(6) = -\frac{97175}{991909534}h^9y^{(9)}(x_n)$, $C_9(\frac{11}{2}) = -\frac{14797895503041912089}{340061911414589398515712}h^9y^{(9)}(x_n)$, and $C_9(6) = -\frac{620140233}{7616709137128}h^9y^{(9)}(x_n)$, respectively.

3.2.7 2-stage 7-value strongly regular SD-GLM ($q = p = 9$)

Similarly, choosing the abscissae values $v = [7, \ \frac{13}{2}]^T$, we obtain the $A(88^0)$-stable strongly regular SD-GLM whose coefficient matrices are defined by

$$A = \begin{bmatrix} \frac{14254502220}{100030266319} & \frac{75866112000}{100030266319} \\ \frac{1136283913848968623545872625}{181502142268875134997030318 08} & \frac{3341214084638105684401365}{8862409290472418701026871} \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} \frac{178252200}{100030266319} & 0 \\ -\frac{3228737706217935721995 4125}{4537553556721878374925757952} & -\frac{289536537145275}{28351128886402 88} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{13880310984540}{88597277770009} & \frac{66263116308480}{88597277770009} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -\frac{54779447520}{88597277770009} & -\frac{628751531520}{88597277770009} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \ U_2 \end{bmatrix}$$

with

$$U_1 = \begin{bmatrix} \frac{113738662800}{100030266319} & -\frac{19164404700}{100030266319} & \frac{7693490000}{100030266319} & -\frac{2949242625}{100030266319} \\ \frac{7834046161420067414433155 8875}{7260085690755005399881212 7232} & -\frac{7622149341098994833160246525}{7260085690755005399881212 7232} & \frac{2581948317929605526804632107}{7260085690755005399881212 7232} & -\frac{1136529071813494172235048 75}{9075107113443756749851515904} \end{bmatrix}$$

$$U_2 = \begin{bmatrix} \frac{857296944}{100030266319} & -\frac{159529300}{100030266319} & \frac{13993200}{100030266319} \\ \frac{25110660145038007764100 3625}{726008569075500539988121 27232} & -\frac{45151219651752217845184875}{726008569075500539988121 27232} & \frac{38641921709366388284030 25}{726008569075500539988121 27232} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{100248496342974}{88597277770009} & -\frac{16234420579944}{88597277770009} & \frac{6452466761510}{88597277770009} & -\frac{2462050052595}{88597277770009} & \frac{713755142394}{88597277770009} & -\frac{132585234124}{88597277770009} & \frac{11615389794}{88597277770009} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\lambda^{(1)} = \frac{1357093993367752794435 23}{2703613257193617459855 36}$, $\lambda^{(2)} = -\frac{4204626974247255756323 5}{3319434626812321038336}$

The corresponding boundary locus plot is given in Figure 2 for $k = 7$. The error constants are $C_{10}(7) = -\frac{5765585}{100030266319}h^{10}y^{(10)}(x_n)$, $C_{10}(\frac{13}{2}) = -\frac{17537155649084807886958 1005}{92929096841664069118479522856 96}h^{10}y^{(10)}(x_n)$, and $C_{10}(7) = -\frac{1520536650503}{28351128886402880}h^{10}y^{(10)}(x_n)$, respectively.

4 Numerical Experiments and Discussion

In this section, consider the implementation of the proposed strongly regular second derivative general linear methods (SR-SD-GLMs) in (19). The new SR-SD-GLMs have the advantage of high order, with a special superconvergence property in which all the stage methods and the output method have uniform order of accuracy, which is known to improve the numerical order of convergence when solving highly stiff problems without suffering from order reduction. The order three SR-SD-GLM (32) is used to illustrate the equilibrium preserving property of the proposed methods in (19), and then compared with the strongly regular GLMs of the same order in [14] and MATLAB ODE solutions.Consider the following problems

Problem 1 [22]

$$y'(x) = -\frac{1}{3}\big(y(x)-1\big)^3 + \frac{20}{3}, \qquad y(0) = 3.5, \qquad x \geq 0. \tag{40}$$

At the critical point $y'(x) = f(\hat{y}(x)) = 0$, this problem has an asymptotic solution of $\hat{y}(x) = \sqrt[3]{20}+1$. We apply the strongly regular SD-GLM (32) to examine the critical solutions using the step size $h = 0.1$. The results obtained are given in Table 1, while the graphical equilibrium solution is given in Figure 3. From Table 1, observe that the result of the strongly regular SD-GLM (32) gives the equilibrium solution at $x = 4$. In contrast, that of [14] achieved equilibrum solution at $x = 5$ and they both remained the same (i.e., preserving the asymptotic solution) for higher values of x. However, we observe that the MATLAB `ode15s` and `ode23tb` only gave an approximate solution to

the exact asymptotic solution, this appear to be so because their underlying methods are not regular or strongly regular.

Table 1: Numerical results for Problem (1) using the strongly regular SD-GLM(32)

x	SRSDGLM (32)	SRGLM [14]	MATLAB ode15s	MATLAB ode23tb	Asymptotic solution $\hat{y}(x) = \sqrt[3]{20} + 1$
0.0	3.500000000000000	3.500000000000000	3.500000000000000	3.500000000000000	3.714417616594907
1.0	3.714266091311190	3.714200832269977	3.712097378424792	3.711616440433633	3.714417616594907
2.0	3.714417516821356	3.714417477879577	3.714067688036725	3.713908504395139	3.714417616594907
3.0	3.714417616529213	3.714417616506191	3.714373750822739	3.714384893718052	3.714417616594907
4.0	3.714417616594907	3.714417616594851	3.714412904399998	3.714410950668706	3.714417616594907
5.0	3.714417616594907	3.714417616594907	3.714417359979546	3.714416507461680	3.714417616594907
6.0	3.714417616594907	3.714417616594907	3.714412802123729	3.714417436298515	3.714417616594907
7.0	3.714417616594907	3.714417616594907	3.714422889525964	3.714417586728544	3.714417616594907
8.0	3.714417616594907	3.714417616594907	3.714416397844337	3.714417610996349	3.714417616594907
9.0	3.714417616594907	3.714417616594907	3.714418420249741	3.714417652792995	3.714417616594907
10.0	3.714417616594907	3.714417616594907	3.714417917987803	3.714417616422044	3.714417616594907

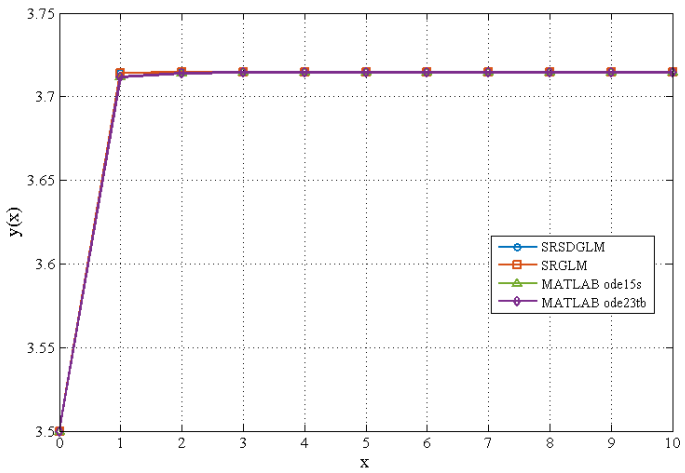


Fig. 3: Graphical solution of the strongly regular SD-GLM (32) comparing with SRGLM [14] and MATLAB ode solutions

Problem 2 [22]

$$\begin{aligned} y_1'(x) &= -\frac{1}{2}y_1(x) + y_2(x), \quad y_1(0) = 0.4, \\ y_2'(x) &= y_1(x) - 2y_2(x), \quad y_2(0) = 0.8, \quad x \geq 0. \end{aligned} \tag{41}$$

Here, the solution at equilibrium point is $\hat{y}_1(x) = 2\hat{y}_2(x)$. Again, we solve the problem using a step size $h = 0.1$. The results are shown in Table 2, and the graphical solutions are given in Figure 4. The result presented in Table 2 shows that the strongly regular SD-GLM(32) yields the asymptotic solution $\hat{y}_1(x) = 0.64$ and $\hat{y}_2(x) = 0.32$ at $x = 12.90$, while, that of [14] yields the asymtotic solution at $x = 20$ (both preserving the asymptotic solution), whereas, the solution of MATLAB `ode15s` and `ode23tb` yields an approximate solution to the solution. Hence, the strongly regular SD-GLM (32) outperforms the existing methods. Note that MATLAB ode15s and ode23t are not regular methods hence they exhibit some difficulties in tracing the asymptotic solution of (2) compare to the results from (32) and [14].

Table 2: Numerical results for Problem (2).

x	y_i(x)	SRSDGLM (32)	SRGLM [14]	MATLAB ode15s	MATLAB ode23tb
0.0	$y_1(x)$	0.400000000000000	0.400000000000000	0.400000000000000	0.400000000000000
	$y_2(x)$	0.800000000000000	0.800000000000000	0.800000000000000	0.800000000000000
5.0	$y_1(x)$	0.639999654450529	0.639985939390262	0.639998690321396	0.639994623851967
	$y_2(x)$	0.320000691098941	0.320028121219476	0.320002619357209	0.320010752296067
10.0	$y_1(x)$	0.639999999999502	0.639999999176247	0.639999998841049	0.640001191864220
	$y_2(x)$	0.320000000000995	0.320000001647506	0.320000002317903	0.319997616263157
12.90	$y_1(x)$	0.640000000000000	0.639999999997982	0.639999999932315	0.640000091865420
	$y_2(x)$	0.320000000000000	0.320000000006739	0.320000000005231	0.319997649576284
15.0	$y_1(x)$	0.640000000000000	0.639999999999952	0.639999999920585	0.640000138911277
	$y_2(x)$	0.320000000000000	0.320000000000097	0.320000000158831	0.319999722177445
20.0	$y_1(x)$	0.640000000000000	0.640000000000000	0.640000000014899	0.640000020015470
	$y_2(x)$	0.320000000000000	0.320000000000000	0.319999999970202	0.319999995996905
25.0	$y_1(x)$	0.640000000000000	0.640000000000000	0.640000000004578	0.639999995562660
	$y_2(x)$	0.320000000000000	0.320000000000000	0.319999999990844	0.320000000874680
30.0	$y_1(x)$	0.640000000000000	0.640000000000000	0.640000000001209	0.639999999977057
	$y_2(x)$	0.320000000000000	0.320000000000000	0.319999999975830	0.320000000045886

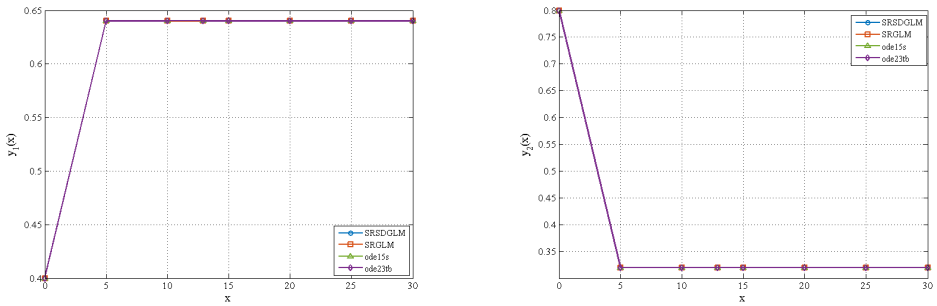


Fig. 4: Graphical solution of the strongly regular SD-GLM (32) comparing with SRGLM [14] and MATLAB ode solutions

Problem 3 [7] Consider the system

$$\begin{aligned} y_1' &= -8y_1 + 7y_2, \qquad y_1(0) = 1, \\ y_2' &= 42y_1 - 43y_2, \qquad y_2(0) = 8, \end{aligned} \tag{42}$$

on the interval $x \in [0, 40]$. The exact solutions are given by

$$y_1(x) = 2e^{-x} - e^{-50x}, \qquad y_2(x) = 2e^{-x} + 6e^{-50x}.$$

Problem 3 has eigenvalues $\lambda_1 = -1,\ \lambda_2 = -50$ and the stiffness ratio is 50. The critical solution is $\hat{y}_1(x) = \hat{y}_2(x) = 0$, whenever $\hat{y}_1' = \hat{y}_2' = 0$. The asymptotic solution obtained using the SR-SD-GLM (32), comparing with existing methods is given in Table 3. The SR-SD-GLM (32) converged to the asymptotic solution at $x = 16.0$, thus outperforming the SRGLM [14], MATLAB ode15s and ode23tb. The SR-SD-GLM (32) converged to the critical solution faster because of its inherent second derivative terms, unlike the traditional SRGLM. It is obvious (See Figures 3, 4, 5) that the proposed SR-SD-GLMs preserve the asymptotic solutions of the ODEs.

Table 3: Numerical results for Problem (3)

x	y_i(x)	SRSDGLM (32)	SRGLM [14]	MATLAB ode15s	MATLAB ode23tb
0.0	$y_1(x)$	1.000000000000000	1.000000000000000	1.000000000000000	1.000000000000000
	$y_2(x)$	8.000000000000000	8.000000000000000	8.000000000000000	8.000000000000000
4.0	$y_1(x)$	0.000000006445659	0.000697478955399	0.036122540945214	0.035289220442850
	$y_2(x)$	0.000000006445659	0.000697482666973	0.036122496186881	0.035289220433127
8.0	$y_1(x)$	0.000000130524592	0.000000208696623	0.000675282527846	0.000619952350109
	$y_2(x)$	0.000000130524592	0.000000208697734	0.000675282523005	0.000619952350109
12.0	$y_1(x)$	0.000000000000172	0.000000000062445	0.000012579998275	0.000010833934458
	$y_2(x)$	0.000000000000172	0.000000000062446	0.000012579998275	0.000010833934458
16.0	$y_1(x)$	0.000000000000000	0.000000000000019	0.000000214532768	0.000000127823373
	$y_2(x)$	0.000000000000000	0.000000000000019	0.000000214532768	0.000000127823373
20.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000006624386	−0.000000001897195
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000006624386	−0.000000001897195
22.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000004508323	0.0000000000084505
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000004508323	0.0000000000084505
24.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000002227672	0.000000000040075
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000002227672	0.000000000040075
26.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000000842687	−0.000000000008967
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000000842687	−0.000000000008967
28.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000000298933	0.000000000000399
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000000298933	0.000000000000399
30.0	$y_1(x)$	0.000000000000000	0.000000000000000	0.000000000034583	0.000000000000198
	$y_2(x)$	0.000000000000000	0.000000000000000	0.000000000034583	0.000000000000198

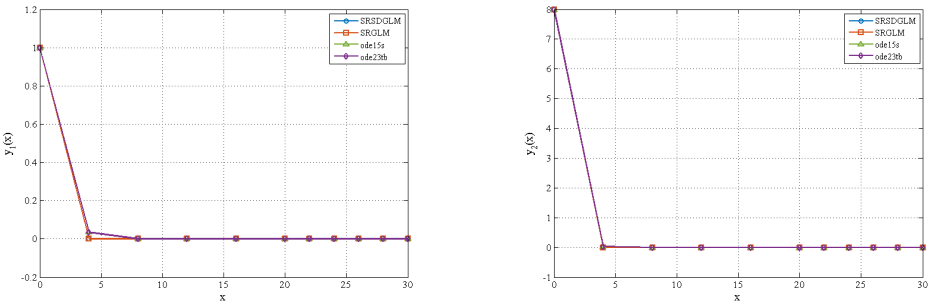


Fig. 5: Graphical solution of the strongly regular SD-GLM (32) comparing with SRGLM [14] and MATLAB ode solutions

5 Conclusion

Regularity ensures that the external stages of GLMs are stable and mimic the asymptotic behaviour of the ODE (1). In contrast, strong regularity ensures that both the external and internal dynamics remain controlled under stiffness, while the method converges to the asymptotic values of the ODE (1). This paper considers a new family of A-stable SD-GLMs (19) derived under regularity and strong regularity conditions in Theorem 3. The results of some numerical experiments presented show that the proposed SD-GLMs are strongly regular. The methods preserve the asymptotic behaviour of the solution of the ODE (1) considered in the numerical experiments.

Declarations

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Conflict of Interest

The authors declare that there is no conflict of interest.

Ethics approval and consent to participate

Not applicable

Consent for publication

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Author contributions

All authors contributed to this work and approved the final manuscript.

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